Oscillation Theorems for Certain Even Order Nonlinear Neutral Functional Differential Equations

Yan Zhao, Tongxing Li, Zhenlai Han*
School of Mathematical Sciences,
University of Jinan, P. R. China

***Abstract*—In this paper, some oscillation theorems of differential equation of the even order are established** $(y(s)+P(s)y(\tau(s)))^{(n)}+Q(s)F(y(\sigma(s)))=0,$ $t\ge t_0$, **here** $n\ge 2$ **is even,** $0\le P(s)\le P_0<\infty$. **The theorems extend and supplement some results in the literature. An example is investigated to illustrate our results.**

Keywords-neutral functional differential equations; oscillation; even order.

I. INTRODUCTION

Neutral differential equations can be applied to numerous applications in engineering technique fields, physics, and so on. They are used for the research of distributed networks, for instance, see [1].

In the last decades, many researches have been made about the oscillation of solutions of neutral functional differential equations, we refer the reader to the papers [2–26].

In the paper, we consider the oscillatory property of nonlinear neutral differential equations of the even order

$$(y(t)+P(t)y(\tau(t)))^{(n)}+Q(t)F(y(\sigma(t)))=0, \quad t\ge t_0, \tag{1}$$

here $n\ge 2$ is even.

Below, we assume that

(H_1) Q and P are continuous functions, $0\le P(t)\le P_0<\infty$,

$$Q(t)>0,\ t\in[t_*,\infty),\ t_*\ge t_0;$$

(H_2) τ' and σ' are continuous functions, $\sigma'(t)\ge 0,\ \tau'(t)=\tau_0>0,\ \sigma(t)\le t,$

$\lim_{t\to\infty}\sigma(t)=\infty$, $\sigma(\tau(t))=\tau(\sigma(t)),$ τ_0 is a constant.

(H_3) $F(u)/u\ge\alpha>0$, F is a continuous functions, $u\ne 0,$ α is a constant.

Grammatikopoulos, Ladas and Meimaridou [6] examined the oscillation of equations of second order

$$[y(s)+P(s)y(s-\tau)]''+Q(s)y(s-\sigma)=0,\ s\ge s_0,$$

here $0\le P(s)<1.$

Baculíková and Lacková [3], Džurina and Hudáková [4] and Liu and Bai [14] examined neutral equations of the second order

$$(R(s)\,|Y'(s)|^{\alpha-1}\,Y'(s))'+Q(s)\,|\,y(\sigma(s))|^{\alpha-1}\,y(\sigma(s))=0, \quad s\ge s_0,$$

here $Y(s)=y(s)+P(s)y(\tau(s)),$ $\sigma(s)\le s,$ $0\le P(s)<1$.

Han et al. [9] and Ye and Xu [22] considered quasilinear equations of the second order

$$(R(s)\psi(y(s))\,|Y'(s)|^{\alpha-1}\,Y'(s))'+Q(s)F(y(\sigma(s)))=0, \quad s\ge s_0,$$

here $Y(s)=y(s)+P(s)y(\tau(s)),\ \tau(s)\le s,\ \sigma(s)\le s,$ $0\le P(s)<1.$

Meng and Xu [15] examined the oscillation of equations of even order

$$[R(s)\,|\,(y(s)+P(s)y(s-\tau))^{(n-1)}\,|^{\alpha-1}\,(y(s)+P(s)y(s-\tau))^{(n-1)}]'+Q(s)F(y(\sigma(s)))=0,$$

$s\ge s_0,$ here $0\le P(s)<1,\ \ \sigma(s)\le s.$

Zafer [23] examined oscillation of the equation

$$[y(s)+P(s)y(\tau(s))]^{(n)}+F(s,y(s),y(\sigma(s)))=0, \quad s\ge s_0,$$

here $1>P(s)>0.$

Zhang, Yan and Gao [24] investigated the oscillation of (1.1), where $0<P(s)<1.$

Clearly, the above oscillation theorems cannot be used to (1.1) when $1<P(s),$ it has few oscillation theorems about (1.1), when $1<P(t).$

Han et al. [7, 8] studied equation (1.1) when $n=2$ and $0\le P(s)\le P_0<\infty.$ Li et al. [11] considered (1.1), when $\infty>P_0\ge P(s)>0$ and $\sigma(s)\ge\tau(s).$

Motivated by [3,4,11,14], we will investigate equation (1),and offer new oscillation theorems for (1) where $\sigma(t) \le \tau(t)$, by employing function Y, and operator T. Our results extend and supplement recent results given by [3] and [11], respectively.

