

On EQ-Fuzzy Logics with Delta Connective

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Abstract

In this paper, extension of the EQ-logic by the Δ -connective is introduced. The former is a new kind of many-valued logic which based on EQ-algebra of truth values, i.e. the algebra in which fuzzy equality is the fundamental operation and implication is derived from it. First, we extend the EQ-algebra by the Δ operation and then introduce axioms and inference rules of EQ $_{\Delta}$ -logic. We also prove the deduction theorem formulated using fuzzy equalities.

Keywords: EQ-algebra, EQ-logic, equational logic, delta connective

1. Introduction

In this paper, we continue development of the theory of a special class of fuzzy logics which are based on EQ-algebras. The latter are special algebras in which the fundamental operation is that of fuzzy equality. The concept of EQ-algebra was introduced in [1] and in more detail elaborated in [2] and [3, 4]. It was motivated by the paper of L. Henkin [5] who introduced type theory (higher-order logic) in which equality is the sole connective. The fuzzy version of type theory introduced in [6] is based on residuated lattice (more specifically, on the IMTL-algebra) in which equivalence (i.e. fuzzy equality on truth values) is defined as biresiduation $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. Since this is a derived operation, a question arises whether it is possible to introduce a special algebra of truth values for fuzzy type theory (FTT) with fuzzy equality being the fundamental operation (it can be demonstrated that it cannot be the sole operation as is the case of classical type theory). A possible answer to this question is the concept of EQ-algebra. The implication operation is derived from the fuzzy equality and is relaxed from the multiplication. Therefore, it is no more the residuation. The role of multiplication in EQ-algebras thus becomes only auxiliary and the multiplication serves as a specific connector. Consequently, EQ-algebras are more general than residuated lattices in the sense that every residuated lattice is an EQ-algebra but not vice-versa. Moreover, the multiplication in EQ-algebra is, in general, non-commutative and can be even non-associative without any influence on the implication which remains unique because of not being tied with the former.

The relation between both kinds of algebras is quite subtle and still needs to be studied in more detail.

After establishing the algebra of truth values, EQ-fuzzy type theory has been introduced in [7]. First, the basic EQ-FTT was established and its generalized completeness property was proved. Further, several extensions were introduced, among them also the above mentioned IMTL-FTT.

All this finally raised the question how EQ-logic based purely on EQ-algebra can be introduced. The first papers on this topic are [8, 9]. EQ-logic can be taken as a step towards realization of Leibniz idea that “a fully satisfactory logical calculus must be an equational one” (cf. [10]). It turned out, however, that the situation is not as straightforward as in the case of residuated (fuzzy) logics (cf. [11]). The equality seems to be more fundamental than implication but it is in a certain sense more restricted. First of all, though the original EQ-algebras allowed also distinct elements to be equal in the degree 1, for logic this is unacceptable and only the, so called, *good* EQ-algebras must be considered, i.e. algebras in which the property $a \sim 1 = a$ holds true. But still, properties of the obtained basic EQ-logic are limited. Namely, despite its completeness, the generalized deduction theorem does not hold. This deficiency has fatal consequences, e.g. for possible development of the predicate EQ-logic since its completeness seems hardly to be proved. This gives rise to introducing the Δ -connective well known already from the residuated fuzzy logics. Unlike the latter, where introducing Δ is an interesting but dispensable option, the role of it in the fuzzy equality-based logics seems to be much deeper. For, it enables us to grasp firmly the classical equality inside fuzzy logics and the resulting system has nicer properties. Philosophically, it demonstrates that classical equality is fundamental principle which can never be fully relaxed and that the mentioned goodness axiom is not sufficient.

In this paper, we extend the basic EQ-logic by the Δ connective and the appropriate axioms and prove several properties of it including the completeness. It is notable that we also prove a slightly modified form of the deduction theorem whose formulation copes well with the character of EQ-logic for which formulations using equivalences and also equational style of proofs is more natural.

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2. EQ-logic: An overview

In this section we recall basic definitions and properties of EQ-algebras and basic EQ-logic.

2.1. EQ-algebras

Definition 1

A non-commutative EQ-algebra \mathcal{E} is an algebra of type $(2, 2, 2, 0)$, i.e.

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle,$$

where for all $a, b, c, d \in E$:

(E1) $\langle E, \wedge, \mathbf{1} \rangle$ is a commutative idempotent monoid (i.e. \wedge -semilattice with top element $\mathbf{1}$). We put $a \leq b$ iff $a \wedge b = a$, as usual.

