upwards closed subsets of W,[3] *and* $u : Frm_{\mathcal{L}} \to V$ *such that* $u(\psi) = \{w \in W \mid v(w, \psi) = t\}$.

1. $\mathcal{V}$ *is an* $\mathcal{L}$*-Gödel valuation.*
2. *If* $a \in D_{\mathcal{V}}(\Gamma \Rightarrow E)$ *then* $\Gamma \Rightarrow E$ *is true in* a.
3. *If* $a \notin u_{min}(\Gamma)$ *then* $\Gamma \Rightarrow$ *is locally true in* a.
4. *Let* **G** *be a canonical Gödel system. If* $\mathcal{W}$ *is* **G***-legal then* $\mathcal{V}$ *is* **G***-legal.*
5. $\mathcal{V}$ *is a model of a hypersequent* H *iff* $\mathcal{W}$ *is a model of* H.

Proof. 1. It is easy to see that $\langle V, \subseteq \rangle$ is a nonempty linearly ordered set with a maximal element $1 = W$ and a minimal element $0 = \emptyset$.

2. Let $s = \Gamma \Rightarrow E$. If $a \in D_{\mathcal{V}}(s)$, either $u_{min}(\Gamma) \subseteq u_{max}(E)$ or $a \in u_{max}(E)$. Consider two cases:

   $E = \emptyset$ In this case $u_{max}(E) = \emptyset$, and so $u_{min}(\Gamma) = \emptyset$. It follows that there exists some $\psi \in \Gamma$ such that $v(b, \psi) = f$ for every $b \in W$. In particular s is true in a.

   $E \neq \emptyset$ Assume $E = \{\varphi\}$, and so $u_{max}(E) = u(\varphi)$. If $a \in u(\varphi)$ then $v(a, \varphi) = t$ and s is true in a. Otherwise, $u_{min}(\Gamma) \subseteq u(\varphi)$. It implies that $u(\psi) \subseteq u(\varphi)$ for some $\psi \in \Gamma$. Therefore, s is locally true in every $b \in W$, and so s is true in a.

3. Since $a \notin u_{min}(\Gamma)$, there exists $\psi \in \Gamma$ such that $a \notin u(\psi)$. It follows from the definition of u, that $v(a, \psi) = f$, and so s is locally true in a.

4. We show that $\mathcal{V}$ respects the rules of **G**. Let $r = \mathcal{S} / \Rightarrow \diamond(p_1, \ldots, p_n)$ $(\mathcal{S}_1, \mathcal{S}_2 / \diamond(p_1, \ldots, p_n) \Rightarrow)$ be a rule in **G** for an n-ary connective $\diamond$. Let σ be an $\mathcal{L}$-substitution.
   First, suppose that r is a right rule. We prove that $D_{\mathcal{V}}(\sigma(\mathcal{S})) \subseteq u(\sigma(\diamond(p_1, \ldots, p_n))$. Let $a \in D_{\mathcal{V}}(\sigma(\mathcal{S}))$. By 2, every $s \in \sigma(\mathcal{S})$ is true in a. It follows that σ fulfils r in a. Since $\mathcal{W}$ is **G**-legal, $v(a, \sigma(\diamond(p_1, \ldots, p_n))) = t$, and so $a \in u(\sigma(\diamond(p_1, \ldots, p_n)))$.
   Second, suppose that r is a left rule. We show that $u(\sigma(\diamond(p_1, \ldots, p_n)) \subseteq D_{\mathcal{V}}(\sigma(\mathcal{S}_1)) \to N_{\mathcal{V}}(\sigma(\mathcal{S}_2))$. This obviously holds if $D_{\mathcal{V}}(\sigma(\mathcal{S}_1)) \subseteq N_{\mathcal{V}}(\sigma(\mathcal{S}_2))$. Assume otherwise. Let $a \in u(\sigma(\diamond(p_1, \ldots, p_n))$. $v(a, \sigma(\diamond(p_1, \ldots, p_n))) = t$, and since $\mathcal{W}$ is **G**-legal, r is not fulfilled in a by σ. By 2 and 3, either $a \notin D_{\mathcal{V}}(\sigma(\mathcal{S}_1))$ or $a \in N_{\mathcal{V}}(\sigma(\mathcal{S}_2))$. Let $b \in W$ such that $b \in D_{\mathcal{V}}(\sigma(\mathcal{S}_1)) \setminus N_{\mathcal{V}}(\sigma(\mathcal{S}_2))$. Again by 2 and 3, every $s \in \sigma(\mathcal{S}_1)$ is true in b, and every $s \in \sigma(\mathcal{S}_2)$ is locally true in b. It follows that σ fulfils r in b, and since $\mathcal{W}$ is **G**-legal, $v(b, \sigma(\diamond(p_1, \ldots, p_n))) = f$. Using the persistence condition, $b < a$. Since $b \in D_{\mathcal{V}}(\sigma(\mathcal{S}_1))$, $a \in D_{\mathcal{V}}(\sigma(\mathcal{S}_1))$ (every element of V is upwards closed). It follows that $a \in N_{\mathcal{V}}(\sigma(\mathcal{S}_2))$, and so $a \in D_{\mathcal{V}}(\sigma(\mathcal{S}_1)) \to N_{\mathcal{V}}(\sigma(\mathcal{S}_2))$.

