On the transitive closure of reciprocal $[0,1]$-valued relations

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Abstract

We build a theoretical framework that enables to extend the concept of transitive closure to the class of complete crisp relations, the class of reciprocal 3-valued relations and the class of reciprocal $[0,1]$-valued relations. We present algorithms to compute the transitive closure of reciprocal $[0,1]$-valued relations, where the type of transitivity is either weak stochastic transitivity or strong stochastic transitivity.

Keywords: Preference relation, reciprocal relation, stochastic transitivity, transitive closure.

1. Introduction

The concept of a preference relation has proven very useful in modelling the preferences which experts express by comparing pairwisely the alternatives from a set of alternatives. Two different representations can be adopted: crisp preference relations or $[0,1]$-valued preference relations, also called fuzzy preference relations. In the latter case, one frequently uses reciprocal $[0,1]$-valued relations Q to express the intensities of preferences, i.e. $Q(x,y)+Q(y,x)=1$ for all couples (x,y) of alternatives [2].

As experts focus on two alternatives at a time, the provided preferences could be inconsistent. In a crisp context, consistency has been defined in terms of acyclicity [8]. This condition is closely related to the transitivity of the associated strict preference relation. In a graded context, the consistency properties that have been proposed for reciprocal preference relations attempt to extend the boolean notion of transitivity. Among the types of transitivity which cover the notion of transitivity in the context of reciprocal $[0,1]$-valued relations, we mention weak stochastic transitivity, moderate stochastic transitivity, strong stochastic transitivity, minimum transitivity and multiplicative transitivity [3, 9]. Recently, some of the present authors have presented the cycle-transitivity framework for describing and studying many types of transitivity of reciprocal relations [5].

Once the consistency property has been chosen, the decision analyst frequently encounters one of the following problems: how to repair the inconsistency of preference relations and how to estimate missing values? In the present paper we focus on the first problem. So far this problem has been primordially dealt with in the context of additive consistency [7, 10], which is a stronger consistency property than any of the usual types of stochastic transitivity. Recently, we have presented a heuristic algorithm to generate a transitive reciprocal relation that approximates a given reciprocal relation for various types of transitivity. In general, however, this approximation is not distance-optimal [6].

In the present paper we investigate the existence of transitive closures within the context of reciprocal $[0,1]$-valued relations. Recall that for a given property P, the P-closure (resp. P-opening) of a crisp or fuzzy relation, if it exists and is unique, can be regarded as the relation having property P that is greater (resp. smaller) than and is closest to the given relation [1]. In the context of fuzzy relations, the existence and uniqueness of the T-transitive closure has been investigated by some of the present authors [4]. Although the class of reciprocal $[0,1]$-valued relations can be regarded as a subclass of the class of fuzzy relations, we cannot simply maintain the standard partial order on the class of fuzzy relations based on subsethood to compare reciprocal relations. Instead, we will define a (partial) order on the class of reciprocal $[0,1]$-valued relations which is compatible with the partial order on the set of crisp complete relations. This order will allow us to formally define the transitive closure of a reciprocal relation for various types of transitivity. The main part of this paper is devoted to the study of the existence and uniqueness of such transitive closures. It will turn out that the transitive closure always exists in the cases of weak stochastic transitivity and strong stochastic transitivity, however not for moderate stochastic transitivity. We also present algorithms for generating these transitive closures.

2. The general problem

Given a set $\mathcal{S}$ of (crisp) relations that share some chosen characteristics, and a certain type of transitivity property $P_{\mathcal{S}}$, we can define the closure with respect to $P_{\mathcal{S}}$ of a given relation R in $\mathcal{S}$.

Definition 1 *Given $R \in \mathcal{S}$, the $P_{\mathcal{S}}$-* closure $\overline{R}^{P_{\mathcal{S}}}$ *of R in $\mathcal{S}$, if it exists, is the relation in $\mathcal{S}$ satisfying*

(i) $\overline{R}^{P_{\mathcal{S}}}$ *is $P_{\mathcal{S}}$-transitive;*
(ii) $R \subseteq \overline{R}^{P_{\mathcal{S}}}$*;*
(iii) For $R' \in \mathcal{S}$, if $R \subseteq R'$ and R' is $P_{\mathcal{S}}$-transitive, then $\overline{R}^{P_{\mathcal{S}}} \subseteq R'$.