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid and \otimes is isotone w.r.t. \leq .

(E3) $a \sim a = \mathbf{1}$

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$

(E7) $a \otimes b \leq a \sim b$

The operation \wedge is called *meet*, \otimes is called *multiplication* and \sim is the *fuzzy equality*.

For all $a, b \in E$ we put

$$a \rightarrow b = (a \wedge b) \sim a \quad (1)$$

and call this operation an *implication*.

The following theorem demonstrates the fundamental properties of the fuzzy equality and implication.

Theorem 1

Let \mathcal{E} be an EQ-algebra. The following holds for all $a, b, c \in E$:

- (a) *Symmetry*: $a \sim b = b \sim a$,
- (b) *Transitivity*: $(a \sim b) \otimes (b \sim c) \leq a \sim c$,
- (c) *Transitivity of implication*:

$$(a \rightarrow b) \otimes (b \rightarrow c) \leq a \rightarrow c.$$

We distinguish several special classes of EQ-algebras (see [2]). Among them, very important for EQ-logics, are *good* EQ-algebras which satisfy the property

$$a \sim \mathbf{1} = a, \quad a \in E.$$

Note that in good EQ-algebras, only one way of the adjunction condition is satisfied:

$$a \leq b \rightarrow c \text{ implies } a \otimes b \leq c.$$

The opposite implication does not hold in general.

2.2. Basic EQ-logic

Basic EQ-logic was introduced in [9]. It has three basic binary connectives $\wedge, \&, \equiv$ and the truth constant \top . Implication is a derived operation defined by

$$A \Rightarrow B := (A \wedge B) \equiv A.$$

The set of all formulas of the given language J is denoted by F_J . The algebra of truth values is formed by a good non-commutative EQ-algebra.

The following formulas are logical axioms of the basic EQ-logic:

$$(EQ1) \quad (A \equiv \top) \equiv A$$

$$(EQ2) \quad A \wedge B \equiv B \wedge A$$

$$(EQ3) \quad (A \circ B) \circ C \equiv A \circ (B \circ C), \quad \circ \in \{\wedge, \&\}$$

$$(EQ4) \quad A \wedge A \equiv A$$

$$(EQ5) \quad A \wedge \top \equiv A$$

$$(EQ6) \quad A \& \top \equiv A$$

$$(EQ7) \quad \top \& A \equiv A$$

$$(EQ8a) \quad ((A \wedge B) \& C) \Rightarrow (B \& C)$$

$$(EQ8b) \quad (C \& (A \wedge B)) \Rightarrow (C \& B)$$

$$(EQ9) \quad ((A \wedge B) \equiv C) \& (D \equiv A) \Rightarrow (C \equiv (D \wedge B))$$

$$(EQ10) \quad (A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (D \equiv B)$$

$$(EQ11) \quad (A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$$

The deduction rules of basic EQ-logic are *equality rule* (EA) and *Leibniz rule* (Leib):

$$(EA) \quad \frac{A, A \equiv B}{B}, \quad (Leib) \quad \frac{A \equiv B}{C[p := A] \equiv C[p := B]}$$

where by $C[p := A]$ we denote a formula (in the proofs also called as a “C-part”) resulting from C by replacing all occurrences of a propositional variable p in C by the formula A . The basic notions of truth evaluation, tautology, theory, etc. are defined as usual.

The main properties of the basic EQ-logic needed in the sequel are summarized in the following lemmas.

Lemma 1

- (a) $A \vdash A \equiv \top$, (rule (T2))
- (b) $\vdash A \equiv A$,
- (c) $A \equiv B \vdash B \equiv A$,
- (d) $A, A \Rightarrow B \vdash B$, (Modus Ponens)
- (e) $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C$,
- (f) $\vdash (A \equiv B) \equiv (B \equiv A)$,
- (g) $A \Rightarrow (B \equiv C), B \equiv D \vdash A \Rightarrow (D \equiv C)$,

- (h) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B)$,
- (i) $\vdash (A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D)$,
- (j) $A \Rightarrow B, C \Rightarrow D \vdash (A \& C) \Rightarrow (B \& D)$,
- (k) $\vdash B \Rightarrow (A \Rightarrow B)$,
- (l) $\vdash (A \equiv B) \& (B \equiv C) \Rightarrow (A \equiv C)$,
- (m) $\vdash (A \& (A \equiv B)) \Rightarrow B$.

