**Theorem 4.4** (*Strong completeness theorem*) Assume $\Theta$ be a set of formulas and $\varphi$ a formula of $\mathcal{L}_R$. The following are equivalent:
(a) $\Theta \vdash \varphi$,
(b) $\Theta \models_R \varphi$ for any RMV-algebra $R$,
(c) $\Theta \models_R \varphi$ for any linearly-ordered RMV-algebra $R$.

**Proof.** The equivalence of (a) and (b) is straightforward. The equivalence with (c) follows by Theorem 2.5. See also [11, Theorem 2.11].

**Theorem 4.5** (*Standard completeness*) For a formula $\varphi$ of $\mathcal{L}_R$, the following are equivalent:
(a) $\vdash \varphi$,
(b) $\models_{[0,1]} \varphi$.

**Proof.** It follows by Theorem 3.4.

As a direct consequence of the standard completeness it follows that the logic of RMV-algebras is a conservative extension of Łukasiewicz logic.

Finally, we prove an approximation result.

**Theorem 4.6** (*Approximation of continuous functions*) Let $n \geq 1$ be a natural number. For any continuous function $h : [0,1]^n \to [0,1]$ there exists a sequence of formulas $(\varphi_n)_n$ of $\mathcal{L}_R$ such that $h$ is the uniform limit of $(f_{\varphi_n})_n$.

**Proof.** If $Form_n$ is the set of the formulas which contain only the variables $v_1, \ldots, v_n$, then $R_n = Form_n/\equiv_\emptyset$ is the free RMV-algebra with $n$-generators. By Theorem 3.7, $R_n$ is a semisimple RMV-algebra. By Theorem 2.11, $R_n$ is dense in $C(X)$ in the *sup*-norm which proves our result.

**Remark 4.7** The logical system briefly presented in this chapter is strongly related with Rational Łukasiewicz Logic developed in [10], where only multiplication by rationals is considered. The algebraic structures of Rational Łukasiewicz Logic are the divisible MV-algebras. Our system is also a conservative extension of Rational Łukasiewicz Logic.

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