In [1], Bandler and Kohout have given the necessary and sufficient conditions for the existence of both closure and opening (or interior) of classical as well as fuzzy relations, with respect to a number of properties, including (min-)transitivity. These results were generalized by De Baets and De Meyer ([4]) by showing the existence of T-transitive closures, where the definition of transitivity is based on a triangular norm T.

In this paper, we will try to generalize the known results for three specific instances of the set $\mathcal{S}$, namely the set $\mathcal{C}_X$ of all complete crisp relations, the set $\mathcal{Q}_X^*$ of all reciprocal 3-valued relations and the set $\mathcal{Q}_X$, which is the extension of $\mathcal{Q}_X^*$ to $[0,1]$-valued relations.

Recall that the existence of closures is dependent on the existence of a largest relation and on whether taking the intersection preserves transitivity. Some basic definitions that will therefore be needed in each of these sets are those of union, intersection and inclusion.

3. Complete crisp relations

3.1. Basic notions

A *(crisp) relation* $R \in \mathcal{P}(X^2)$ on a universe X can be represented either by a set or by its characteristic mapping $R : X^2 \to \{0,1\}$. By convention, if $(x,y) \in R$, then $R(x,y) = 1$, else $R(x,y) = 0$. If the universe X is finite, the relation R can also be represented by a matrix $[r_{ij}]$ with entries $r_{ij} = R(x_i, x_j)$. This matrix representation is dependent on the indexing of the given universe. Although most results can be generalized, we will assume the universe X to be finite.

A crisp relation R is *complete* when it satisfies

$$(\forall (x,y) \in X^2)(R(x,y) = 0 \Rightarrow R(y,x) = 1)\,.$$

We will denote the class of all complete crisp relations on a universe X by $\mathcal{C}_X$, i.e.

$$\mathcal{C}_X = \{R \in \mathcal{P}(X^2) \mid R \text{ is complete}\}\,.$$

When using the functional representation, $R_1 \in \mathcal{C}_X$ is said to be *smaller* than $R_2 \in \mathcal{C}_X$, or R_2 to be *larger* than R_1, denoted as $R_1 \subseteq R_2$, when

$$(\forall (x,y) \in X^2)(R_1(x,y) \le R_2(x,y))\,.$$

There is a unique largest relation in $\mathcal{C}_X$, namely the relation for which all entries have a value equal to 1.

The *union* and *intersection* of two relations R_1 and R_2 on the same universe X are given by

$$(R_1 \cup R_2)(x,y) = \max(R_1(x,y), R_2(x,y))$$

and

$$(R_1 \cap R_2)(x,y) = \min(R_1(x,y), R_2(x,y))\,.$$

Note that $R_1 \cap R_2 \subseteq R_i \subseteq R_1 \cup R_2$ for $i = 1,2$.

It is obvious that the union is an internal operator on $\mathcal{C}_X$. However, the intersection of two complete relations is not necessarily a complete relation. A necessary condition for the completeness of the intersection of two complete relations is that both relations are 'like-minded'. More formally, we introduce the notion of compatibility.

Definition 2 *Two relations $R_1, R_2 \in \mathcal{C}_X$ are said to be* compatible*, denoted $R_1 \approx R_2$, if*

$$\begin{aligned}(\forall (x,y) \in X^2)((&R_1(x,y) = R_2(x,y) = 1)\\ &\vee (R_1(y,x) = R_2(y,x) = 1)\\ &\vee (R_1(x,y) = R_1(y,x) = 1)\\ &\vee (R_2(x,y) = R_2(y,x) = 1))\end{aligned}$$

Note that compatibility is a tolerance relation on $\mathcal{C}_X$, since it is reflexive and symmetric. Any two relations that are comparable, are compatible. However, compatible relations are not necessarily comparable.

Proposition 1 *For $R_1, R_2, R_3 \in \mathcal{C}_X$ the following hold:*

(i) If $R_1 \subseteq R_2$ then $R_1 \approx R_2$.
(ii) If $R_3 \subseteq R_1$ and $R_3 \subseteq R_2$ then $R_1 \approx R_2$.

The compatibility of two relations guarantees the completeness of their intersection. In general, the following proposition can be proven.

