On generalizations of weighted means and OWA operators

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Abstract

In this paper we analyze two classes of functions proposed in the literature to simultaneously generalize weighted means and OWA operators: WOWA operators and HWA operators. Since, in some cases, the results provided by these operators may be questionable, we introduce functions that also generalize both operators and characterize those that satisfy a condition imposed to maintain the relationship among the weights.

Keywords: Weighted means, OWA operators, WOWA operators, HWA operators.

1. Introducción

Weighted means and ordered weighted averaging (OWA) operators (Yager [12]) are well-known functions widely used in the aggregation processes. Although both are defined through a weighting vector, their behavior is quite different: The weighted means allow to weight each information source in relation to their reliability while OWA operators allow to weight the values according to their ordering.

The need to combine both functions has been reported by several authors (see, among others, Torra [6] and Torra and Narukawa [9]). For this reason, two classes of functions have appeared in the literature with the intent of simultaneously generalizing weighted means and OWA operators: the weighted OWA (WOWA) operator (Torra [6]) and the hybrid weighted averaging (HWA) operator (Xu and Da [11]).

The aim of this paper is to analyze WOWA operators and HWA operators. Moreover, since, in some cases, the results provided by these operators may be questionable, we propose to use functions that maintain the relationship among the weights of a weighting vector when the non-zero components of the other weighting vector are equal. In this way, we obtain a class of functions that have been previously introduced by Engemann *et al.* [2] in a framework of decision making under risk and uncertainty.

The paper is organized as follows. In Section 2 we introduce weighted means, OWA operators and the two classes of functions proposed in the literature to simultaneously generalize weighted means and OWA operators: WOWA operators and HWA operators. Section 3 shows some drawbacks of both generalizations. In Section 4 we propose a condition to maintain the relationship among the weights and characterize the functions that satisfy this condition. The paper concludes in Section 5.

2. Preliminares

Throughout the paper we will use the following notation: vectors will be denoted in bold; $\boldsymbol{\eta}$ will denote the vector $(1/n, \ldots, 1/n)$; $\boldsymbol{x} \geq \boldsymbol{y}$ will mean $x_i \geq y_i$ for all $i \in \{1, \ldots, n\}$; given σ a permutation of $\{1, \ldots, n\}$, $\boldsymbol{x}_\sigma$ will denote the vector $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

In the following definition we present some well-known properties usually demanded to the functions used in the aggregation processes.

Definition 1. *Let $F : \mathrm{R}^n \longrightarrow \mathrm{R}$ be a function.*

1. *F is symmetric if for all $\boldsymbol{x} \in \mathrm{R}^n$ and for all permutation σ of $\{1, \ldots, n\}$ the following holds:*
$$F(\boldsymbol{x}_\sigma) = F(\boldsymbol{x}).$$
2. *F is monotonic if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{R}^n$ the following holds:*
$$\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; F(\boldsymbol{x}) \geq F(\boldsymbol{y}).$$
3. *F is idempotent if for all $x \in \mathrm{R}$ the following holds:*
$$F(x, \ldots, x) = x.$$
4. *F is compensative (also called internal) if for all $\boldsymbol{x} \in \mathrm{R}^n$ the following holds:*
$$\min(\boldsymbol{x}) \leq F(\boldsymbol{x}) \leq \max(\boldsymbol{x}).$$
5. *F is homogeneous of degree 1 if for all $\boldsymbol{x} \in \mathrm{R}^n$ and for all $\lambda > 0$ the following holds:*
$$F(\lambda \boldsymbol{x}) = \lambda F(\boldsymbol{x}).$$

2.1. Weighted means and OWA operators

Weighted means and OWA operators are defined by vectors with non-negative components whose sum is 1.