Following [14], $\phi = \phi(t,s,l)$ belongs to the function class Y, denote by $\phi \in Y$, if ϕ continuous functions, which satisfies $\phi(t,s,l) > 0$, $\phi(t,t,l) = 0$, $\phi(t,l,l) = 0$, $l < s < t$, $\partial\phi/\partial s$ is locally integrable with respect to s. By choosing ϕ, it is possible to derive some oscillation theorems to another equations.

Define $T[\cdot;l,t]$,

$$T[g;l,t] = \int_l^t \phi(t,s,l) g(s) \mathrm{d}s, \tag{2}$$

$g \in C([t_0,\infty), R)$, $t \ge s \ge l \ge t_0$. $\varphi = \varphi(t,s,l)$ is defined by

$$\frac{\partial\phi(t,s,l)}{\partial s} = \varphi(t,s,l) \quad \phi(t,s,l). \tag{3}$$

$T[\cdot;l,t]$ is a linear operator,

$$T[g';l,t] = -T[g\varphi;l,t], \tag{4}$$

g' is continuous functions.

II. MAIN RESULT

In the section, some oscillation theorems of neutral functional differential equation of the even order (1) are derived. The following lemmas are needed.

Lemma 2.1. [16] Suppose $g \in C^n([t_0,\infty), R^+)$, $g^{(n)}(s)$ is eventually of one sign for s, so there are $s_x > s_0$, integer a, $0 \le a \le n$, with $n+a$ odd for $g^{(n)}(s) \le 0$, or $n+a$ even for $g^{(n)}(s) \ge 0$, such that $a > 0$ implies that $(-1)^{l+k} g^k(s) > 0$, for $t > t_x$, $k = a, a+1, \cdots, a-1$, and $a \le n-1$ implies that $g^{(k)}(s) > 0$ for $s > s_x, k = 0,1,2,\cdots,a-1$.

Lemma 2.2. [16] Suppose g is as in Lemma 2.1, $g^{(n-1)}(s) g^{(n)}(s) \le 0$ for $s > s_x$, so, for every $0 < \lambda < 1$, exists $M > 0$, the following inequality holds, $g(\lambda s) \ge M s^{n-1} | g^{(n-1)}(s) |$.

Lemma 2.3. [15] Suppose y is an eventually positive solutions of equations of the even order (1), then exists a $s_1 \ge s_0$, the following inequality holds,

$$B^{(n)}(s) \le 0, \; B^{(n-1)}(s) > 0, \; B'(s) > 0, \; B(s) > 0, \tag{5}$$

for $s \ge s_1$, where $B(s) = y(s) + P(s) y(\tau(s))$.

Theorem 2.1. If

$$\int_{t_0}^{\infty} Q_1(s) \mathrm{ds} = \infty, \tag{6}$$

here $Q_1(s) = \min\{\alpha Q(s), \alpha Q(\tau(s))\}$, then every solution of functional differential equations of the even order (1) is oscillatory.

Proof. Suppose y be a nonoscillatory solutions of equations of the even order (1), then exists a t_1, $y(t) \ne 0$, $t \ge t_1$.

Assume that $y(t) > 0$, $y(\sigma(t)) > 0$, $y(\tau(t)) > 0$, $t \ge t_1$, without loss of generality. From Lemma 2.3, exists a $t_2 \ge t_1$, (2.1) holds. Using definition of $B(t) = y(t) + P(t) y(\tau(t))$ and applying (1), we get for sufficiently large t, the following inequality holds,

$$B^{(n)}(t) + \alpha Q(t) y(\sigma(t)) + \alpha P_0 Q(\tau(t)) y(\sigma(\tau(t))) + \alpha P_0 B^{(n)}(\tau(t)) \le 0.$$

So, the following inequality holds,

$$B^{(n)}(t) + Q_1(t) B(\sigma(t)) + \alpha P_0 B^{(n)}(\tau(t)) \le 0, \tag{7}$$

here $Q_1(t) = \min\{\alpha Q(t), \alpha Q(\tau(t))\}$. Integrating (7) from t_3 $(\ge t_2)$ to t, and using $\tau'(t) \ge \tau_0$, we obtain

$$\begin{aligned} \int_{t_3}^{t} Q_1(s) B(\sigma(s)) \mathrm{ds} &\le -\alpha P_0 \int_{t_3}^{t} B^{(n)}(\tau(s)) \mathrm{ds} - \int_{t_3}^{t} B^{(n)}(s) \mathrm{d}s \\ &\le B^{(n-1)}(t_3) - B^{(n-1)}(t) + \frac{\alpha P_0}{\tau_0} B^{(n-1)}(\tau(t_3)) - \frac{\alpha P_0}{\tau_0} B^{(n-1)}(\tau(t)). \end{aligned} \tag{8}$$

Since $B^{(n)}(t) \le 0$, $B^{(n-1)}(t) > 0$, $B'(t) > 0$, we have $\int_{t_3}^{\infty} Q_1(s) \mathrm{ds} < \infty$, due to (8), which is a contradiction to (6).