Proposition 2 *Let $R_i \in \mathcal{C}_X, i \in I$, then $\bigcap_{i \in I} R_i \in \mathcal{C}_X$ if and only if $R_i \approx R_j$ for all $i,j \in I$.*

3.2. Transitivity and transitive closure in $\mathcal{C}_X$

The classical definition of transitivity is of course still valid when restricting the set of relations to $\mathcal{C}_X$.

A crisp relation R is said to be *transitive* when for any $(x,y,z) \in X^3$

$$((R(x,y) = 1 \wedge R(y,z) = 1) \Rightarrow R(x,z) = 1)\,.$$

For a general relation $R \in \mathcal{P}(X^2)$, its transitive closure can be constructed by taking the intersection of all transitive relations that are larger than the relation R itself. Due to Propositions 1 and 2, when repeating this construction for $R \in \mathcal{C}_X$, the resulting relation will still be complete. Furthermore, as an extension of the result for general crisp relations, transitivity will be preserved when taking the intersection of compatible relations in $\mathcal{C}_X$.

Proposition 3 *Let $R_1, R_2 \in \mathcal{C}_X$. If $R_1 \approx R_2$ and both are transitive, then $R_1 \cap R_2 \in \mathcal{C}_X$ is transitive.*

From the above, we may conclude that the existence of the transitive closure is guaranteed on $\mathcal{C}_X$.

Proposition 4 *For any $R \in \mathcal{C}_X$, its transitive closure in $\mathcal{C}_X$ exists and is given by*

$$\overline{R} = \bigcap\{R' \in \mathcal{C}_X \mid R' \supseteq R \text{ and } R' \text{ is transitive}\}\,.$$

In practice, the construction of the transitive closure of a (complete) relation as given by the formula in Proposition 4, can be done by means of the Floyd-Warshall algorithm. Note that, in order to change a given relation $R \in \mathcal{C}_X$ into a transitive relation, we need to change the values of some of the entries from 0 to 1, thus in a sense adding entries to the relation. It is well known that the complexity of this algorithm is $\Theta(|X|^3)$.

Algorithm FWR
Input: X, $R \in \mathcal{C}_X$
Output: $\overline{R}$
for all $x \in X$ **do**
  for all $y \in X$ **do for all** $z \in X$ **do**
    if $R(y,x) = 1$ **and** $R(x,z) = 1$ **and** $R(y,z) = 0$
    then $R(y,z) := 1$

The dual case of finding the transitive opening suffers from the same problem as occurs in the classical case, namely the union of complete relations, although always complete, does not preserve transitivity, hence the transitive opening in $\mathcal{C}_X$ does not always exist.

4. Reciprocal 3-valued relations

In the following section we will illustrate how the class of complete crisp relations can be converted into the class of reciprocal 3-valued relations.

4.1. Basic notions

A 3-valued relation $Q : X^2 \to \{0, \frac{1}{2}, 1\}$ is called *reciprocal* if

$$(\forall (x,y) \in X^2)(Q(x,y) + Q(y,x) = 1)\,.$$

We will denote the class of all reciprocal 3-valued relations on the universe X by $\mathcal{Q}_X^*$.

Any complete crisp relation R on a universe X can be converted into a 3-valued reciprocal relation Q_R on X [5], i.e.

$$Q_R(x,y) = \frac{1 + R(x,y) - R(y,x)}{2}\,.$$

Vice versa, any reciprocal 3-valued relation Q on X can be converted into a complete crisp relation R_Q on the same universe, by means of

$$R_Q(x,y) = \begin{cases} 1 & \text{if } Q(x,y) \geq \frac{1}{2} \\ 0 & \text{if } Q(x,y) < \frac{1}{2} \end{cases}$$

These two definitions provide a 1-to-1 correspondence between $\mathcal{C}_X$ and $\mathcal{Q}_X^*$.

Proposition 5 *For $Q \in \mathcal{Q}_X^*$ and $R \in \mathcal{C}_X$, the following hold:*

(i) $Q_{R_Q} = Q$,
(ii) $R_{Q_R} = R$,
(iii) $Q_R = Q \Leftrightarrow R_Q = R$

The advantage of a reciprocal representation over the crisp representation is that the information concerning $R(x,y)$ and $R(y,x)$ is contained in $Q(x,y)$ alone, or equivalently, in $Q(y,x)$ alone. When defining inclusion, union and intersection on $\mathcal{Q}_X^*$ in accordance with the classical definition, this reciprocal character calls for a two-part definition. Due to the correspondence between $\mathcal{C}_X$ and $\mathcal{Q}_X^*$ we can extend the definition of compatibility to $\mathcal{Q}_X^*$.