For the proof, see [9]. Let us emphasize that the formal proofs proceed mainly in an equational style which is more natural for EQ-logics. This style is also used in the proof below.

Lemma 2

- (a) $\vdash (A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$,
- (b) $\vdash ((A \equiv B) \& (C \equiv D)) \Rightarrow ((A \wedge C) \equiv (B \wedge D))$.

PROOF: (a)

$$((A \wedge C) \equiv (A \wedge C)) \& (B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C)) \quad (\text{EQ9})$$

$$\Leftrightarrow \langle (\text{Leib}) + \text{Lemma 1(b)} + \text{rule (T2)}; \text{“C-part”}: \mathbf{p} \& (B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C)) \rangle$$

$$\top \& (B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$$

$$\Leftrightarrow \langle (\text{Leib}) + (\text{EQ7}); \text{“C-part”}: \mathbf{p} \Rightarrow ((A \wedge C) \equiv (B \wedge C)) \rangle$$

$$(B \equiv A) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$$

$$\Leftrightarrow \langle (\text{Leib}) + \text{Lemma 1(f)}; \text{“C-part”}: \mathbf{p} \Rightarrow ((A \wedge C) \equiv (B \wedge C)) \rangle$$

$$(A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$$

(b) First we use Lemma 2(a) $\vdash (A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$. Then we use Lemma 2(a) again and also (EQ2) and the Leibniz rule to obtain $\vdash (C \equiv D) \Rightarrow ((B \wedge C) \equiv (B \wedge D))$. We use both formulas as assumptions in Lemma 1(j) to get $\vdash ((A \equiv B) \& (C \equiv D)) \Rightarrow ((A \wedge C) \equiv (B \wedge C)) \& ((B \wedge C) \equiv (B \wedge D))$. Finally using Lemma 1(l) and (e) we obtain Lemma 2(b). \square

The proof of the following theorems can be found in [9].

Theorem 2 (Soundness)

The basic EQ-fuzzy logic is sound, i.e. $\vdash A$ implies $\models A$.

Theorem 3 (Completeness)

The following is equivalent for every formula A :

- (a) $\vdash A$,
- (b) $e(A) = \mathbf{1}$ for every good non-commutative EQ-algebra \mathcal{E} and a truth evaluation $e : F_J \rightarrow E$.

3. EQ $_{\Delta}$ -logic

In this section we introduce propositional EQ-logic with Δ connective*) which we will call basic EQ $_{\Delta}$ -logic. Its semantics is based on a good non-commutative EQ $_{\Delta}$ -algebra.

3.1. EQ $_{\Delta}$ -algebra

Definition 2

An EQ $_{\Delta}$ -algebra is an algebra

$$\mathcal{E}_{\Delta} = \langle E, \wedge, \otimes, \sim, \Delta, \mathbf{1} \rangle \quad (2)$$

which is a good non-commutative EQ-algebra \mathcal{E} extended by a unary additional operation $\Delta : E \rightarrow E$ fulfilling the following axioms:

$$(E\Delta 1) \Delta a \leq \Delta \Delta a$$

$$(E\Delta 2) \Delta(a \sim b) \leq \Delta a \sim \Delta b$$

$$(E\Delta 3) \Delta(a \wedge b) = \Delta a \wedge \Delta b$$

$$(E\Delta 4) \Delta a = \Delta a \otimes \Delta a.$$

$$(E\Delta 5) \Delta(a \sim b) \leq (a \otimes c) \sim (b \otimes c)$$

$$(E\Delta 6) \Delta(a \sim b) \leq (c \otimes a) \sim (c \otimes b)$$

Axioms (E Δ 1)–(E Δ 4) have already been introduced in [7]. Axioms (E Δ 5) and E Δ 6) characterize behavior of the multiplication with respect to the crisp equality. They turn out to be necessary for further development of EQ-logic, especially of its predicate version (which is not yet existing).

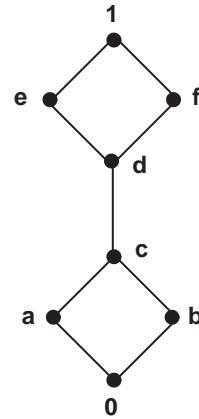


Figure 1: Eight element EQ $_{\Delta}$ -algebra.