Remark 2.1. Theorem 2.1 involves [3].

Theorem 2.2. Suppose $\sigma(t) \le \tau(t)$, exist $\phi \in Y$, $k \in C^1([t_0,\infty), R^+)$, such that for some $0 < \lambda < 1$, and for every $M > 0$,

$$\limsup_{t\to\infty} T\left[k(s) Q_1(s) - \frac{(1 + \frac{\alpha P_0}{\tau_0})\left(\varphi + \frac{k'(s)}{k(s)}\right)^2}{4\lambda M} \frac{k(s)}{\sigma^{n-2}(s)\sigma'(s)}; l, t \right] > 0, \tag{9}$$

here $Q_1(s) = \min\{\alpha Q(s), \alpha Q(\tau(s))\}$, T is defined by (2), $\varphi = \varphi(t,s,l)$ is defined as in (3), so, equations of the even order (1) is oscillatory.

Proof. Suppose y be a nonoscillatory solutions of equations of the even order (1), then exists t_1, such that $y(t) \neq 0, t \geq t_1$.

Let $y(t) > 0$, $y(\sigma(t)) > 0$, $y(\tau(t)) > 0$, $t \geq t_1$. There is a $t_2 \geq t_1$, (5), (7) hold, by Lemma 2.3 and (1). We use Lemma 2.2 for $g = B'$, exist $M > 0$, $t_3 \geq t_2$, the inequality holds,

$$B'(\lambda\sigma(t)) \geq M\sigma^{n-2}(t)B^{(n-1)}(\sigma(t)) \geq M\sigma^{n-2}(t)B^{(n-1)}(t), \quad (10)$$

So, we define

$$Z(t) = \frac{k(t)B^{(n-1)}(t)}{B(\lambda\sigma(t))}, \quad (11)$$

So, $Z(t) > 0$ and

$$Z'(t) = k'(t)\frac{B^{(n-1)}(t)}{B(\lambda\sigma(t))} + k(t)\frac{B^{(n)}(t)B(\lambda\sigma(t)) - \lambda B^{(n-1)}(t)B'(\lambda\sigma(t))\sigma'(t)}{B^2(\lambda\sigma(t))}. \quad (12)$$

In view of (10), (11) , (12), we have

$$Z'(t) \leq k(t)\frac{B^{(n)}(t)}{B(\lambda\sigma(t))} + \frac{k'(t)}{k(t)}Z(t) - \lambda M\frac{\sigma^{n-2}(t)\sigma'(t)}{k(t)}Z^2(t). \quad (13)$$

Define another function

$$A(t) = \frac{k(t)B^{(n-1)}(\tau(t))}{B(\lambda\sigma(t))}. \quad (14)$$

So, $A(t) > 0$ and

$$A'(t) = \frac{k'(t)B^{(n-1)}(\tau(t))}{B(\lambda\sigma(t))} + k(t)\frac{B^{(n)}(\tau(t))\tau'(t)B(\lambda\sigma(t)) - \lambda B^{(n-1)}(\tau(t))B'(\lambda\sigma(t))\sigma'(t)}{B^2(\lambda\sigma(t))}. \quad (15)$$

From (10), (14), (15) and $\sigma(t) \leq \tau(t)$, we get

$$A'(t) \leq \frac{k(t)\tau'(t)B^{(n)}(\tau(t))}{B(\lambda\sigma(t))} + \frac{k'(t)A(t)}{k(t)} - \frac{\lambda M\sigma^{n-2}(t)\sigma'(t)A^2(t)}{k(t)}. \quad (16)$$

Then, by (13) , (16), we have

$$Z'(t) + \frac{\alpha p_0}{\tau'(t)}A'(t) \leq \frac{k(t)B^{(n)}(t)}{B(\lambda\sigma(t))} + \frac{\alpha p_0 k(t)B^{(n)}(\tau(t))}{B(\lambda\sigma(t))} + \frac{k'(t)Z(t)}{k(t)} - \frac{\lambda M\sigma^{n-2}(t)\sigma'(t)Z^2(t)}{k(t)} + \frac{\alpha p_0}{\tau'(t)}\frac{k'(t)}{k(t)}A(t) - \frac{\alpha p_0}{\tau'(t)}\lambda M\frac{\sigma^{n-2}(t)\sigma'(t)}{k(t)}A^2(t).$$