Definition 3 *Two relations $Q_1, Q_2 \in \mathcal{Q}_X^*$ are said to be compatible, denoted $Q_1 \approx Q_2$, if the corresponding complete relations in $\mathcal{C}_X$ are compatible, i.e. if $R_{Q_1} \approx R_{Q_2}$.*

According to the classical ordering on $\mathcal{C}_X$, the greatest complete relation is that for which all entries have value 1. It follows that, when choosing an ordering on $\mathcal{Q}_X^*$, the greatest reciprocal relation must be the one for which all entries have value $\frac{1}{2}$. This leads to the following definition.

Definition 4 *For $Q_1, Q_2 \in \mathcal{Q}_X^*$, Q_1 is said to be* smaller *than Q_2, denoted as $Q_1 \sqsubseteq Q_2$, if*

$$(\forall (x,y) \in X^2)\left(\left(Q_1(x,y) \geq Q_2(x,y) \geq \frac{1}{2}\right)\right.$$
$$\left.\vee \left(Q_1(x,y) \leq Q_2(x,y) \leq \frac{1}{2}\right)\right)\,.$$

Definition 5 *Let $Q_1, Q_2 \in \mathcal{Q}_X^*$. The* union *of Q_1 and Q_2 is the reciprocal 3-valued relation $Q_1 \sqcup Q_2$ defined by*

$$Q_1 \sqcup Q_2(x,y) =$$
$$\begin{cases} \min(Q_1(x,y), Q_2(x,y)) & \textit{if } \min(Q_1(x,y), Q_2(x,y)) \geq \frac{1}{2} \\ \max(Q_1(x,y), Q_2(x,y)) & \textit{if } \max(Q_1(x,y), Q_2(x,y)) \leq \frac{1}{2} \\ \frac{1}{2} & \textit{else} \end{cases}$$

Note that this definition is meaningful as $Q_1 \sqcup Q_2$ always yields a reciprocal relation. Recall that the intersection of complete relations is only meaningful when both relations are compatible. Similarly, we can only consistently define the intersection of reciprocal 3-valued relations if the relations considered are compatible.

Definition 6 *Let $Q_1, Q_2 \in \mathcal{Q}_X^*$ and $Q_1 \approx Q_2$. The* intersection *of Q_1 and Q_2 is the reciprocal 3-valued relation $Q_1 \sqcap Q_2$ defined by*

$$Q_1 \sqcap Q_2(x,y) =$$
$$\begin{cases} \max(Q_1(x,y), Q_2(x,y)) & \textit{if } \min(Q_1(x,y), Q_2(x,y)) \geq \frac{1}{2} \\ \min(Q_1(x,y), Q_2(x,y)) & \textit{else} \end{cases}$$

If follows directly from this definition that, whenever compatibility is satisfied, $Q_1 \sqcap Q_2 = Q_2 \sqcap Q_1$. Furthermore, it is easily verified that the same connection between union and intersection and inclusion exists as in the case of crisp relations, namely $Q_1 \sqcap Q_2 \sqsubseteq Q_i \sqsubseteq Q_1 \sqcup Q_2$ for $i = 1, 2$. Note that this implies that both the union $Q_1 \sqcup Q_2$ and, on the condition of compatibility, the intersection $Q_1 \sqcap Q_2$ are compatible with Q_1 and Q_2.

Regarding the associativity of the intersection on $\mathcal{Q}_X^*$, the following result can be proven.

Proposition 6 *Let $Q_1, Q_2, Q_3 \in \mathcal{Q}_X^*$. If $Q_1 \approx Q_2$, $Q_1 \approx Q_3$, $Q_2 \approx Q_3$ then*

(i) $Q_1 \approx Q_2 \sqcap Q_3$, $Q_2 \approx Q_1 \sqcap Q_3$, $Q_3 \approx Q_1 \sqcap Q_2$,
(ii) $(Q_1 \sqcap Q_2) \sqcap Q_3 = Q_1 \sqcap (Q_2 \sqcap Q_3)$.

This proposition can be generalized to the case of more than three reciprocal relations Q_i, $i \in I$, that are pairwise compatible. For such relations, the intersection $\bigsqcap_{i \in I} Q_i$ is uniquely defined.