Definition 2. *A vector $\boldsymbol{\mu} \in \mathrm{R}^n$ is a weighting vector if $\boldsymbol{\mu} \in [0, 1]^n$ and $\sum_{i=1}^{n} \mu_i = 1$.*

Definition 3. *Let* $\boldsymbol{p}$ *be a weighting vector. The weighted mean associated with* $\boldsymbol{p}$ *is the function* $F_{\boldsymbol{p}} : \mathrm{R}^n \longrightarrow \mathrm{R}$ *given by*

$$F_{\boldsymbol{p}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i.$$

The weighted means are monotonic, idempotent, compensative and homogeneous of degree 1 functions.

Yager [12] introduced OWA operators as a tool for aggregation procedures in multicriteria decision making. An OWA operator is similar to a weighted mean, but with the values of the variables previously ordered in a decreasing way. Thus, contrary to the weighted means, the weights are not associated with concrete variables. Consequently, OWA operators satisfy symmetry. Moreover, OWA operators also exhibit some other interesting properties such as monotonicity, idempotence, compensativeness and homogeneity of degree 1.

Definition 4. *Let* $\boldsymbol{w}$ *be a weighting vector. The OWA operator associated with* $\boldsymbol{w}$ *is the function* $F^{\boldsymbol{w}} : \mathrm{R}^n \longrightarrow \mathrm{R}$ *given by*

$$F^{\boldsymbol{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{\sigma(i)},$$

where σ *is a permutation of* $\{1, \ldots, n\}$ *such that* $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$.

One of the most important issues in the theory of OWA operators is the determination of associated weights (see, for instance, Xu [10] and Fullér [3]). In [13], Yager relates the OWA operators weights to quantifiers.

Definition 5. *A function* $Q : [0,1] \longrightarrow [0,1]$ *is a quantifier if it satisfies the following properties:*

1. $Q(0) = 0$.
2. $Q(1) = 1$.
3. $x > y \;\Rightarrow\; Q(x) \geq Q(y)$*; i.e., it is a non-decreasing function.*

Given a quantifier Q, the OWA operator weights can be obtained from the following expression (Yager [13]):

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \ldots, n.$$

From this relation follows:

$$Q\left(\frac{i}{n}\right) = \sum_{j=1}^{i} w_j, \quad i = 1, \ldots, n;$$

i.e., the same weighting vector can be obtained through any quantifier interpolating the points $\left(i/n, \sum_{j=1}^{i} w_j\right)$, $i = 1, \ldots, n$.

2.2. Generalizations of the weighted means and OWA operators

Two classes of functions have been proposed in the literature to simultaneously generalize weighted means and OWA operators: WOWA operators and HWA operators.

WOWA operators were introduced by Torra [6] in order to consider situations where both the importance of information sources and the importance of values had to be taken into account.

Definition 6. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors. The WOWA operator associated with* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *is the function* $W_{\boldsymbol{p}}^{\boldsymbol{w}} : \mathrm{R}^n \longrightarrow \mathrm{R}$ *given by*

$$W_{\boldsymbol{p}}^{\boldsymbol{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \mu_i x_{\sigma(i)},$$

where σ *is a permutation of* $\{1, \ldots, n\}$ *such that* $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ *and the weight* μ_i *is defined as*

$$\mu_i = f\left(\sum_{j=1}^{i} p_{\sigma(j)}\right) - f\left(\sum_{j=1}^{i-1} p_{\sigma(j)}\right),$$

where f *is a non-decreasing function that interpolates the points* $\left(i/n, \sum_{j=1}^{i} w_j\right)$ *together with the point* $(0,0)$*. Moreover,* f *is the identity when the points can be interpolated in this way.*

Different interpolation functions provide different results (on this, see Torra and Lv [8]). On the other hand, it is worth noting that any quantifier generating the weighting vector $\boldsymbol{w}$ satisfies the required properties of f given in the previous definition (under the assumption that the quantifier is the identity when $\boldsymbol{w} = \boldsymbol{\eta}$). For this reason, it is possible to give an alternative definition of WOWA operators using quantifiers (Torra and Godo [7]).