Example 1

An example of a finite non-trivial EQ $_{\Delta}$ -algebra is the following: we take the lattice structure in Figure 1. The multiplication and fuzzy equality are defined

*)This is also called Baaz delta.

as follows:

$$\otimes \begin{array}{c} \mathbf{0} \\ a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array}$$

$$\sim \begin{array}{c} \mathbf{0} \\ a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array}$$

Using (1) we obtain the implication:

$$\rightarrow \begin{array}{c} \mathbf{0} \\ a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array} \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ \mathbf{1} \end{array}$$

The Δ operation is defined by

$$\Delta(x) = \begin{cases} \mathbf{0} & \text{if } x = \mathbf{0}, a, b, c \\ d & \text{if } x = d, e, f \\ \mathbf{1} & \text{if } x = \mathbf{1} \end{cases}$$

Note that the multiplication \otimes is not commutative. Indeed, for example $c \otimes f = a$ but $f \otimes c = \mathbf{0}$. Moreover, this algebra is also non-residuated since, e.g., $\mathbf{0} = c \otimes e \leq \mathbf{0}$, but $c \not\leq e \rightarrow \mathbf{0}$.

It is easy to verify the following properties of EQ_Δ -algebra:

Lemma 3

Let \mathcal{E}_Δ be an EQ_Δ -algebra. For all $a, b \in E$ it holds that

- (a) $\Delta \mathbf{1} = \mathbf{1}$,
- (b) $\Delta a \leq a$,
- (c) If $a \leq b$ then $\Delta a \leq \Delta b$,
- (d) $\Delta(a \rightarrow b) \leq \Delta a \rightarrow \Delta b$.

3.2. Basic EQ_Δ -logic

Let J be a language of EQ_Δ -logic, F_J a set of all formulas in the language J and

$$\mathcal{E}_\Delta = \langle E, \wedge, \otimes, \sim, \Delta, \mathbf{1} \rangle$$

be a good non-commutative EQ_Δ -algebra. The language of EQ_Δ -logic is the language of basic EQ-logic expanded by the unary connective Δ .

A truth evaluation of formulas is a mapping $e : F_J \rightarrow E$ defined as follows: if A is a propositional variable p then $e(p) \in E$. Otherwise,

$$\begin{aligned} e(\top) &= \mathbf{1}, \\ e(A \wedge B) &= e(A) \wedge e(B), \\ e(A \& B) &= (e(A) \otimes e(B)), \\ e(A \equiv B) &= e(A) \sim e(B), \\ e(\Delta A) &= \Delta e(A) \end{aligned}$$

for all formulas $A, B \in F_J$.

3.2.1. Logical axioms and inference rules

Definition 3

Axioms of EQ_Δ -logic are those of basic EQ-logic plus the following ones:

- (EQ Δ 1) $\Delta A \Rightarrow \Delta \Delta A$
- (EQ Δ 2) $\Delta(A \equiv B) \Rightarrow (\Delta A \equiv \Delta B)$
- (EQ Δ 3) $\Delta(A \wedge B) \equiv (\Delta A \wedge \Delta B)$
- (EQ Δ 4) $\Delta A \equiv (\Delta A \& \Delta A)$
- (EQ Δ 5) $\Delta(A \equiv B) \Rightarrow ((A \& C) \equiv (B \& C))$
- (EQ Δ 6) $\Delta(A \equiv B) \Rightarrow ((C \& A) \equiv (C \& B))$

Deduction rules of EQ_Δ -logic are *equanimity rule*, *Leibniz rule* and *Necessitation rule*:

$$(N) \quad \frac{A}{\Delta A}.$$

3.2.2. Main properties

Lemma 4

All axioms of EQ_Δ -logic are tautologies.

PROOF: This is straightforward using the axioms and properties of EQ_Δ -algebra. \square

Lemma 5

The deductive rules of EQ_Δ -logic are sound in the following sense. Let $e : F_J \rightarrow E$ be a truth evaluation:

- (a) If $e(A) = \mathbf{1}$ and $e(A \equiv B) = \mathbf{1}$ then $e(B) = \mathbf{1}$.
- (b) If $e(B \equiv C) = \mathbf{1}$ then $e(A[p := B] \equiv A[p := C]) = \mathbf{1}$ for any formula A .
- (c) If $e(A) = \mathbf{1}$ then $e(\Delta A) = \mathbf{1}$.

PROOF: By straightforward verification. \square

The following formulas are provable in EQ Δ -logic.