4.2. Transitivity in $\mathcal{Q}_X^*$

The transitivity condition for crisp relations is easily translated to a condition for reciprocal 3-valued relations. An entry equal to 1 in the crisp relation, means an entry equal to $\frac{1}{2}$ or 1 in the 3-valued relation.

Definition 7 *A relation $Q \in \mathcal{Q}_X^*$ is said to be* transitive *if for every $(x, y, z) \in X^3$*

$$\left(Q(x, y) \geq \frac{1}{2} \wedge Q(y, z) \geq \frac{1}{2}\right) \Rightarrow Q(x, z) \geq \frac{1}{2}.$$

Proposition 7 *For $Q \in \mathcal{Q}_X^*$, Q is transitive if and only if R_Q is transitive.*

From the correspondence between $\mathcal{C}_X$ and $\mathcal{Q}_X^*$, the existence of the transitive closure in $\mathcal{Q}_X^*$ follows.

Proposition 8 *For any $Q \in \mathcal{Q}_X^*$, its transitive closure exists and is given by*

$$\overline{Q} = \bigsqcap \{Q' \in \mathcal{Q}_X^* \mid Q \sqsubseteq Q' \text{ and } Q' \text{ is transitive}\}.$$

A simple adaptation of the Floyd-Warshall algorithm yields a practical solution to constructing the transitive closure of a given reciprocal 3-valued relation. Whereas in the crisp case, we needed to set some additional entries of the given relation equal to the value 1, in the case of a relation $Q \in \mathcal{Q}_X^*$ the corresponding operation consists of setting some of the entries of Q to the value $\frac{1}{2}$. Note that the complexity of the modified algorithm is still $\Theta(|X|^3)$.

Algorithm FWQ
Input: X, $Q \in \mathcal{Q}_X^*$
Output: $\overline{Q}$
for $x \in X$ **do**
 for $y \in X$ **do for** $z \in X$ **do**
  if $Q(y, x) \geq \frac{1}{2}$ **and** $Q(x, z) \geq \frac{1}{2}$ **and** $Q(y, z) < \frac{1}{2}$
  then $Q(y, z) := \frac{1}{2}, Q(z, y) := \frac{1}{2}$

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5. Generalization to reciprocal $[0, 1]$-valued relations

We now generalize the discussion of Section 4 to $[0, 1]$-valued relations.

5.1. Basic notions

A $[0, 1]$-valued relation $Q : X \to [0, 1]$ is called reciprocal if

$$(\forall (x, y) \in X^2)(Q(x, y) + Q(y, x) = 1).$$

We will denote the class of all reciprocal $[0, 1]$-valued relations on a universe X by $\mathcal{Q}_X$. Clearly, $\mathcal{Q}_X^* \subset \mathcal{Q}_X$.

For any $Q \in \mathcal{Q}_X$ and any $\alpha \in]0, 1]$, the α-cut $(Q)_\alpha$ of Q is given by

$$(Q)_\alpha(x, y) = \begin{cases} 1 & \text{if } Q(x, y) \geq \alpha \\ 0 & \text{if } Q(x, y) < \alpha \end{cases}$$

When we take $\alpha = \frac{1}{2}$ the resulting crisp relation, due to the reciprocity of Q, is a complete relation. For $Q \in \mathcal{Q}_X^*$, it is easily seen that $(Q)_{\frac{1}{2}} = R_Q$.

Proposition 9 *For $Q \in \mathcal{Q}_X$, $Q_{(Q)_{\frac{1}{2}}} \in \mathcal{Q}_X^*$ and $Q_{(Q)_{\frac{1}{2}}} = Q$ if and only if $Q \in \mathcal{Q}_X^*$.*

The notion of compatibility can be generalized to $\mathcal{Q}_X$ in the same way as it was generalized to $\mathcal{Q}_X^*$ in Definition 3.

Definition 8 *Two relations $Q_1, Q_2 \in \mathcal{Q}_X$ are said to be compatible, denoted $Q_1 \approx Q_2$, if the corresponding complete relations in $\mathcal{C}_X$ are compatible, i.e. if $R_{Q_1} \approx R_{Q_2}$.*

The definitions regarding union, intersection and inclusion as formulated in the previous section on $\mathcal{Q}_X^*$, can be extended to $\mathcal{Q}_X$ without difficulty, namely by taking Q_1 and Q_2 in $\mathcal{Q}_X$ instead of in $\mathcal{Q}_X^*$. It can be shown that both union and intersection on $\mathcal{Q}_X$ are still commutative and associative operations. Note that, if needed, min and max can be replaced by inf and sup, respectively.