Definition 7. *Let* $\boldsymbol{p}$ *be a weighting vector and let* Q *be a quantifier. The WOWA operator associated with* $\boldsymbol{p}$ *and* Q *is the function* $W_{\boldsymbol{p}}^{Q} : \mathrm{R}^n \longrightarrow \mathrm{R}$ *given by*

$$W_{\boldsymbol{p}}^{Q}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \mu_i x_{\sigma(i)},$$

where σ *is a permutation of* $\{1, \ldots, n\}$ *such that* $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ *and the weight* μ_i *is defined as*

$$\mu_i = Q\left(\sum_{j=1}^{i} p_{\sigma(j)}\right) - Q\left(\sum_{j=1}^{i-1} p_{\sigma(j)}\right).$$

WOWA operators are monotonic, idempotent, compensative and homogeneous of degree 1 functions. Moreover, $W_{\boldsymbol{p}}^{\boldsymbol{\eta}} = F_{\boldsymbol{p}}$ and $W_{\boldsymbol{\eta}}^{\boldsymbol{w}} = F^{\boldsymbol{w}}$ (Torra [6]).

The second class of function that simultaneously generalize weighted means and OWA operators were introduced by Xu and Da [11].

Definition 8. *Let* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *be two weighting vectors. The HWA operator associated with* $\boldsymbol{p}$ *and* $\boldsymbol{w}$ *is the function* $H_{\boldsymbol{p}}^{\boldsymbol{w}} : \mathrm{R}^n \longrightarrow \mathrm{R}$ *given by*

$$H_{\boldsymbol{p}}^{\boldsymbol{w}}(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i (np_{\sigma(i)} x_{\sigma(i)}),$$

where σ *is a permutation of* $\{1, \dots, n\}$ *such that* $p_{\sigma(1)} x_{\sigma(1)} \geq \cdots \geq p_{\sigma(n)} x_{\sigma(n)}$.

As we can see in the previous definition, the HWA operator associated with $\boldsymbol{p}$ and $\boldsymbol{w}$ is the composition of the OWA operator associated with $\boldsymbol{w}$, $F^{\boldsymbol{w}}$, with the function $H : \mathrm{R}^n \longrightarrow \mathrm{R}^n$ defined by

$$H(x_1, \dots, x_n) = (np_1 x_1, \dots, np_n x_n).$$

It is easy to check that $H_{\boldsymbol{p}}^{\boldsymbol{\eta}} = F_{\boldsymbol{p}}$ and $H_{\boldsymbol{\eta}}^{\boldsymbol{w}} = F^{\boldsymbol{w}}$ (Xu and Da [11]). Moreover, it is also straightforward to check that HWA operators are monotonic and homogeneous of degree 1 functions.

3. Analysis of WOWA operators and HWA operators

In this section we illustrate with examples some questionable behaviors of WOWA operators and HWA operators.

3.1. WOWA operators

As we have seen in the previous section, WOWA operators satisfy many interesting properties. However, they do not always provide the expected result as we show in the following examples.

Example 1. Suppose we have five sensors to measure a certain physical property. The sensors are of different quality and precision, so they are weighted according to the weighting vector $\boldsymbol{p} = (0.3, 0.2, 0.2, 0.2, 0.1)$. Moreover, to prevent a faulty sensor alter the measurement, we take the weighting vector $\boldsymbol{w} = (0, 1/3, 1/3, 1/3, 0)$; thus, the maximum and minimum values are not considered.

Given $\boldsymbol{w} = (0, 1/3, 1/3, 1/3, 0)$, we have to choose a quantifier interpolating the points $(0, 0)$, $(0.2, 0)$, $(0.4, 1/3)$, $(0.6, 2/3)$, $(0.8, 1)$ and $(1, 1)$. We consider the quantifier given by

$$Q(x) = \begin{cases} 0 & \text{if } x \leq 0.2, \\ \frac{5}{3}x - \frac{1}{3} & \text{if } 0.2 < x < 0.8, \\ 1 & \text{if } x \geq 0.8, \end{cases}$$

which is depicted in Figure 1.

