Exact Wave functions of Time-dependent Mesoscopic LC Circuit with an External Power Source

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Abstract. Quantization of the mesoscopic circuit with an external power source has been studied in this paper. The exact quantum invariant and wave function of the time-dependent mesoscopic LC circuit with an external power source, has been obtained via the field invariant method. The auxiliary functions in the quantum invariant operator and wave function corresponds to the external power source in the Hamiltonian of the mesoscopic circuit.

Introduction

As the miniaturization of integrated circuit size, quantum mechanical effects of circuit must be considered[1-2]. Therefore, it is urgently needed to establish a quantum theory of the mesoscopic circuit[2]. Up to now, many works have been done to solve the mesoscopic circuit whose Hamiltonians are explicitly time-dependent[3-7]. There are mainly two kinds of methods: path integral method and invariant operator method. In 1969, via the so-called invariant operator method, Lewis and Riesenfeld presented the exact wave function of the time-dependent harmonic oscillator by finding out the quantum invariant operator of the time-dependent Hamiltonian[3]. The quantization process of a L-C design mesoscopic circuit is similar to harmonic oscillator's[4]. In 1992, Pedrosa et al. calculated the exact wave function and quantum fluctuations of the harmonic oscillator with time-dependent mass and frequency by quantum invariant operator method[5]. Later, they used the quantum invariant operator method to calculate the mesoscopic RLC circuit[6-7]. However, when an external power source applies to the mesoscopic circuit, the new quantum effect of the mesoscopic circuit will be produced[8-10]. Hereby, with the aid of quantum invariant operator method, we will study the time-dependent mesoscopic LC circuit with an external power source and derive its exact quantum invariant and wave function.

Calculation of exact quantum invariant and wave function

The Hamiltonian of the time-dependent LC mesoscopic circuit with an external power source can be written as[11]

$$H(t)=\frac{p^2}{2L(t)}+\frac{1}{2}L(t)\omega^2(t)q^2-\xi(t)q \tag{1}$$

Where $\xi(t)$ is the electromotive force of external power source, p denotes the generalized current, q denotes the charge, L stands for the inductance and ω stands for frequency.

According to the canonical transformation,

$$\begin{cases}\dot{q}=\dfrac{\partial H}{\partial p}\\[2ex] \dot{p}=-\dfrac{\partial H}{\partial q}\end{cases} \tag{2}$$

the equation of motion can be expressed as

$$\ddot{q}+\frac{\dot{L}}{L}\dot{q}+\omega^2 q=\frac{\xi(t)}{L} \tag{3}$$

In the bidimensional complex plane, Eq. (2) can be rewritten as

$$\ddot{r}+\frac{\dot{L}}{L}\dot{r}+\omega^2 r=\frac{\xi(t)}{L} \tag{4}$$

Where $r(t)=\rho(t)e^{i\theta}$. We find that

$$\begin{cases}\dot{r}=\dot{\rho}e^{i\theta}+i\rho e^{i\theta}\dot{\theta}\\ \ddot{r}=\ddot{\rho}e^{i\theta}+2i\dot{\rho}e^{i\theta}\dot{\theta}-\rho e^{i\theta}\dot{\theta}^2+i\rho e^{i\theta}\ddot{\theta}\end{cases} \tag{5}$$

Substituting (4) into (3),

$$\ddot{\rho}e^{i\theta}+2i\dot{\rho}e^{i\theta}\dot{\theta}-\rho e^{i\theta}\dot{\theta}^2+i\rho e^{i\theta}\ddot{\theta}+\frac{\dot{L}}{L}\dot{\rho}e^{i\theta}+\frac{\dot{L}}{L}i\rho e^{i\theta}\dot{\theta}+\omega^2\rho e^{i\theta}=\frac{\xi(t)}{L} \tag{6}$$

$$\begin{cases}\ddot{\rho}+\frac{\dot{L}}{L}\dot{\rho}+\left(\omega^2-\dot{\theta}^2\right)\rho=\frac{\xi(t)}{L}e^{-i\theta}\\ 2\dot{\rho}\dot{\theta}+\rho\ddot{\theta}+\frac{\dot{L}}{L}\rho\dot{\theta}=0\end{cases} \tag{7}$$

$$\dot{\theta}=\frac{C}{L\rho^2} \tag{8}$$

$$\ddot{\rho}+\frac{\dot{L}}{L}\dot{\rho}+\omega^2\rho=\frac{\xi(t)}{L}e^{-i\theta}+\frac{C^2}{L^2\rho^3} \tag{9}$$

Then we can obtain the following auxiliary equations:

$$\begin{cases}\ddot{\rho}+\frac{\dot{L}}{L}\dot{\rho}+\omega^2\rho=\frac{1}{L^2\rho^3}\\ \ddot{\alpha}+\frac{\dot{L}}{L}\dot{\alpha}+\omega^2\alpha=\frac{\xi(t)}{L}\end{cases} \tag{10}$$