5.2. Transitivity on $\mathcal{Q}_X$

The wider scope of values of a $[0, 1]$-valued relation makes it possible to define the transitivity in $\mathcal{Q}_X$ in different ways. A commonly used class of transitivity is that of g-stochastic transitivity, as defined by De Baets and De Meyer in [5]. We recall the definition.

Definition 9 *Let g be an increasing $[\frac{1}{2},1]^2 \to [0,1]$ mapping such that $g(\frac{1}{2},\frac{1}{2}) \le \frac{1}{2}$. A relation $Q \in \mathcal{Q}_X$ is called g-*stochastic transitive *if for any $(x,y,z) \in X^3$ it holds that:*

$$\left(Q(x,y) \ge \frac{1}{2} \wedge Q(y,z) \ge \frac{1}{2}\right) \Rightarrow \left(Q(x,z) \ge g\left(Q(x,y),Q(y,z)\right)\right).$$

The most commonly used types of stochastic transitivity are:

(i) Weak stochastic (WS) transitivity when $g = \frac{1}{2}$,
(ii) Moderate stochastic (MS) transitivity when $g = \min$,
(iii) Strong stochastic (SS) transitivity when $g = \max$.

Note that when restricted to $\mathcal{Q}_X^*$, these four types of transitivity reduce to Definition 7. In general, the restriction of g-stochastic transitivity to $\mathcal{Q}_X^*$ will coincide with Definition 7 if $g \ge \frac{1}{2}$.

5.3. Weak stochastic transitivity

The simplest case to consider is the case of WS-transitivity, since this uses the exact same definition for transitivity as was used on $\mathcal{Q}_X^*$. Examining the procedure of constructing the transitive closure in $\mathcal{Q}_X^*$, one can remark that the entries of the relation are only evaluated to verify whether they are equal to the value $\frac{1}{2}$ are not. If they are not, their precise value is irrelevant. This gives the opportunity of easily constructing the WS-transitive closure for a relation in $\mathcal{Q}_X$, namely by first reducing the relation to its $\frac{1}{2}$-cut, then constructing the closure in $\mathcal{Q}_X^*$ and finally reimposing the values of the relevant entries of the relation.

Proposition 10 *For any $Q \in \mathcal{Q}_X$, the weak stochastic transitive closure $\overline{Q}^{\mathrm{ws}}$ always exists. Furthermore if $Q^* = Q_{(Q)_{\frac{1}{2}}}$ then*

$$\overline{Q}^{\mathrm{ws}} = \overline{Q^*} \sqcup Q\,.$$

Corollary 1 *If $Q \in \mathcal{Q}_X^*$, then $\overline{Q}^{\mathrm{ws}} = \overline{Q}$.*

Note that for the construction of $\overline{Q}^{\mathrm{ws}}$, we can use **FWQ**, since the only changes made by this algorithm are setting the value of some entries equal to $\frac{1}{2}$.

5.4. Moderate stochastic transitivity

Now let us assume that $Q_1, Q_2 \in \mathcal{Q}_X$ are compatible and that both are MS-transitive. Furthermore, assume that Q_1 and Q_2 are such that for some $x,y,z \in X$ we have

$$\frac{1}{2} \le Q_1(y,z) \le Q_1(x,z) \le Q_2(x,y) \le Q_2(x,z) < Q_1(x,y) \le Q_2(y,z)\,.$$

Note that this specific ordering is in accordance with the MS-transitivity of both relations. For the intersection $Q_1 \sqcap Q_2$ of Q_1 and Q_2 we find in this case:

$$Q_1 \sqcap Q_2(x,y) = \max(Q_1(x,y),Q_2(x,y)) \ge \frac{1}{2}$$
$$Q_1 \sqcap Q_2(y,z) = \max(Q_1(y,z),Q_2(y,z)) \ge \frac{1}{2}$$

This means that, under the given assumptions, for the intersection $Q_1 \sqcap Q_2$ to be MS-transitive, it is necessary that

$$Q_1 \sqcap Q_2(x,z) \ge g(Q_1 \sqcap Q_2(x,y), Q_1 \sqcap Q_2(y,z))$$

with $g = \min$. However, evaluating this inequality with the given ordering for the entries of both relations, we find:

$$Q_2(x,z) < \min(Q_1(x,y), Q_2(y,z))\,.$$

Hence we find that MS-transitivity is not preserved by intersection of compatible relations on $\mathcal{Q}_X$. This means that a construction of the transitive closure similar to the one for WS-transitivity cannot be repeated. This leads to the fact that in general the MS-transitive closure in $\mathcal{Q}_X$ does not exist.

