Effective Discretization Scheme for the Heston Model: Implicit Solution Based Approach

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***Abstract*—Four novel discretization schemes that built on the implicit solution of the variance, are proposed to effectively simulate the Heston model. The idea behind these schemes is fundamentally different from those of the Euler and Milstein based discretization schemes, and the so-called nearly exact simulation approaches. Numerical experiments show that these new schemes are of both accuracy and fast convergence in option pricing, and seem almost comparable to the famous quadratic exponential scheme.**

Keywords-discretization scheme; Heston; implicit solution; exact simulation; QE scheme

I. INTRODUCTION

The stochastic volatility (SV) model has become the de facto standard model in the derivatives market (Grzelak, Oosterlee & Weeren, 2011**0**[1]; Haastrecht, Lord & Pelsser, 2014**[2]**). The Heston model, introduced by Heston(1993), now is one of the most widely used SV models, due to its nice market fit.

However, effective simulation to the Heston model is not a trivial problem. The traditional Euler and Milstein based discretization schemes, generally relying on the truncation techniques to deal with 'negative-variance' problem, cannot meet the accuracy and robust requirements in financial practice. Broadie & Kaya(2006) [3] propose an exact simulation scheme to the Heston model, building on the known conditional distribution of the variance process, to say, non-central Chi-squared distribution, but this exact approach is quite impractical due to its low efficiency.

Recently, interests have been focused on nearly exact simulation to the Heston model (see Broadie & Kaya(2006)[3]; Andersen (2008)[4]; Haastrecht & Pelsser (2010)[5]; Glasserman & Kim(2011)[6]; Tsea & Wana (2013)[7]; Fabrice Rouah (2013)[8]; etc). These 'nearly exact simulation' literature generally apply some simple distributions (or functions) with good computability to approximate the non-central Chi-squared distribution to sample the variance and the asset price, with trade-off of efficiency and accuracy. For example, Andersen(2008)[4] suggests the quadratic exponential function, Glasserman & Kim(2011)[6] use Gamma expansion, while Tsea & Wana(2013) [7] hire an inverse Gaussian distribution, to act as the approximation.

In this research, we keep moving forward, but along a quite different direction. We apply the implicit solution of the variance (Hanson, 2010) [9] to effectively simulate the Heston model. We develop four novel discretization schemes in this direction and show their accuracy and convergence by numerical experiments and comparisons with the famous quadratic exponential scheme[4].

II. AN IMPLICIT SOLUTION OF THE VARIANCE TO THE HESTON MODEL

The Heston model (Broadie & Kaya(2006), Glasserman & Kim(2011), Tsea & Wana(2013)) writes

$$\begin{cases} d\left(\ln S_t\right) = \left(r - \tfrac{1}{2}V_t\right)dt + \sqrt{V_t}\,dW_t^s \\ dV_t = \kappa\left(\theta - V_t\right)dt + \sigma_v\sqrt{V_t}\,dW_t^v \end{cases} \tag{1}$$

where $< dW_t^s, dW_t^v >= \rho dt$.

Hanson(2010)[9] provides an implicit solution of the variance in Equation (1), that is

$$V_t = e^{-\kappa t}\left(\frac{Y(t)}{2}\right)^2 \tag{2}$$

where $Y(t) = 2\sqrt{V_0} + \int_0^t e^{\frac{\kappa}{2}s}\left[\frac{\kappa\theta - \frac{1}{4}\sigma_v^2}{\sqrt{V(s)}}ds + \sigma_v dW_v(s)\right]$.

By changing the form $\int_0^t f(\cdot)ds$ in $Y(t)$ to $\int_t^{t+\Delta t} f(\cdot)ds$, we can rewrite:

$$Y(t+\Delta t) = 2e^{\frac{\kappa}{2}t}\sqrt{V_t} + \int_t^{t+\Delta t} e^{\frac{\kappa}{2}u}\left[\frac{\kappa\theta - \frac{1}{4}\sigma_v^2}{\sqrt{V(u)}}du + \sigma_v dW_v(u)\right]. \tag{3}$$

III. THE DISCRETIZATION SCHEME

A. Sampling the Variance Process

1) *For a small variance*: Since there exists a singular integral term in Equation (3), we need to carefully sample the variance near zero. Our method is: given ε_v a small enough positive value (for example, a critical value that corresponds to 0.1% the cumulative distribution value of the non-central Chi-squared distribution), when $\Delta t \to 0^+$, $\Delta t/\varepsilon_v \to 0^+$ and $\hat{V}_t < \varepsilon_v$, the variance at $t+\Delta t$ is then sampled as follows

$$\hat{V}_{t+\Delta t} = \max\left\{\varepsilon_v, e^{-\kappa\Delta t}\left[\sqrt{\hat{V}_t} + 0.5\left(\tfrac{\kappa\theta - \frac{1}{4}\sigma_v^2}{\sqrt{\hat{V}_t}}\Delta t + \sigma_v\Delta W_t^v\right)\right]^2\right\}. \tag{4}$$