5. One direction easily follows from 2. The converse is easy and left to the reader.

□

[3]A subset A of W is upwards closed if $u \in A$ whenever $w \in A$ and $w \leq u$.

Corollary 39 (Soundness and Completeness)**.** *Every coherent canonical Gödel system* **G** *is strongly sound and complete with respect to the semantics of* **G***-legal Gödel valuations. In other words:* $\mathcal{H} \vdash_{\mathbf{G}} H$ *iff* $\mathcal{H} \models_{\mathbf{G}} H$.

Proof. Assume $\mathcal{H} \vdash_{\mathbf{G}} H$. By Theorem 34 $\mathcal{H} \models^K_{\mathbf{G}} H$. Let $\mathcal{V}$ be a **G**-legal $\mathcal{L}$-Gödel valuation which is a model of $\mathcal{H}$. We show that it is a model of H. Let $\mathcal{W} = \langle V, \geq, v \rangle$, where v is defined as in Lemma 37. By Lemma 37, $\mathcal{W}$ is a **G**-legal $\mathcal{L}$-Gödel frame which is a model of $\mathcal{H}$. It follows that $\mathcal{W}$ is a model of H, and Lemma 37 again implies that $\mathcal{V}$ is a model of H. The converse is analogous (using Lemma 38). □

Acknowledgements

This research was supported by The Israel Science Foundation (grant no. 280-10).

References

[1] A. Avron, Using Hypersequents in Proof Systems for Non-classical Logics, *Annals of Mathematics and Artificial Intelligence*, 4, 225–248, 1991.

[2] A. Avron, A Simple Proof of Completeness and Cut-admissibility for Propositional Gödel Logic, *Journal of Logic and Computation*, doi: 10.1093/logcom/exp055, 2009.

[3] A. Avron and I. Lev, Non-deterministic Multi-valued Structures, *Journal of Logic and Computation*, vol. 15, 241–261, 2005. A conference version: Canonical Propositional Gentzen-Type Systems, In *Goré, R., Leitsch, A., Nipkow, T., (eds.) IJCAR 2001*, LNCS (LNAI), vol. 2083, pp. 529–544. Springer, Heidelberg, 2001.

[4] A. Avron and O. Lahav, On Constructive Connectives and Systems, *Journal of Logical Methods in Computer Science*, 6, 2010.

[5] M. Baaz, A. Ciabattoni and C. G. Fermüller, Hypersequent Calculi for Gödel Logics - a Survey, *Journal of Logic and Computation*, 13:835–861, 2003.

[6] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, 1998.

[7] G. Metcalfe, N. Olivetti, D. Gabbay, *Proof Theory for Fuzzy Logics*, Springer, 2009.

[8] G. Sundholm, Proof theory and Meaning, in *H*andbook of Philosophical Logic (D. M. Gabbay and F. Guenthner, eds), 2nd ed., Vol 9, 165–198, 2002.