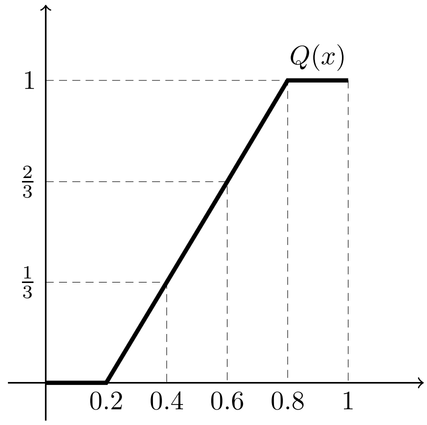

Figure 1: Quantifier associated to the weighting vector $\boldsymbol{w} = (0, 1/3, 1/3, 1/3, 0)$.

Suppose the values obtained by the sensors are $\boldsymbol{x} = (10, 4, 5, 6, 3)$. If σ is a permutation ordering these values in a decrease way, then, in this case, $\boldsymbol{p}_\sigma = \boldsymbol{p} = (0.3, 0.2, 0.2, 0.2, 0.1)$. The weighting vector $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_1 &= Q(0.3) - Q(0) &&= 1/6, \\ \mu_2 &= Q(0.5) - Q(0.3) &&= 1/3, \\ \mu_3 &= Q(0.7) - Q(0.5) &&= 1/3, \\ \mu_4 &= Q(0.9) - Q(0.7) &&= 1/6, \\ \mu_5 &= Q(1) \quad - Q(0.8) &&= 0, \end{aligned}$$

and the value returned by the WOWA operator is

$$W_{\boldsymbol{p}}^{\boldsymbol{w}}(10, 4, 5, 6, 3) = \frac{10}{6} + 2 + \frac{5}{3} + \frac{4}{6} = 6.$$

However, our intention is not to consider the maximum and minimum values and only take into account the values 4, 5 and 6; which have been provided by sensors with the same weight. Therefore, it seems logical to make the average of these values, in which case we would get 5 as final value.

Example 2. Consider again the situation of the previous example and suppose now that $\boldsymbol{p} = (0.4, 0.2, 0.2, 0.1, 0.1)$ and $\boldsymbol{x} = (10, 3, 5, 6, 7)$. If σ is a permutation ordering these values from the largest to the smallest element, then $\boldsymbol{p}_\sigma = (0.4, 0.1, 0.1, 0.2, 0.2)$. The weighting vector $\boldsymbol{\mu}$ is

$$\begin{aligned} \mu_1 &= Q(0.4) - Q(0) &&= 1/3, \\ \mu_2 &= Q(0.5) - Q(0.4) &&= 1/6, \\ \mu_3 &= Q(0.6) - Q(0.5) &&= 1/6, \\ \mu_4 &= Q(0.8) - Q(0.6) &&= 1/3, \\ \mu_5 &= Q(1) \quad - Q(0.8) &&= 0, \end{aligned}$$

and the value returned by the WOWA operator is

$$W_{\boldsymbol{p}}^{\boldsymbol{w}}(10, 3, 5, 6, 7) = \frac{10}{3} + \frac{7}{6} + 1 + \frac{5}{3} = \frac{43}{6}.$$

As in the previous example, we do not want to consider the maximum and minimum values and to aggregate the remaining ones, in this case the values 5, 6, and 7. However, the WOWA operator returns a value greater than the three aggregate values because it weights the maximum (10 in this case) with 1/3.