2) *For a large variance*: When $\hat{V}_t \geq \varepsilon_v$, we apply proper numerical integration methods, such as the Euler method, the trapezoidal method and the Simpson method, to sample the variance $\hat{V}_{t+\Delta t}$. The common discretization scheme for $\hat{V}_{t+\Delta t}$ is

$$\hat{V}_{t+\Delta t} = e^{-\kappa(t+\Delta t)}\left(\frac{\hat{Y}_{t+\Delta t}}{2}\right)^2$$

where $\hat{Y}_{t+\Delta t}$ is sampled by the following three ways.

1) Euler method (for simplicity, hereinafter, we denote the implicit solution based discretization scheme built on the Euler method as **IS$_1$)**

$$\hat{Y}_{t+\Delta t} = e^{\frac{\kappa}{2}t}\left\{2\sqrt{\hat{V}_t} + \frac{\kappa\theta - \frac{1}{4}\sigma_v^2}{\sqrt{\hat{V}_t}}\Delta t + \sigma_v \Delta W_v(t)\right\}. \tag{5}$$

2)The trapezoidal method (IS$_2$)

$$\hat{Y}_{t+\Delta t} = 2e^{\frac{\kappa}{2}t}\sqrt{\hat{V}_t} + \frac{1}{2}e^{\frac{\kappa}{2}t}\left[\left(\kappa\theta - \tfrac{1}{4}\sigma_v^2\right)\left(\tfrac{1}{\sqrt{\hat{V}_t}} + \tfrac{e^{\frac{\kappa}{2}\Delta t}}{\sqrt{\hat{V}_{t+\Delta t}}}\right)\Delta t + \sigma_v\left(1 + e^{\frac{\kappa}{2}\Delta t}\right)\Delta W_v(t)\right] \tag{6}$$

Hence, $\sqrt{\hat{V}_{t+\Delta t}}$ can be sampled as the non-negative solution to equation

$$a_1 x^2 + b_1 x + c_1 = 0 \tag{7}$$

with coefficients $a_1 = \pm e^{\frac{\kappa}{2}\Delta t}$; $b_1 = -\left(\sqrt{\hat{V}_t} + \frac{h\Delta t}{4\sqrt{\hat{V}_t}} + \frac{1}{4}g\right)$; $c_1 = -\frac{1}{4}e^{\frac{\kappa}{2}\Delta t}h\Delta t$, $h = \kappa\theta - \frac{1}{4}\sigma_v^2$ and $g = \sigma_v\left(1 + \exp(\frac{\kappa}{2}\Delta t)\right)\Delta W_v(t)$.

It is easy to show the quadratic equation (7) always has a non-negative solution. This guarantees that the sampling of $\hat{V}_{t+\Delta t}$ by the **IS$_2$** is effective.

3)The Simpson rule

$$\hat{Y}_{t+\Delta t} = 2e^{\frac{\kappa}{2}t}\sqrt{\hat{V}_t} + \frac{1}{6}e^{\frac{\kappa}{2}t}\begin{bmatrix}\left(\kappa\theta - \frac{1}{4}\sigma_v^2\right)\left(\frac{1}{\sqrt{\hat{V}_t}} + 4\frac{e^{\frac{\kappa}{4}\Delta t}}{\sqrt{\hat{V}_{t+0.5\Delta t}}} + \frac{e^{\frac{\kappa}{2}\Delta t}}{\sqrt{\hat{V}_{t+\Delta t}}}\right)\Delta t \\ +\sigma_v\left(1 + 4e^{\frac{\kappa}{4}\Delta t} + e^{\frac{\kappa}{2}\Delta t}\right)\Delta W_v(t)\end{bmatrix} \tag{8}$$

We have two ways to approximate to $\sqrt{\hat{V}_{t+0.5\Delta t}}$ in Equation (8): $\frac{\sqrt{\hat{V}_t} + \sqrt{\hat{V}_{t+\Delta t}}}{2}$ and $\sqrt{\frac{\hat{V}_t + \hat{V}_{t+\Delta t}}{2}}$. If the first way is taken, $\sqrt{\hat{V}_{t+\Delta t}}$ can be sampled as one of non-negative solutions to equation

