

Aggregation operators of general aimed information

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Abstract

The aim of this paper is to introduce, by axiomatic way, the measure of the *general* (i.e. without probability) *aimed information* for crisp sets.

We translate the problem in a system of functional equation and we present a class of solutions; in the case of independence we characterize the solution.

Finally, we give some aggregation operators for this form of information measures.

Keywords: Information, functional equations, aggregation operators.

1. Introduction

In the setting of *general* (i.e. without probability) information, we shall characterize a measure of information of a crisp variable event, which has the aim to achieve a fixed crisp event. We call this new measure *aimed information*.

Example: the measure of information about sets of men from different states, who achieve the same diploma.

In the sect. 2 we shall recall some preliminaries following the information theory of Kampé De Fériet and Forte.

The sect. 3 is devoted to the introduction of the aimed information, by axiomatic way, and the statement of the problem in a system of functional equations. For this system we shall find a class of solutions; the solution has been found in the case of independence.

Then, in the sect. 4 some aggregation operators for the aimed information are proposed; finally, in the sect. 5 we give the conclusion.

2. Preliminaires

Let Ω be an abstract space and \mathcal{A} a σ -algebra of all not-empty subsets of Ω , $\mathcal{A} \subset \mathcal{P}(\Omega)$, such that (Ω, \mathcal{A}) is measurable. We refer to [7] for all knowledges and operations about crisp sets. We recall the definition of measure of information as in [8, 5].

Definition 2.1 The measure $J(\cdot)$ of the general information is a map

$$J(\cdot) : \mathcal{A} \rightarrow [0, +\infty]$$

such that $\forall A_1, A_2 \in \mathcal{A}$:

- (i) $A_1 \supset A_2 \Rightarrow J(A_1) \leq J(A_2)$,
- (ii) $J(\emptyset) = +\infty$, $J(X) = 0$,
- (iii) $J(A_1 \cap A_2) = J(A_1) + J(A_2)$, if $A_1 \cap A_2 \neq \emptyset$.

As regard to (iii), we assume it as the definition of independence with respect to the information J (shortly J -independence).

If J satisfies the previous properties [(i)-(iii)], we call the triple (Ω, \mathcal{A}, J) *information space* [2, 5].

3. General aimed information

Let $S \in \mathcal{A}$ be a fixed set, which represents the object of the events $A \in \mathcal{A}$.

Now, we are ready to give, by axiomatic way, the definition of general aimed information.

Definition 3.1 The *general information* of $A \in \mathcal{A}$, *aimed* by a fixed event $S \in \mathcal{A}$ is a map

$$I(\cdot \rightarrow S) : \mathcal{A} \rightarrow [0, +\infty]$$

such that

- (j) $A \supset A' \Rightarrow I(A \rightarrow S) \leq I(A' \rightarrow S)$,
 $\forall A, A' \in \mathcal{A}$,
- (jj) $I(\emptyset \rightarrow S) = +\infty$, $I(\Omega \rightarrow S) = 0$,
- (jjj) $I((A \cap B) \rightarrow S) = I(A \rightarrow S) + I(B \rightarrow S)$,
if $A \cap B \neq \emptyset$.

As regard to (jjj), we assume it as the definition of aimed-independence with respect to the information I (shortly I -aimed-independence).

Using these properties, we shall give a class of general aimed information. Therefore, we plan out a system of functional equations [1], for which we shall present a class of solutions. It is possible to find a solution of the system if the condition of independence has been added.

4. Statement of the problem

Let (Ω, \mathcal{A}, J) be an information space.

Taking into account the previous axiomatic statement, fixed $S \in \mathcal{A}$, we shall looking for a class of aimed measures $I(A \rightarrow S)$, by supposing that $I(A \rightarrow S)$ depends only on $J(A \cap S)$, $J(A)$, $J(S)$. For this reason, fixed S , we restrict our definition to the following family:

$$\mathcal{A}_S = \{A \in \mathcal{A} / A \cap S \neq \emptyset\};$$

at least \mathcal{A}_S contains the whole space Ω , so this family is not empty. We are looking for the aimed information $I(\cdot \rightarrow S)$ as a function:

$$I(\cdot \rightarrow S) : \mathcal{A}_S \rightarrow [0, +\infty]$$

and a function $\Phi : D \rightarrow [0, +\infty]$, such that

$$I(A \rightarrow S) = \Phi \left(J(A \cap S), J(A), J(S) \right). \quad (1)$$

The domain D will be specified later.