Where $\rho(t)$ and $\alpha(t)$ are all auxiliary functions of time. Through the method similar with that in reference [12], we can obtain

$$V(q,t)=-\xi(t)q+\frac{1}{2}L\omega^2q^2+\frac{1}{\rho^2}\frac{1}{L}U\left(\frac{q-\alpha}{\rho}\right) \tag{11}$$

$$\tilde{G}\left(\frac{q-\alpha}{\rho}\right)=U\left(\frac{q-\alpha}{\rho}\right)+\frac{1}{2}\left(\frac{q-\alpha}{\rho}\right)^2 \tag{12}$$

$$I(q,p,t)=\frac{1}{2}\left[\rho(p-L\dot{\alpha})-\dot{\rho}(Lq-L\alpha)\right]^2+\tilde{G}\left(\frac{q-\alpha}{\rho}\right) \tag{13}$$

Substituting (7) into (8), we can get the exact quantum invariant of the time-dependent LC mesoscopic circuit with an external power source:

$$I=\frac{1}{2}\left\{\left[\rho\left(p-L\dot{\alpha}\right)-L\dot{\rho}\left(q-\alpha\right)\right]^2+\left(\frac{q-\alpha}{\rho}\right)^2\right\} \tag{14}$$

By analogy with the time-dependent LC mesoscopic circuit without external power source, we can obtain

$$\begin{cases}P=p-L\dot{\alpha}\\ Q=q-\alpha\end{cases} \tag{15}$$

Q , P still satisfy the original commutation relation[13]

$$\left[Q,P\right]=i\hbar \tag{16}$$

Through appropriate unitary transformation, the invariant operator can be simplified in forms of harmonic oscillator. First, one can choose unitary transformation operator as follow

$$U = \exp\left[\frac{-iL\dot{\rho}}{2\hbar\rho}(q-\alpha)^2\right] \tag{17}$$

So commutation relation can be obtained

$$[A, I] = \frac{1}{2}L\rho\dot{\rho}(PQ+QP) - L^2\dot{\rho}^2Q^2 \tag{18}$$

$$[A,[A,I]] = L^2\dot{\rho}^2Q^2 \tag{19}$$

where $A = \frac{-iL\dot{\rho}}{2\hbar\rho}(q-\alpha)^2$.

Based on the unitary transformation,

$$I' = UIU^+ = e^{-A}Ie^A = I + [A,I] + \frac{1}{2}[A,[A,I]] \tag{20}$$

the invariant operator can be changed into

$$I' = UIU^{-1} = \frac{1}{2}\rho^2P^2 + \frac{1}{2}\frac{Q^2}{\rho^2} \tag{21}$$

The corresponding eigenequation is

$$I'\varphi_n(q,t) = \lambda_n\varphi_n(q,t) \tag{22}$$

By analoging with the classical harmonic oscillator, the new eigenstate can be derived as

$$\varphi_n(q,t) = \left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{\frac{1}{2}} \exp\left[-\left(\frac{q-\alpha}{\rho}\right)^2 \cdot \frac{1}{2\hbar}\right] H_n\left[\left(\frac{1}{\hbar}\right)^{\frac{1}{2}}\left(\frac{q-\alpha}{\rho}\right)\right] \tag{23}$$

According to $\varphi_n(q,t) = U\Psi_n(q,t)$, the wave function of the time-dependent LC mesoscopic circuit with an external power source is derived as

$$\Psi_n(q,t) = \left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{\frac{1}{2}} \exp\left[-\left(\frac{q-\alpha}{\rho}\right)^2 \cdot \frac{1}{2\hbar}\right] H_n\left[\left(\frac{1}{\hbar}\right)^{\frac{1}{2}}\left(\frac{q-\alpha}{\rho}\right)\right] \exp\left[\frac{iL\dot{\rho}}{2\hbar\rho}(q-\alpha)^2\right] \tag{24}$$

If the electromotive force $\xi(t) = 0$, namely auxiliary functions $\alpha(t) = 0$, Eq. (9) and Eq. (18) will become

$$I = \frac{1}{2}\left\{(\rho p - L\dot{\rho}q)^2 + \left(\frac{q}{\rho}\right)^2\right\} \tag{25}$$

$$\Psi_n(q,t) = \left[\frac{1}{\pi^{1/2}\hbar^{1/2}n!2^n}\right]^{\frac{1}{2}} \exp\left[-\left(\frac{q}{\rho}\right)^2 \cdot \frac{1}{2\hbar}\right] H_n\left[\left(\frac{1}{\hbar}\right)^{\frac{1}{2}}\left(\frac{q}{\rho}\right)\right] \exp\left[\frac{iL\dot{\rho}}{2\hbar\rho}q^2\right] \tag{26}$$

According to reference [3], (19) and (20) are the exact quantum invariant and wave function of the time-dependent mesoscopic LC circuit without any external power source. So, the auxiliary functions $\alpha(t)$ in the Quantum invariant operator and wave function is corresponding with the electromotive force of the external source $\xi(t)$ in the Hamiltonian of the quantum system. Therefore, the time-dependent mesoscopic RLC circuit with an external power source also can be studied by the means of quantum invariant operator method.

Conclusion

In this paper, we use the quantum invariant operator method to calculate the exact wave function of the time-dependent mesoscopic LC circuit with an external power source. Firstly, we obtain the exact quantum invariant of the Hamiltonian of the quantum systems with the method by Lewis and Leach. Secondly, through appropriate unitary transformation, the invariant operator can be simplified into a form of harmonic oscillator, whose eigenstate can be easily written. Finally, the wave function of the time-dependent LC mesoscopic circuit with an external power source has been obtained.

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