Lemma 6

- (a) $\Delta A \Rightarrow A$,
- (b) $\Delta \top \equiv \top$,
- (c) $\Delta(A \Rightarrow B) \Rightarrow (\Delta A \Rightarrow \Delta B)$,
- (d) $\Delta(A \& B) \Rightarrow (\Delta A \& \Delta B)$,
- (e) $\Delta((A \equiv B) \& (C \equiv D)) \Rightarrow ((A \& C) \equiv (B \& D))$.

PROOF: (a)

$$\Delta(A \equiv \top) \Rightarrow ((A \& \top) \equiv (\top \& \top)) \quad (\text{EQ}\Delta 5)$$

$$\Leftrightarrow \langle (\text{Leib}) + (\text{EQ6}); \text{“C-part”}: \Delta(A \equiv \top) \Rightarrow (\mathbf{p} \equiv (\top \& \top)) \rangle$$

$$\Delta(A \equiv \top) \Rightarrow (A \equiv (\top \& \top))$$

$$\Leftrightarrow \langle (\text{Leib}) + (\text{EQ6}); \text{“C-part”}: \Delta(A \equiv \top) \Rightarrow (A \equiv \mathbf{p}) \rangle$$

$$\Delta(A \equiv \top) \Rightarrow (A \equiv \top)$$

$$\Leftrightarrow \langle (\text{Leib}) + (\text{EQ1}); \text{“C-part”}: \Delta \mathbf{p} \Rightarrow \mathbf{p} \rangle$$

$$\Delta A \Rightarrow A$$

(b) Follows from (EQ1) in the form $\vdash (\Delta \top \equiv \top) \equiv \Delta \top$ and from $\vdash \Delta \top$ using (EA).

(c)

$$\Delta((A \wedge B) \equiv A) \Rightarrow (\Delta(A \wedge B) \equiv \Delta A) \quad (\text{EQ}\Delta 2)$$

$$\Leftrightarrow \langle (\text{Leib}) + \text{EQ}\Delta 3; \text{“C-part”}: \Delta((A \wedge B) \equiv A) \Rightarrow (\mathbf{p} \equiv \Delta A) \rangle$$

$$\Delta((A \wedge B) \equiv A) \Rightarrow ((\Delta A \wedge \Delta B) \equiv \Delta A) \quad (\text{i.e. (c)})$$

(e) First we use Lemma 6(d)

$$\vdash \Delta((A \equiv B) \& (C \equiv D)) \Rightarrow (\Delta(A \equiv B) \& \Delta(C \equiv D)).$$

Using assumptions (EQ Δ 5) and $\vdash \Delta(C \equiv D) \Rightarrow \Delta(C \equiv D)$ in Lemma 1(j) we obtain

$$\vdash (\Delta(A \equiv B) \& \Delta(C \equiv D)) \Rightarrow (((A \& C) \equiv (B \& C)) \& \Delta(C \equiv D)).$$

In the same way (but using (EQ Δ 6)) we get

$$\vdash (((A \& C) \equiv (B \& C)) \& \Delta(C \equiv D)) \Rightarrow (((A \& C) \equiv (B \& C)) \& ((B \& C) \equiv (B \& D))).$$

Finally from the previous formulas and Lemma 1(l) in the form

$$\vdash (((A \& C) \equiv (B \& C)) \& ((B \& C) \equiv (B \& D))) \Rightarrow ((A \& C) \equiv (B \& D))$$

using Lemma 1(e) we find desired formula. \square

The following is the deduction theorem formulated in the style natural for EQ-logic.

Theorem 4 (Deduction theorem)

For each theory T and formulas A, B, C it holds that

$$T \cup \{A \equiv B\} \vdash C \quad \text{iff} \quad T \vdash \Delta(A \equiv B) \Rightarrow C.$$

PROOF: [Hilbert style] Let $T \cup \{A \equiv B\} \vdash C$. The proof proceeds by induction on the length of the proof of C .