It is important to emphasize that when $\boldsymbol{p} = (0.4, 0.2, 0.2, 0.1, 0.1)$ and $\boldsymbol{w} = (0, 1/3, 1/3, 1/3, 0)$ (and regardless of the quantifier used), the weight assigned by the WOWA operator to the first sensor is 1/3 when its value is the maximum or the minimum. Therefore, the intended purpose of using the weighting vector $\boldsymbol{w}$ (not considering the maximum and minimum values) is not reached. On the other hand, the weight assigned by the WOWA operator to the first sensor can be up to 2/3 (for instance, when its value is the median of the values). In this case, it may be that three values given by the sensors are not taken into account (for instance, when $\boldsymbol{p}_\sigma = (0.1, 0.1, 0.4, 0.2, 0.2)$).

On the other hand, there are other interesting properties that the WOWA operator does not satisfy:

1. The value returned by the WOWA operator does not always lie between the values returned by the weighted mean and the OWA operator:

$$\begin{aligned} F^{\boldsymbol{w}}(10,3,5,6,7) &= \frac{7}{3} + 2 + \frac{5}{3} = 6, \\ F_{\boldsymbol{p}}(10,3,5,6,7) &= 4 + 0.6 + 1 + 0.6 + 0.7 \\ &= 6.9, \end{aligned}$$

but $W_{\boldsymbol{p}}^{\boldsymbol{w}}(10,3,5,6,7) = 43/6$.

2. The value returned by the WOWA operator does not always coincide with the values returned by the weighted mean and the OWA operator when both are equal:

$$\begin{aligned} F^{\boldsymbol{w}}(8,2.5,5,6,7) &= \frac{7}{3} + 2 + \frac{5}{3} = 6, \\ F_{\boldsymbol{p}}(8,2.5,5,6,7) &= 3.2 + 0.5 + 1 + 0.6 + 0.7 \\ &= 6, \end{aligned}$$

but

$$W_{\boldsymbol{p}}^{\boldsymbol{w}}(8,2.5,5,6,7) = \frac{8}{3} + \frac{7}{6} + 1 + \frac{5}{3} = 6.5.$$

3.2. HWA operators

Although HWA operators are monotonic and homogeneous of degree 1 functions, they are neither idempotent nor compensative, as we show in the following example.

Example 3. Consider again the situation of Example 2, where $\boldsymbol{p} = (0.4, 0.2, 0.2, 0.1, 0.1)$ and $\boldsymbol{w} = (0, 1/3, 1/3, 1/3, 0)$. If 10 is the value returned by all sensors, then the vector of components $np_{\sigma(i)}x_{\sigma(i)}$ is $(20, 10, 10, 5, 5)$ and

$$H_{\boldsymbol{p}}^{\boldsymbol{w}}(10,10,10,10,10) = \frac{25}{3} \neq 10;$$

that is, $H_{\boldsymbol{p}}^{\boldsymbol{w}}$ is not idempotent. On the other hand, if the values obtained by the sensors are $\boldsymbol{x} = (10, 5, 5, 8, 6)$, then the vector of components $np_{\sigma(i)}x_{\sigma(i)}$ is $(20, 5, 5, 4, 3)$, and

$$H_{\boldsymbol{p}}^{\boldsymbol{w}}(10,5,5,8,6) = \frac{14}{3} < \min\{10,5,5,8,6\};$$

i.e., $H_{\boldsymbol{p}}^{\boldsymbol{w}}$ is not compensative.

4. Choosing functions to maintain the relationship among the weights

As we have seen in the previous section, the HWA operators are neither idempotent nor compensative. For their part, WOWA operators have good properties but they do not always return the expected value. If we consider again Example 2, we want to aggregate the values 5, 6, and 7, which are the values given by the sensors with weights 0.2, 0.1, and 0.1, respectively. One possibility is to weight these values by means of the weighting vector $(0.5, 0.25, 0.25)$. In this way, it is possible to maintain the relationship among the initial weights. The returned value in this case is 23/4.