5.5. Strong stochastic transitivity

If, instead of MS-transitive, we assume two compatible relations $Q_1, Q_2 \in \mathcal{Q}_X$ to be SS-transitive, we find that the operation used to take the intersection of two relations commutes with the mapping $g = \max$. This is rather obvious since both are the same. In this case, the inequality

$$Q_1 \sqcap Q_2(x,z) \ge g(Q_1 \sqcap Q_2(x,y), Q_1 \sqcap Q_2(y,z))$$

can be rewritten as

$$Q_1 \sqcap Q_2(x,z) \ge \max\left(g(Q_1(x,y),Q_1(y,z)), g(Q_2(x,y),Q_2(y,z))\right).$$

The validity of this equality simply follows from the SS-transitivity of both relations. Thus, like WS-transitivity and unlike MS-transitivity, SS-transitivity is preserved by taking the intersection. A similar result as for complete crisp relations and reciprocal 3-valued relations can be proven.

Proposition 11 *For $Q \in \mathcal{Q}_X$, its strong stochastic transitive closure $\overline{Q}^{\mathrm{ss}}$ always exists and is given by*

$$\overline{Q}^{\mathrm{ss}} = \bigsqcap\{Q' \in \mathcal{Q}_X \mid Q \sqsubseteq Q' \text{ and } Q' \text{ is SS-transitive}\}\,.$$

The next question is how to construct $\overline{Q}^{\mathrm{ss}}$ in practice. First note that strong stochastic transitivity implies weak stochastic transitivity. Therefore, a first step in the construction is to apply **FWQ**. The next step would then consist of lowering the values of $Q(x, y)$ and $Q(y, z)$ to that of $Q(x, z)$, for every $(x, y, z) \in X^3$ for which

$$\min(Q(x, y), Q(y, z)) \geq \frac{1}{2}$$

and

$$Q(x, z) < \max(Q(x, y), Q(y, z)) \, .$$

When adapting Q for the triple $(x, y, z) \in X^3$, it can happen that for some other triples in X^3 the transitivity condition is broken. It can, however, be proven that doing this adjustment twice for every triple, the resulting relation is indeed the SS-transitive closure of the original relation, providing us with the algorithm **SSC** for generating the SS-transitive closure of a given reciprocal $[0, 1]$-valued relation. Since the outer loop in the algorithm is independent of the cardinality of the universe, the complexity is still $\Theta(|X|^3)$.

Algorithm SSC
Input: X, $Q \in \mathcal{Q}_X$
Output: $\overline{Q}^{\mathrm{SS}}$
Use **FWQ**
for $h := 1$ **to** 2 **do**
for $x \in X$ **do for** $y \in X$ **do**
if $Q(x, y) > \frac{1}{2}$ **then for** $z \in X$ **do**
if $Q(y, z) \geq \frac{1}{2}$ **then**
$Q(x, y) := \min(Q(x, y), Q(x, z))$
$Q(y, x) := 1 - Q(x, y)$
if $Q(z, x) \geq \frac{1}{2}$ **then**
$Q(x, y) := \min(Q(x, y), Q(z, y))$
$Q(y, x) := 1 - Q(x, y)$

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6. Illustrating examples

Example 1 *Consider the complete crisp relation R_1. This relation is not transitive due to $R_1(x, u) = 0$, $R_1(y, x) = 0$ and $R_1(u, y) = 0$. Adjusting these values yields the transitive closure of R_1.*

R_1	x	y	z	u
x	1	1	1	0
y	0	1	1	1
z	0	0	1	0
u	1	0	1	1

$\overline{R_1}$	x	y	z	u
x	1	1	1	1
y	1	1	1	1
z	0	0	1	0
u	1	1	1	1

(1)

Example 2 *Checking WS-transitivity for the following reciprocal relation Q_1, it is seen that the values of entries $Q_1(x, u)$, $Q_1(y, x)$ and $Q_1(u, y)$ need adjustment.*

Q_1	x	y	z	u
x	0.5	0.9	0.65	0.2
y	0.1	0.5	0.6	1
z	0.35	0.4	0.5	0.4
u	0.8	0	.6	0.5

Note that $R_{Q_1} = R_1$ and that the entries that need adjustment to construct the closures are the same for Q_1 as for R_1, regardless of the specific values of Q_1.