$$a_2 x^3 + b_2 x^2 + c_2 x + d_2 = 0 \tag{9}$$

with coefficients $a_2 = \pm e^{\frac{\kappa}{2}\Delta t}$; $b_2 = \pm e^{\frac{\kappa}{2}\Delta t}\sqrt{\hat{V}_t} - \left[\sqrt{\hat{V}_t} + \frac{1}{12}\left(h\Delta t\frac{1}{\sqrt{\hat{V}_t}} + g\right)\right]$; $c_2 = -\hat{V}_t - \frac{1}{12}\cdot$ $(h\Delta t + k\sqrt{\hat{V}_t}) - \frac{2}{3}e^{\frac{\kappa}{4}\Delta t}h\Delta t - \frac{1}{12}e^{\frac{\kappa}{2}\Delta t}h\Delta t; d_2 = -\frac{1}{12}e^{\frac{\kappa}{2}\Delta t}\sqrt{\hat{V}_t}h\Delta t$; $h = \left(\kappa\theta - \frac{1}{4}\sigma_v^2\right)$ and $g = \sigma_v\cdot\ (1 + 4e^{\frac{\kappa}{4}\Delta t} + e^{\frac{\kappa}{2}\Delta t})\Delta W_v(t)$.

For convenience of notation, we denotes this scheme as **IS$_3$.**

If the second approximation is taken, i.e., $\sqrt{\hat{V}_{t+0.5\Delta t}} \approx \sqrt{\frac{\hat{V}_t + \hat{V}_{t+\Delta t}}{2}}$, $\sqrt{\hat{V}_{t+\Delta t}}$ can be sampled as one of non-negative solutions to the following equation

$$\left(a_3 x^2 - b_3 x - c_3\right)^2\left(\hat{V}_t + x^2\right) = d_3 x^2 \tag{10}$$

with coefficients $a_3 = \pm e^{\frac{\kappa}{2}\Delta t}$; $b_3 = \sqrt{\hat{V}_t} + \frac{1}{12}h\Delta t\frac{1}{\sqrt{\hat{V}_t}} + \frac{1}{12}g$; $c_3 = \frac{1}{12}e^{\frac{\kappa}{2}\Delta t}h\Delta t$; $d_3 = \frac{2}{9}e^{\frac{\kappa}{2}\Delta t}\cdot(h\Delta t)^2$, and $h = \left(\kappa\theta - \frac{1}{4}\sigma_v^2\right); g = \sigma_v\left(1 + 4e^{\frac{\kappa}{4}\Delta t} + e^{\frac{\kappa}{2}\Delta t}\right)\Delta W_v(t)$.

We denotes this approach as **IS$_4$.**

Remark: During the sampling process, if the above equations (7), (9) and (10) have more than one non-negative roots, we may take the maximum one, the minimum one, the nearest neighbour to $\hat{V}_t$, or take one randomly. In this research, we use the third way. On the other hand, if any one of these equations has none of non-negative roots, we can use another $\Delta W_v(t)$ to make it has.

B. Sampling the Asset Price

Let $W_t^{(1)}$ and $W_t^{(2)}$ be two independent standard Weiner processes such that $W_t^s = \rho W_t^{(1)} + \sqrt{1-\rho^2}W_t^{(2)}; W_t^v = W_t^{(2)}$. Then Equation (1) can be uncoupled as

$$\begin{cases} d\left(\ln S_t\right) = \left(r - \frac{1}{2}V_t\right)dt + \sqrt{V_t}\left[\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}\right] \\ dV_t = \kappa(\theta - V_t)dt + \sigma_v\sqrt{V_t}dW_t^{(1)} \end{cases} \tag{11}$$

Integrating Equation (11), yields

$$\ln S_{t+\Delta t} = \ln S_t + r\Delta t - \frac{1}{2}\int_t^{t+\Delta t}V_s dt + \int_t^{t+\Delta t}\rho\sqrt{V_s}dW_s^{(1)} + \sqrt{1-\rho^2}\int_t^{t+\Delta t}\sqrt{V_s}dW_s^{(2)} \tag{12}$$

and

$$\int_t^{t+\Delta t}\sqrt{V_s}dW_s^{(1)} = \frac{1}{\sigma_v}\left(V_{t+\Delta t} - V_t - \kappa\theta\Delta t + \kappa\int_t^{t+\Delta t}V_s ds\right) \tag{13}$$

Substituting Equation (13) into Equation (12), yields

$$\ln S_{t+\Delta t} = \ln S_t + r\Delta t + \frac{\rho}{\sigma_v}(V_{t+\Delta t} - V_t - \kappa\theta\Delta t) + \left(\frac{\rho\kappa}{\sigma_v} - \frac{1}{2}\right)\int_t^{t+\Delta t}V_s ds + \sqrt{1-\rho^2}\int_u^{t+\Delta t}\sqrt{V_s}dW_s^{(2)} \tag{14}$$