Putting $x = J(A \cap S)$, $y = J(A)$, $z = J(S)$, $y' = J(A')$, $t = J(B)$, $x \geq y$, $x \geq z$, $x, y, z, x', t \in [0, +\infty]$, with $B \in \mathcal{A}_S$, we recognize that the domain D of the function Φ is the following

$$D = \{(x, y, z) / x \geq y, x \geq z\}.$$

The (1) becomes

$$I(A \rightarrow S) = \Phi(x, y, z). \quad (2)$$

From the properties (j), (jj), (jjj) above, we get the following system of functional equations:

$$\begin{cases} (a) & \Phi(x, y, z) \leq \Phi(x', y', z) \quad x \leq x', y \leq y', \\ (b) & \Phi(+\infty, +\infty, z) = +\infty, \\ (c) & \Phi(z, 0, z) = 0, \\ (d) & \Phi(y + t + z, y + t, z) = \\ & \Phi(y + z, y, z) + \Phi(t + z, t, z). \end{cases}$$

For the (d), fixed S , and given $A, B \in \mathcal{A}_S$, we suppose that A, B and S are J -independent, from (iii) $J(A \cap B \cap S) = J(A) + J(B) + J(S)$. Moreover, by (1), we get

$$\begin{aligned} I((A \cap B) \rightarrow S) &= \Phi \left(J(A \cap B \cap S), J(A \cap B), J(S) \right) \\ &= \Phi \left(J(A) + J(B) + J(S), J(A) + J(B), J(S) \right) \end{aligned} \quad (3)$$

and

$$I(B \rightarrow S) = \Phi \left(J(B \cap S), J(B), J(S) \right). \quad (4)$$

Taking into account (2), we obtain the equation (d) by (1), (3) and (4).

We are looking for an universal solution of the system [(a)-(d)], i.e. a solution of the equations and the inequality in every proper space.

First, we can give the following result:

Proposition 4.1 *A class of solutions of the system [(a)-(c)] is*

$$\Phi_h(x, y, z) = h^{-1} \left[h(x) + h(y) - h(z) \right], \quad (5)$$

where h is any continuous, positive, increasing function with $h(0) = 0$ and $h(+\infty) = +\infty$.

Proof. The (a) is valid as h is increasing and $x \leq x', y \leq y'$.

The (b) and (c) are satisfied from the values of h .

□

By using the J - aimed independence and the form (5) with the hypothesis (on h) of the Proposition (4.1), the system [(a)-(d)] becomes:

$$\begin{cases} (a') & h^{-1} \left[h(x) + h(y) - h(z) \right] \leq \\ & h^{-1} \left[h(x') + h(y') - h(z) \right] \quad x \leq x', y \leq y', \\ (b') & h^{-1} \left[h(+\infty) + h(+\infty) - h(z) \right] = +\infty, \\ (c') & h^{-1} \left[h(z) + h(0) - h(z) \right] = 0, \\ (d') & h^{-1} \left[h(y + t + z) + h(y + t) - h(z) \right] = \\ & h^{-1} \left[h(y + z) + h(y) - h(z) \right] + \\ & h^{-1} \left[h(t + z) + h(t) - h(z) \right]. \end{cases}$$

Proposition 4.2 *The solution of the system [(a')-(d')] is*

$$h(x) = c \quad x \quad c > 0. \quad (6)$$

Proof. It is easy to see that (6) is solution of the system [(a')-(d')].

In fact, (a'), (b'), (c') are identically satisfied by (6) through the hypothesis of h . As regards to (d'), putting

$$\varphi(y, z) = h^{-1} \left[h(y + z) + h(y) - h(z) \right], \quad (7)$$

(d') is

$$\varphi(y + t, z) = \varphi(y, z) + \varphi(t, z). \quad (8)$$

Fixed $z = z^*$ the (8) is the classical Cauchy equation [1], whose solution is

$$\varphi(y, z^*) = \lambda(z^*)y. \quad (9)$$

From (7), we get

$$\lambda(z^*)y = h^{-1} \left[h(y + z^*) + h(y) - h(z^*) \right], \text{ i.e.}$$

$$h \left(\lambda(z^*)y \right) = h(y + z^*) + h(y) - h(z^*). \quad (10)$$

If $z^* = 0$, as $h(0) = 0$, from (10) it is $h \left(\lambda(0)y \right) = 2 h(y)$, but h is injective, so $\lambda(0) = 2$ and it implies $\lambda(z) = 2 \quad \forall z$. The function h satisfies the following condition:

$$h(2y) = 2 h(y). \quad (11)$$

From (7), we have obtained

$$\varphi(y, z) = 2 \ y = h^{-1} \left[h(y + z) + h(y) - h(z) \right], \text{ i.e.}$$

$$h(2 \ y) = h(y + z) + h(y) - h(z). \quad (12)$$

The two conditions (11) and (12) imply

$$h(y + z) = h(y) + h(z). \quad (13)$$

We have found again the Cauchy equation, whose solution is $h(x) = c \ x$, with c real positive number, moreover, the solution of the system [(a')-(d')] is (6). \square

Taking into account the Propositions (4.1) and (4.2), we have the following theorem:

Theorem 4.3 *The solution of the system [(a)-(d)] is*

$$\Phi(x, y, z) = x + y - z. \quad (14)$$

\square

So, we have found a class of I -aimed information: from (5) we get

$$I_h(A \rightarrow S) =$$

$$h^{-1} \left(h(J(A \cap S)) + h(J(A)) - h(J(S)) \right),$$

where h is any continuous, positive, increasing function with $h(0) = 0$ and $h(+\infty) = +\infty$.