According to the above remarks, we look for a function $F_{\boldsymbol{p}}^{\boldsymbol{w}} : \mathrm{R}^n \longrightarrow \mathrm{R}$ given by

$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \rho_i x_{\sigma(i)},$$

where σ is a permutation of $\{1, \ldots, n\}$ such that $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ and the weight ρ_i is defined as

$$\rho_i = \frac{f(w_i, p_{\sigma(i)})}{\sum_{j=1}^{n} f(w_j, p_{\sigma(j)})},$$

where $f : [0,1]^2 \longrightarrow [0,1]$. In this way the weights ρ_i depend on the weights w_i and $p_{\sigma(i)}$.

In order to maintaining the relationship among the weights of a vector ($\boldsymbol{p}$ or $\boldsymbol{w}$) when the nonzero components of the other vector are equal, it is necessary that f satisfies the following condition:

$$f(tx, y) = f(x, ty) = tf(x, y),$$

for all $x, y \in [0,1]$ and $t \in [0, \infty)$ with $tx, ty \in [0,1]$. In the next proposition we characterize the functions that satisfy this condition.

Proposition 1. *Let $f : [0,1]^2 \longrightarrow [0,1]$ be a function such that*

$$f(tx, y) = f(x, ty) = tf(x, y)$$

for all $x, y \in [0,1]$ and $t \in [0, \infty)$ with $tx, ty \in [0,1]$. Then $f(x, y) = cxy$, where $c \in [0,1]$.

PROOF:

Given $x, y \in [0,1]$,

$$\begin{aligned} f(x, y) &= f(x \cdot 1, y \cdot 1) = xf(1, y \cdot 1) \\ &= xyf(1,1). \quad \blacksquare \end{aligned}$$

If $f(x, y) = cxy$, with $c \in [0,1]$, then

$$\rho_i = \frac{w_i p_{\sigma(i)}}{\sum_{j=1}^{n} w_j p_{\sigma(j)}};$$

that is,

$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(x_1, \ldots, x_n) = \frac{\sum_{i=1}^{n} w_i p_{\sigma(i)} x_{\sigma(i)}}{\sum_{j=1}^{n} w_j p_{\sigma(j)}}.$$

It is worth noting that this function has been used by Engemann *et al.* [2] in a framework of decision making under risk and uncertainty (in this case, $\boldsymbol{p}$ is the vector of probabilities of the states of nature).

In order to ensure that $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ is well defined, we need that $w_j p_{\sigma(j)}$ be non-zero for some $j \in \{1, \ldots, n\}$. This requirement is guaranteed by any of the following conditions:

1. The number of non-zero weights in each vector $\boldsymbol{p}$ and $\boldsymbol{w}$ is greater than $n/2$.
2. All the components of $\boldsymbol{p}$ are non-zero.

It is important to point out that the last condition can be assumed without loss of generality (if any component of $\boldsymbol{p}$ is zero, this means that the weight of this information source is null; so it is not necessary to take it into account).

In addition to this, $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ has another problem in your definition: sometimes, the vector $\boldsymbol{p}_\sigma$ is not unique and $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ may return different values according to the vector $\boldsymbol{p}_\sigma$ used. This fact is illustrated in the following example.

Example 4. Consider $\boldsymbol{p} = (0.5, 0.2, 0.3)$, $\boldsymbol{w} = (0.3, 0.4, 0.3)$, and $\boldsymbol{x} = (7, 5, 7)$. When $\boldsymbol{x}$ is ordered from greatest to least, then we have the vector $(7, 7, 5)$. In this vector, the first component can be associated to the weight 0.5 or 0.3. In the first case, the components of the weighting vector $\boldsymbol{\rho}$ are

$$\rho_1 = \frac{5}{11}, \qquad \rho_2 = \frac{4}{11}, \qquad \rho_3 = \frac{2}{11},$$

and

$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(7, 5, 7) = \frac{35}{11} + \frac{28}{11} + \frac{10}{11} = \frac{73}{11}.$$