- (a) If $C := (A \equiv B)$ then $T \vdash \Delta(A \equiv B) \Rightarrow (A \equiv B)$ by Lemma 6(a).
- (b) C is an axiom of T then $T \vdash \Delta(A \equiv B) \Rightarrow C$ using Lemma 1(k) and Lemma 1(d).
- (c) Let C have been obtained using rule (EA) by the proof

$$\dots, D, D \equiv C, C.$$

Then

$$(L.1) \quad T \vdash \Delta(A \equiv B) \Rightarrow (\Delta(A \equiv B) \& \Delta(A \equiv B)) \quad ((\text{EQ}\Delta 4), \text{Lemma 1(h), (d)})$$

$$(L.2) \quad T \vdash (\Delta(A \equiv B) \& \Delta(A \equiv B)) \Rightarrow (D \& (D \equiv C)) \quad (2x \text{ inductive assumption, Lemma 1(j)})$$

$$(L.3) \quad T \vdash (D \& (D \equiv C)) \Rightarrow C \quad (\text{Lemma 1(m)})$$

$$(L.4) \quad T \vdash \Delta(A \equiv B) \Rightarrow C \quad (L.1, L.2, L.3 \text{ and Lemma 1(e)})$$

- (d) Let $C := D[p := E] \equiv D[p := F]$ have been obtained using rule (Leib) by the proof

$$\dots, E \equiv F, D[p := E] \equiv D[p := F].$$

Then the proof proceeds by induction on the complexity of the formula C .

- (i) If D is either \top or q (other than p) then $D[p := E] \equiv D[p := F] = D \equiv D$ and $T \vdash \Delta(A \equiv B) \Rightarrow (D \equiv D)$ follows from Lemma 1(b), Lemma 1(k) and Lemma 1(d).

- (ii) If D is p then it follows immediately from the inductive assumption.

- (iii) Let D be $G \circ H$, where $\circ \in \{\wedge, \&, \equiv\}$. Then we need to prove $T \vdash \Delta(A \equiv B) \Rightarrow ((G \circ H)[p := E] \equiv (G \circ H)[p := F])$

thus

$$T \vdash \Delta(A \equiv B) \Rightarrow ((G[p := E] \circ H[p := E]) \equiv (G[p := F] \circ H[p := F]))$$

and thus shortly

$$T \vdash \Delta(A \equiv B) \Rightarrow ((G' \circ H') \equiv (G'' \circ H'')),$$

where $G' := G[p := E]$, $H' := H[p := E]$, $G'' := G[p := F]$ and $H'' := H[p := F]$.

Let first D be $G \wedge H$. Then

$$(L.1) \quad T \vdash \Delta(A \equiv B) \Rightarrow (\Delta(A \equiv B) \& \Delta(A \equiv B)) \quad ((\text{EQ}\Delta 4), \text{Lemma 1(h), (d)})$$

$$(L.2) \quad T \vdash (\Delta(A \equiv B) \& \Delta(A \equiv B)) \Rightarrow ((G' \equiv G'') \& (H' \equiv H''))$$

(2x inductive assumption,
Lemma 1(j))

$$(L.3) \quad T \vdash ((G' \equiv G'') \& (H' \equiv H'')) \Rightarrow ((G' \wedge H') \equiv (G'' \wedge H''))$$

(Lemma 2(b))

$$(L.4) \quad T \vdash \Delta(A \equiv B) \Rightarrow ((G' \wedge H') \equiv (G'' \wedge H''))$$

(L.1, L.2, L.3 and Lemma 1(e))

(iv) Let D be $G \equiv H$. Then

$$(L.1) \quad T \vdash \Delta(A \equiv B) \Rightarrow (\Delta(A \equiv B) \& \Delta(A \equiv B))$$

((EQ Δ 4), Lemma 1(h), (d))

$$(L.2) \quad T \vdash (\Delta(A \equiv B) \& \Delta(A \equiv B)) \Rightarrow ((G' \equiv G'') \& (H' \equiv H''))$$

(2x inductive assumption,
Lemma 1(j))

$$(L.3) \quad T \vdash ((G' \equiv G'') \& (H' \equiv H'')) \Rightarrow ((G' \equiv H') \equiv (G'' \equiv H''))$$

(Lemma 1(i))

$$(L.4) \quad T \vdash \Delta(A \equiv B) \Rightarrow ((G' \equiv H') \equiv (G'' \equiv H''))$$

(L.1, L.2, L.3 and Lemma 1(e))

(v) Let D be $G \& H$.

$$(L.1) \quad T \vdash (\Delta(A \equiv B) \Rightarrow ((G' \equiv G'') \& (H' \equiv H''))) \quad (\text{see above})$$

$$(L.2) \quad T \vdash \Delta(A \equiv B) \Rightarrow \Delta\Delta(A \equiv B)$$

((EQ Δ 1))

$$(L.3) \quad T \vdash \Delta\Delta(A \equiv B) \Rightarrow \Delta((G' \equiv G'') \& (H' \equiv H''))$$