From (7), the following inequality holds,

$$Z'(t) + \frac{\alpha p_0}{\tau_0}A'(t) \leq -k(t)Q(t) + \frac{k'(t)}{k(t)}Z(t) - \frac{\lambda M\sigma^{n-2}(t)\sigma'(t)Z^2(t)}{k(t)} + \frac{\alpha p_0}{\tau_0}\frac{k'(t)A(t)}{k(t)} - \frac{\alpha p_0}{\tau_0}\lambda M\frac{\sigma^{n-2}(t)\sigma'(t)}{k(t)}A^2(t) \quad (17)$$

By $T[\cdot;l,t]$ into (17), we have

$$T\left[Z'(s) + \frac{\alpha p_0 A'(s)}{\tau_0};l,t\right] \leq$$

$$T\left[-k(s)Q(s) + \frac{k'(s)Z(s)}{k(s)} - \lambda M\frac{\sigma^{n-2}(s)\sigma'(s)Z^2(s)}{k(s)} + \frac{\alpha p_0}{\tau_0}\frac{k'(s)A(s)}{k(s)} - \frac{\alpha p_0\lambda M}{\tau_0}\frac{\sigma^{n-2}(s)\sigma'(s)A^2(s)}{k(s)};l,t\right].$$

By (4) , the inequality of about, the following inequality holds,

$$T\left[k(s)Q(s);l,t\right] \leq T\left[\left(\varphi + \frac{k'(s)}{k(s)}\right)Z(s) - \lambda M\frac{\sigma^{n-2}(s)\sigma'(s)}{k(s)}Z^2(s) + \frac{\alpha P_0}{\tau_0}\left(\varphi + \frac{k'(s)}{k(s)}\right)A(s) - \frac{\alpha P_0}{\tau_0}\lambda M\frac{\sigma^{n-2}(s)\sigma'(s)}{k(s)}A^2(s);l,t\right]. \quad (18)$$

So, by (18), the following inequality holds,

$$T\left[k(s)Q(s);l,t\right] \leq T\left[\left(\frac{\left(\varphi + \frac{k'(s)}{k(s)}\right)^2 + \frac{\alpha P_0}{\tau_0}\left(\varphi + \frac{k'(s)}{k(s)}\right)^2}{4\lambda M}\right)\frac{k(s)}{\sigma^{n-2}(s)\sigma'(s)};l,t\right],$$

then,

$$T\left[k(s)Q(s) - \frac{\left(1 + \frac{\alpha P_0}{\tau_0}\right)\left(\varphi + \frac{k'(s)}{k(s)}\right)^2}{4\lambda M}\frac{k(s)}{\sigma^{n-2}(s)\sigma'(s)};l,t\right] \leq 0.$$

The following results are obtained,

$$\limsup_{t\to\infty} T\left[k(s)Q(s) - \frac{\left(1 + \frac{\alpha P_0}{\tau_0}\right)\left(\varphi + \frac{k'(s)}{k(s)}\right)^2}{4\lambda M}\frac{k(s)}{\sigma^{n-2}(s)\sigma'(s)};l,t\right] \leq 0,$$

which contradicts (9).

Remark 2.2. Theorem 2 extends [3].

Remark 2.3. The results got in this paper complement the results got in [11].

Example 2.1 Investigate the even order equations

$$[y(s) + 2y(s+\pi)]^{(n)} + y\left(s - \frac{n\pi}{2}\right) = 0. \quad (19)$$

Let

$$Q(s) = 1,\ P(s) = 2,\ \sigma(s) = s - \frac{n\pi}{2},\ \tau(s) = s + \pi,$$

Then, we have every solution of the even order neutral equations (19) is oscillatory by Theorem 2.1. In fact, $y(s)=\sin s$ is an oscillatory solution of (19).

III. ACKNOWLEDGEMENTS

This research is supported by the NSF of China (61374074), and supported by NSF of Shandong Provincial (ZR2012AM009, ZR2013AL003).

*Corresponding author: Zhen-Lai Han, e-mail: hanzhenlai@163.com.

REFERENCES

[1] Hale, J. K. Theory of functional differential equations, Springer-Verlag, New York, 1977.

[2] Agarwal, R. P., Grace, S. R., O'Regan, D. Oscillation theory for difference and functional differential equations, Kluwer Academic Publishers, Dordrecht, 2000.

[3] Baculíková, B., Lacková, D. Oscillation criteria for second order retarded differential equations. *Studies of the University of Zilina, Mathematical Series,* 20, pp.11–18, 2006.