$\overline{Q_1}$	x	y	z	u
x	0.5	0.5	0.65	0.5
y	0.5	0.5	0.6	0.5
z	0.35	0.4	0.5	0.4
u	0.5	0.5	0.6	0.5

Further checking shows that $\overline{Q_1}$ is MS-transitive too. SS-transitivity is not satisfied due to

$$\overline{Q_1}(y, z) < \max(\overline{Q_1}(y, x), \overline{Q_1}(x, z)) \, ,$$
$$\overline{Q_1}(u, z) < \max(\overline{Q_1}(u, x), \overline{Q_1}(x, z)) \, .$$

This is however solved by adjusting the value of $\overline{Q_1}(x, z))$, which leads to $\overline{Q_1}^{\mathrm{SS}}$.

$\overline{Q_1}^{\mathrm{SS}}$	x	y	z	u
x	0.5	0.5	0.6	0.5
y	0.5	0.5	0.6	0.5
z	0.3	0.4	0.5	0.4
u	0.5	0.5	0.6	0.5

Example 3 *Consider $Q_2, Q_3 \in \mathcal{Q}_X$.*

Q_2	x	y	z
x	0.5	0.9	0.65
y	0.1	0.5	0.6
z	0.35	0.4	0.5

Q_3	x	y	z
x	0.5	0.7	0.8
y	0.3	0.5	1
z	0.2	0	0.5

(2)

Both relations are MS-transitive, however $Q_2 \sqcap Q_3$ is not, due to

$$Q_2 \sqcap Q_3(x, z) < \min(Q_2 \sqcap Q_3(x, y), Q_2 \sqcap Q_3(y, z)) \, .$$

$Q_2 \sqcap Q_3$	x	y	z
x	0.5	0.9	0.8
y	0.1	0.5	1
z	0.2	0	0.5

It can be seen that there are two relations that are minimal in the fact that they are MS-transitive and larger than $Q_2 \sqcap Q_3$.

Q_{23}	x	y	z
x	0.5	0.8	0.8
y	0.2	0.5	1
z	0.2	0	0.5

Q_{32}	x	y	z
x	0.5	0.9	0.8
y	0.1	0.5	0.8
z	0.2	0.2	0.5

Finally, it is easily verified that $\overline{Q_2 \sqcap Q_3}^{\mathrm{SS}}$ is the relation given as

$\overline{Q_2 \sqcap Q_3}^{\mathrm{SS}}$	x	y	z
x	0.5	0.8	0.8
y	0.2	0.5	0.8
z	0.2	0.2	0.5

7. Conclusion

We have set up a framework for reciprocal $[0, 1]$-valued relations, based on the class of all complete crisp relations. Within this framework we have considered three types of g-stochastic transitivity: weak stochastic transitivity, moderate stochastic transitivity and strong stochastic transitivity. We have shown that for both weak and strong stochastic transitivity, the corresponding transitive closure always exists, but in the case of moderate stochastic transitivity the transitive closure does not exist in general.

A similar analysis can be done with regard to the concept dual to that of transitive closure, namely transitive opening, or interior. Supposing, for $g \geq \frac{1}{2}$, a g-stochastic transitive opening exists for every $Q \in \mathcal{Q}_X$, then it must also exist for any $Q \in \mathcal{Q}_X^* \subset \mathcal{Q}_X$. But in that case the g-stochastic transitivity is reduced to Definition 7, so according to that definition, any reciprocal 3-valued relation will have a transitive opening in $\mathcal{Q}_X^*$. Finally, this would imply that any complete crisp relation has a transitive opening in $\mathcal{C}_X$. It has been noted that the union of complete relations does not preserve transitivity. Furthermore, it can be seen that a unique smallest complete relation does not exist. Thus, it can be concluded that in general the g-stochastic transitive opening, with $g \geq \frac{1}{2}$, does not exist.

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