In the case of independence, from (14), we have:

$$I(A \rightarrow S) = J(A \cap S) + J(A) - J(S).$$

5. Aggregation operators

We know [3, 4, 6, 9, 10] that an aggregation operator is a procedure by which a unique value can be associated to the result obtained through different tests or different data base. The unique value is kind of arithmetic mean. We think that it can be useful for the applications to give some aggregation operators for aimed general information, too.

From now on, we fix $S \in \mathcal{A}$. Let $\mathcal{I}_S \subset [0, +\infty]^n$ be the family of the aimed general information $I(\cdot \rightarrow S)$.

The aggregation operator L of n aimed general information $I(A_k \rightarrow S), A_k \in \mathcal{A}, k = 1, \dots, n$, is a map $L : \mathcal{I}_S \rightarrow [0, +\infty]$ [3, 6, 10] defined by the following properties:

(I) *idempotence*:

if $I(A_k \rightarrow S) = \lambda, \forall k = 1, \dots, n$,

then $L(\underbrace{\lambda, \dots, \lambda}_{n \text{ times}}) = \lambda$;

(II) *monotonicity*:

$$I(A_1 \rightarrow S) \leq I(A_1' \rightarrow S) \implies$$

$$L \left(I(A_1 \rightarrow S), \dots, I(A_k \rightarrow S), \dots, I(A_n \rightarrow S) \right) \leq L \left(I(A_1' \rightarrow S), \dots, I(A_k \rightarrow S), \dots, I(A_n \rightarrow S) \right);$$

(III) *continuity from below*:

$$I(A_{1/m} \rightarrow S) \nearrow I(A_1 \rightarrow S) \implies$$

$$L \left(I(A_{1/m} \rightarrow S), \dots, I(A_k \rightarrow S), \dots, I(A_n \rightarrow S) \right) \nearrow L \left(I(A_1 \rightarrow S), \dots, I(A_k \rightarrow S), \dots, I(A_n \rightarrow S) \right).$$

Putting $I(A_k \rightarrow S) = x_k, k = 1, \dots, n, I(A_1' \rightarrow S) = x_1', I(A_{1/m} \rightarrow S) = x_{1/m}$, we obtain:

$$\begin{cases} (I') \ L(\underbrace{\lambda, \dots, \lambda}_{n \text{ times}}) = \lambda, \\ (II') \ x_1 \leq x_1' \implies \\ \quad L(x_1, \dots, x_k, \dots, x_n) \leq L(x_1', \dots, x_k, \dots, x_n) \\ (III') \ x_{1/m} \nearrow x_1 \implies \\ \quad L(x_{1/m}, \dots, x_k, \dots, x_n) \nearrow L(x_1, \dots, x_k, \dots, x_n). \end{cases}$$

For the solutions of the system [(I')-(III')], we propose the following propositions, whose proofs are immediate:

Proposition 5.1 *Two natural solutions of the system [(I')-(III')] are*

$$L(x_1, \dots, x_k, \dots, x_n) = \bigvee_{k=1}^n x_k;$$

$$L(x_1, \dots, x_k, \dots, x_n) = \bigwedge_{k=1}^n x_k.$$

\square

Proposition 5.2 *A class of solution of the system [(I')-(III')] is*

$$L(x_1, \dots, x_k, \dots, x_n) = h^{-1} \left(\frac{h(x_1) + \dots + h(x_n)}{n} \right)$$

where $h : [0, +\infty] \rightarrow [0, +\infty]$ is any continuous, increasing function $h(0) = 0, h(+\infty) = +\infty$. \square

Remark.

1) If $h(x) = x$, the aggregation operator L is the arithmetic mean.

2) These results are valid also in fuzzy setting.

6. Conclusion

In this paper, we have presented a definition, by axiomatic way, of general aimed information $I(A \rightarrow S)$ in the crisp setting.

If we suppose that this information is linked to $J(A \cap S), J(A), J(S)$, we found a class of solution of a system of functional equations. If we consider the independence, we obtained the solution.

Then, we have given some classes of aggregation operators of these general aimed information.

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