[4] Džurina, J., Hudáková, D. Oscillation of second order neutral delay differential equations. *Math. Bohem*., 134, pp. 31–38, 2009.

[5] Grace, S. R. Oscillation theorems for nonlinear differential equations of second order. *J. Math. Anal. Appl*., 171, pp.220–241, 1992.

[6] Grammatikopoulos, M. K., Ladas, G., Meimaridou, A. Oscillation of second order neutral delay differential equation. *Rat. Mat*., 1, pp. 267–274, 1985.

[7] Han, Z. L., Li, T. X., Sun, S. R., Sun, Y. B. Remarks on the paper [Appl. Math. Comput. 207 (2009) 388–396]. *Appl. Math. Comput*., 215, pp.3998–4007, 2010.

[8] Han, Z. L., Li, T. X., Sun, S. R., Chen, W. S. On the oscillation of second-order neutral delay differential equations. *Adv. Differ. Equ*., 2010, pp. 1–8, 2010.

[9] Han, Z. L., Li, T. X., Sun, S. R., Chen, W. S. Oscillation criteria for second-order nonlinear neutral delay differential equations. *Adv. Differ. Equ*., 2010, 1–23 (2010).

[10] Karpuz, B., Manojlović, J. V., Öcalan, Ö, Shoukaku, Y. : Oscillation criteria for a class of second-order neutral delay differential equations. *Appl. Math. Comput*., 210, 303–312 (2009).

[11] Li, T. X., Han, Z. L., Zhao, P., Sun, S. R. Oscillation of even-order neutral delay differential equations. *Adv. Differ. Equ*., 2010, pp.1–9, 2010.

[12] Li, H. J., Yeh, C. C. Oscillation criteria for second order neutral delay equations. *Comput. Math. Appl*., 36, pp.123–132, 1998.

[13] Lin, X. Y., Tang, X. H. Oscillation of solutions of neutral differential equations with a superlinear neutral term. *Appl. Math. Lett*., 20, pp.1016–1022, 2007.

[14] Liu, L. H., Bai, Y. Z. New oscillation criteria for second-order nonlinear neutral delay differential equations. *J. Comput. Appl. Math*., 231, pp.657–663, 2009.

[15] Meng, F. W., Xu, R. Oscillation criteria for certain even order quasi-linear neutral differential equations with deviating arguments. *Appl. Math. Comput*., 190, pp.458–464, 2007.

[16] Philos, Ch. G. A new criteria for the oscillatory and asymptotic behavior of delay differential equations. *Bull. Acad. Pol. Sci. Ser. Sci. Mat.*, 39, pp.61–64, 1981.

[17] Rath, R. N., Misra, N., Panhy, L. N. Oscillatory and asymptotic behaviour of a nonlinear second order neutral differential equations. *Math. Slovaca*, 57, pp.157–170, 2007.

[18] Şahiner, Y. On oscillation of second order neutral type delay differential equations. *Appl. Math. Comput.*, 150, pp.697–706, 2004.

[19] Sun, Y. G., Meng, F. W. Note on the paper of Džurina and Stavroulakis. *Appl. Math. Comput*., 174, pp.1634–1641, 2006.

[20] Xu, R., Meng, F. W. Oscillation criteria for second order quasi-linear neutral delay differential equations. *Appl. Math. Comput.*, 192, pp.216–222, 2007.

[21] Xu, Z. T., Liu, X. X. Philos-type oscillation criteria for Emden-Fowler neutral delay differential equations. *J. Comput. Appl. Math*., 206, pp.1116–1126, 2007.

[22] Ye, L. H., Xu, Z. T. Oscillation criteria for second order quasilinear neutral delay differential equations. *Appl. Math. Comput*., 207, pp.388–396, 2009.

[23] Zafer, A. Oscillation criteria for even order neutral differential equations. *Appl. Math. Lett.*, 11, pp.21–25, 1998.

[24] Zhang, Q. X., Yan, J. R., Gao, L. Oscillation behavior of even-order nonlinear neutral differential equations with variable coefficients. *Comput. Math. Appl*., 59, pp.426–430, 2010.

[25] Han, Z. L., Li, T. X., Sun, S. R., Chen, W. S. Oscillation of second order quasilinear neutral delay differential equations. *J. Appl. Math. Comput*., 40, pp.143–152, 2012.

[26] Li, T. X., Han, Z. L., Zhang, C. H., Sun, S. R. On the oscillation of second-order Emden-Fowler neutral differential equations. *J. Appl. Math. Comput*., 37(1-2), pp.601–610, 2011.