Then we can easily obtain by Equation (14)

$$
\begin{cases}
\mathrm{E}\left(\ln S_{t+\Delta t}\middle|S_t,V_{t+\Delta t},V_t\right)=\ln S_t+r\Delta t+\dfrac{\rho}{\sigma_v}(V_{t+\Delta t}-V_t-\kappa\theta\Delta t)+\left(\dfrac{\rho\kappa}{\sigma_v}-\dfrac{1}{2}\right)\int_t^{t+\Delta t}V_s ds; \\
\mathrm{V}_{ar}\left(\ln S_{t+\Delta t}\middle|S_t,V_{t+\Delta t},V_t\right)=\left(1-\rho^2\right)\int_t^{t+\Delta t}V_s ds
\end{cases}
\tag{15}
$$

where $\int_t^{t+\Delta t}V_s ds\approx\omega V_{t+\Delta t}+(1-\omega)V_t$, $\omega\in[0,1]$.

Hence, the asset price can be sampled via the following formula

$$
\hat{S}_{t+\Delta t}=\exp\left\{\mathrm{E}\left(\ln S_{t+\Delta t}\middle|\hat{S}_t,\hat{V}_{t+\Delta t},\hat{V}_t\right)+Z\cdot\sqrt{\mathrm{V}_{ar}\left(\ln S_{t+\Delta t}\middle|\hat{S}_t,\hat{V}_{t+\Delta t},\hat{V}_t\right)}\right\}
\tag{16}
$$

where Z is a standard Weiner process.

IV. NUMERICAL EXPERIMENT

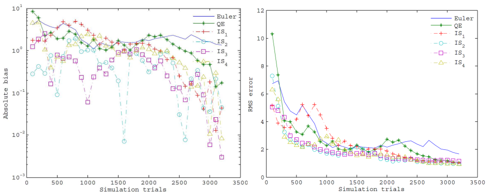
To test the accuracy and convergence of our newly proposed schemes, we take the same parameters (listed in Table 1) with BK's in [3] with spot price 100 and strike 100. For the limit of length, we only select a difficult case in the sense that the Feller condition is very dissatisfied with $F_{eller}=\frac{2\kappa\theta}{\varepsilon^2}-1=-0.64$. Generally, if parameters in the Heston model are such that the Feller condition is not satisfied, this case may act as a benchmark test for discretization schemes.

TABLE I .PARAMETERS IN THE HESTON MODEL AND THE REFERENCE VALUE OF AN EUROPEAN CALL OPTION.

κ	θ	ε	ρ	r	$V(0)$	T **(y)**	$C(0)$
2.00	0.09	1.00	-0.30	0.05	0. 09	5. 0	34.9998

We compare our numerical results with those simulated by the Euler and the QE scheme. In experiments, we take $\Delta t=1/4$. For the QE scheme, $\gamma_1=0.5$, $\gamma_2=0.5$ and $\Psi_c=1.5$. The implementation codes of all discretization schemes, are programmed with Matlab language and executed on a PC equipped with Win7(64bit) and Intel(R) Xeon(R) CPU E5-1620 v2 @3.70GHz 3.70GHz RAM 8.00GB.

We plot the absolute bias curve in Figure 1(a) for all discretization schemes, i.e., the Euler scheme, QE scheme and our four discretization schemes. To show how well these schemes works, we also provide the root-mean-squared (RMS) error curve in Figure 1(b) with definition $RMS=\sqrt{bias^2+\sigma_f^2}$ (wher σ_f denotes as the standard error of payoff samples $f(\hat{S}_T^i), i=1,2,\cdots,N_{paths}$).



(a) Absolute bias (b) Convergence

FIGURE I..ACCURACY AND CONVERGENCE OF DISCRETIZATION SCHEMES. IN FIGURE, IS_I (I=1,2,3,4) MEANS THE IMPLICIT SOLUTION BASED DISCRETIZATION SCHEME PROPOSED IN THIS RESEARCH.

Figure 1 (a) and (b) presents that our newly proposed discretization schemes can achieve good accuracy and fast convergence in option pricing under the Heston model with parameters such that the Feller condition is very dissatisfied, just as well as the famous QE scheme does, while the Euler scheme almost fails to rightly price the European call option in this experiment. Further, according to the absolute bias plotted in Figure 1 (b), the convergence processes of $\mathbf{IS_2}$, $\mathbf{IS_3}$ and $\mathbf{IS_4}$ scheme seem more stable than that of $\mathbf{IS_1}$ scheme. This can be explained by the accuracy order of numerical method applied in the treatment to the singular integral term in Equation (3).

V. CONCLUSIONS

The newly proposed implicit-solution based discretization schemes in this article can successfully simulate the Heston model with good accuracy and convergence that can almost be comparable to the quadratic exponential scheme's. However, we also notice the RMS of our discretization schemes are notable and should be enhanced. Behind the RMS, there may be a challenging problem: how to select a 'good' root when the sampling equation has more than one non-negative solutions.

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