(L.1, rule (N),
Lemma 6(c), Lemma 1(d))

$$(L.4) \quad T \vdash \Delta((G' \equiv G'') \& (H' \equiv H'')) \Rightarrow ((G' \& H') \equiv (G'' \& H''))$$

(Lemma 6(e))

$$(L.5) \quad T \vdash \Delta(A \equiv B) \Rightarrow ((G' \& H') \equiv (G'' \& H''))$$

(L.2, L.3, L.4 and Lemma 1(e))

(e) Let $C := \Delta D$ have been obtained using rule (N) by the proof

$$\dots, D, \Delta D.$$

Then $T \vdash \Delta(A \equiv B) \Rightarrow \Delta D$ follows from inductive assumption using rule (N), Lemma 6(c), 1(d), (EQ Δ 1) and Lemma 1(e).

The converse implication follows from rule (N) and Lemma 1(d). \square

Remark 1

If we put $B := \top$ in this theorem then it is easy to deduce the “standard” form of the delta deduction theorem:

$$T \cup \{A\} \vdash C \quad \text{iff} \quad T \vdash \Delta A \Rightarrow C.$$

Definition 4

Put

$$A \approx B \quad \text{iff} \quad \vdash A \equiv B, \quad A, B \in F_J.$$

It follows from Lemmas 1(b), (f) and (l) that \approx is an equivalence on F_J . Let us denote by $[A]$ an equivalence class of A and put $\bar{E} = \{[A] \mid A \in F_J\}$. Finally we define

$$\begin{aligned} \mathbf{1} &= [\top], \\ [A] \wedge [B] &= [A \wedge B], \\ [A] \otimes [B] &= [A \& B], \\ [A] \sim [B] &= [A \equiv B], \\ \Delta[A] &= [\Delta A]. \end{aligned}$$

Lemma 7

The algebra $\bar{\mathcal{E}}_\Delta = \langle \bar{E}, \wedge, \otimes, \sim, \Delta, \mathbf{1} \rangle$ is a good non-commutative EQ Δ -algebra.

PROOF: We have to verify axioms of EQ Δ -algebra: for axioms (E1)–(E7) and “goodness property” see the proof of Lemma 10 in [9]. For axioms (E Δ 1)–(E Δ 6) see (EQ Δ 1)–(EQ Δ 6). \square

Theorem 5 (Soundness)

The basic EQ Δ -fuzzy logic is sound.

PROOF: This follows from Lemma 4 and 5. \square

Theorem 6 (Completeness)

The following is equivalent for every formula A :

- (a) $\vdash A$,
- (b) $e(A) = \mathbf{1}$ for every good non-commutative EQ Δ -algebra \mathcal{E}_Δ and a truth evaluation $e : F_J \longrightarrow E$.

PROOF: The implication (a) to (b) is soundness.

(b) to (a): By Lemma 7 the algebra $\bar{\mathcal{E}}_\Delta$ of equivalence classes of formulas is a good non-commutative EQ Δ -algebra. Thus, if (b) holds then it holds also for $e : F_J \longrightarrow \bar{E}$. If $e(A) = \mathbf{1}$ then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$ and so, $\vdash A$ by rule (T1). \square

4. Conclusion

In this paper, we continue the development of EQ-logics, i.e. logics based on EQ-algebra of truth values. We introduced the Δ -connective whose role in fuzzy equality-based logics seems to be more important than in residuated fuzzy logics. Besides others, it enables us to prove the deduction theorem which in basic EQ-logic without Δ does not hold. The completeness theorem has been proved in its basic form. With respect to the results from [3] it is clear that the representability of prelinear EQ-algebras can be extended also to prelinear

EQ $_{\Delta}$ -algebras. Consequently, extension of the completeness theorem to hold with respect to all linearly ordered EQ $_{\Delta}$ -algebras will be possible.

It is an open question whether there is an axiomatic system with modus ponens and necessitation as the only deduction rules in EQ $_{\Delta}$ -logics. In the case of positive answer, however, equanimity and Leibniz rules are more natural than modus ponens for equality based logic, because the latter inference rule is based on implication — derived connective in EQ-logics.

The future work will focus on further development of all introduced kinds of EQ-logics, study of their relation to residuated fuzzy logics and also to the development of predicate EQ-logics. It seems to be impossible to develop predicate EQ-logics without Δ .

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