In the second case, the components of the weighting vector $\boldsymbol{\rho}$ are

$$\rho_1 = \frac{9}{35}, \qquad \rho_2 = \frac{4}{7}, \qquad \rho_3 = \frac{6}{35},$$

and

$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(7, 5, 7) = \frac{9}{5} + 4 + \frac{6}{7} = \frac{233}{35}.$$

A similar problem arises in the IOWA operators, introduced by Yager and Filev [14]. The solution proposed by these authors, applied to our framework, is to replace the weights associated to equal values by the average of them. In the previous example we replace the weights $p_1 = 0.5$ and $p_3 = 0.3$ by 0.4. In this case the components of the weighting vector $\boldsymbol{\rho}$ are

$$\rho_1 = \frac{6}{17}, \qquad \rho_2 = \frac{8}{17}, \qquad \rho_3 = \frac{3}{17},$$

and

$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(7, 5, 7) = \frac{42}{17} + \frac{56}{17} + \frac{15}{17} = \frac{113}{17}.$$

It is worth noting that the behavior of this function is similar to a weighted mean, but where the weighting vector varies depending on $\boldsymbol{x}$[1]. For this reason, $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ is idempotent and compensative. Moreover, it is easy to check that $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ is homogeneous of degree 1 and that $F_{\boldsymbol{p}}^{\boldsymbol{\eta}} = F_{\boldsymbol{p}}$ (Engemann *et al.* [2]) and $F_{\boldsymbol{\eta}}^{\boldsymbol{w}} = F^{\boldsymbol{w}}$.

Nevertheless, as noted by Liu [5], $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ is not monotonic. In fact, as we show in the following example, $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ is not monotonic although the non-zero components of the weighting vector $\boldsymbol{w}$ are equal.

Example 5. Consider again the vectors $\boldsymbol{p} = (0.4, 0.2, 0.2, 0.1, 0.1)$ and $\boldsymbol{w} = (0, 1/3, 1/3, 1/3, 0)$. Then, we have:

$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(6.5, 3, 5, 6, 7) = \frac{26}{7} + \frac{6}{7} + \frac{10}{7} = 6,$$
$$F_{\boldsymbol{p}}^{\boldsymbol{w}}(10, 3, 5, 6, 7) = \frac{7}{4} + \frac{6}{4} + \frac{10}{4} = \frac{23}{4}.$$

On the other hand, similar to WOWA operators, there are other interesting properties that $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ does not satisfy:

1. The value returned by $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ does not always lie between the values returned by the weighted mean and the OWA operator:
$$F^{\boldsymbol{w}}(10, 3, 5, 6, 7) = 6,$$
$$F_{\boldsymbol{p}}(10, 3, 5, 6, 7) = 6.9,$$
but $F_{\boldsymbol{p}}^{\boldsymbol{w}}(10, 3, 5, 6, 7) = 23/4$.
2. The value returned by $F_{\boldsymbol{p}}^{\boldsymbol{w}}$ does not always coincide with the value returned by the weighted mean and the OWA operator when both values are the same:
$$F^{\boldsymbol{w}}(8, 2.5, 5, 6, 7) = 6,$$
$$F_{\boldsymbol{p}}(8, 2.5, 5, 6, 7) = 6,$$
but $F_{\boldsymbol{p}}^{\boldsymbol{w}}(8, 2.5, 5, 6, 7) = 23/4$.

5. Concluding remarks

In this paper we have analyzed the functions proposed in the literature to simultaneously generalize weighted means and OWA operators. The HWA operators are neither idempotent nor compensative, and the WOWA operators do not always provide the expected result. Due to the questionable behavior of these operators, we have imposed a condition to maintain the relationship among the weights and we have characterized the functions that satisfy this condition. However, the obtained functions are not monotonic. So, we can conclude that none of the analyzed functions is fully convincing.

[1] This behavior is also seen in OWA operators and *mixture operators*. These last functions were introduced by Marques Pereira and Pasi [4] and they are a particular case of Bajraktarević means [1].

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