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Recurrence Relations for Single and Product Moments of Dual Generalized Order Statistics from a General Class of Distributions

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In this paper we derive some general recurrence relations between single and product moments of dual generalized order statistics from a general class of distributions, thus generalizing and unifying the earlier results in this direction due to several authors.

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1. Introduction

Kamps (1995a) has introduced generalized order statistics as a unified approach to several models of ascendingly ordered random variables, e.g., ordinary order statistics, sequential order statistics, progressively type II censored order statistics, upper records and upper Pfeifer records (see Kamps, 1995b or Cramer and Kamps, 2001). However, random variables that are descendingly ordered can not be integrated into this framework. Burkschat et al. (2003) introduced dual generalized order statistics (dgos) as a unification of several models of descendingly ordered random variables like,

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reversed order statistics $X_{n:n},\ldots,X_{1:n}$, lower k-records and lower Pfeifer records. For any result in the initial model of generalized order statistics, there exists a corresponding one in the dual model. Several authors like Ahsanullah (2004), Barakat and El-Adll (2009), Arslan (2010), Jaheen and Al Harbi(2011) and Saran and Pandey (2012) have worked on dgos.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Let $X_{j:n}$ denote the j^{th} order statistic of a sample $(X_1, X_2, \ldots, X_n)$. Assume that $n \in N$, $k \geq 1$, $m \in R$ be the parameters such that $\gamma_r = k+n-r+M_r > 0$, $M_r = \sum_{j=r}^{n-1} m_j \ \forall\ 1 \leq r \leq n$. By the dual generalized order statistics from an absolutely continuous distribution $F(x)$ with density function $f(x)$ we mean random variables $X'(1,n,\tilde{m},k),\ldots,X'(n,n,\tilde{m},k)$ having the joint density function of the form

$$k\Big(\prod_{j=1}^{n-1}\gamma_j\Big)\Big(\prod_{i=1}^{n-1}(F(x_i))^{m_i}f(x_i)\Big)(F(x_n))^{k-1}f(x_n), \tag{1.1}$$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

For convenience, let us define $X'(0,n,\tilde{m},k) = 0$. It may be mentioned that for $m_1 = \cdots = m_{n-1} = 0,\ k = 1, i.e., \gamma_j = n-j+1;\ 1 \leq j \leq n-1$, the dual generalized order statistics coincide in the distribution theoretical sense with reversed order statistics. Similarly, for $m_1 = \ldots = m_{n-1} = -1,\ k \in N,\ i.e., \gamma_j = k;\ 1 \leq j \leq n-1$, the joint distribution of dual generalized order statistics equals that of the lower k-records.

We may consider two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

Case II: $\gamma_i \neq \gamma_j, i \neq j,\ i,j = 1,2,\ldots,n-1$.

For **Case I**, the r^{th} dual generalized order statistic will be denoted by $X'(r,n,m,k)$, and the pdf of $X'(r,n,m,k)$ is given by

$$f^{X'(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}[F(x)], \qquad x \in R. \tag{1.2}$$

Also, the joint pdf of $X'(r,n,m,k)$ and $X'(s,n,m,k),\ 1 \leq r < s \leq n$, is given by

$$\begin{aligned} f^{X'(r,n,m,k),X'(s,n,m,k)}(x,y) &= \frac{c_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}[F(x)] \\ &.[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y), \quad x > y, \end{aligned} \tag{1.3}$$

where

$$g_m(x) = h_m(x) - h_m(1), \qquad x \in [0,1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & \text{if} \quad m \neq -1 \\ -\log x, & \text{if} \quad m = -1. \end{cases}$$

Let $\mu_{(r,n,m,k)}^{(i)}$ denote the i^{th} moment of the r^{th} dual generalized order statistic $X'(r,n,m,k)$. Similarly, $\mu_{(r,s,n,m,k)}^{(i,j)}$ denotes the $(i,j)^{th}$ product moment of the r^{th} and s^{th} dual generalized order statistics.

For **Case II**, the r^{th} dual generalized order statistic will be denoted by $X'(r,n,\tilde{m},k)$, and the pdf of $X'(r,n,\tilde{m},k)$ is given by

$$f^{X'(r,n,\tilde{m},k)}(x) = c_{r-1} f(x) \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1}, \quad x \in R. \tag{1.4}$$

Also, the joint pdf of $X'(r,n,\tilde{m},k)$ and $X'(s,n,\tilde{m},k), 1 \le r < s \le n$, is given by

$$f^{X'(r,n,\tilde{m},k),X'(s,n,\tilde{m},k)}(x,y) = c_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \Big[\frac{F(y)}{F(x)}\Big]^{\gamma_i} \sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)},\ x > y, \tag{1.5}$$

where

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \qquad \gamma_j = k + n - j + M_j, \qquad r = 1, 2, \ldots, n,$$

$$a_i(r) = \prod_{j(\neq i)=1}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \le i \le r \le n$$

and

$$a_i^{(r)}(s) = \prod_{j(\neq i)=r+1}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \quad r+1 \le i \le s \le n.$$

In addition, for $m_i = m_j = m$, it can be shown that

$$\sum_{i=1}^{r} a_i(r)(F(x))^{\gamma_i} = \frac{(F(x))^{\gamma_r}}{(r-1)!} g_m^{r-1}(F(x)), \tag{1.6}$$

and

$$\sum_{i=r+1}^{s} a_i^{(r)}(s) \Big(\frac{F(y)}{F(x)}\Big)^{\gamma_i} = \frac{1}{(s-r-1)!} \Big(\frac{F(y)}{F(x)}\Big)^{\gamma_s} \Big(\frac{1}{F(x)}\Big)^{(m+1)(s-r-1)} \cdot \Big[h_m(F(y)) - h_m(F(x))\Big]^{s-r-1}. \tag{1.7}$$

In the above case II, let $\mu_{(r,n,\tilde{m},k)}^{(p)}$ denote the p^{th} moment of the r^{th} dual generalized order statistic $X'(r,n,\tilde{m},k)$. Similarly, $\mu_{(r,s,n,\tilde{m},k)}^{(p,q)}$ denotes the $(p,q)^{th}$ product moment of the r^{th} and s^{th} dual generalized order statistics.

Burkschat et. al (2003) defined dual generalized order statistics, alternatively, as

$$X'(r,n,\tilde{m},k) \sim F^{-1}(W_r), \ \ r = 1, 2, \ldots, n,$$

where $W_r = \prod_{j=1}^{r} B_j$, B_j being independent random variables distributed as $\beta_1(\gamma_j, 1)$ having c.d.f. $F(t) = t^{\gamma_j}, t \in [0,1]$.

In this paper, we derive some general recurrence relations satisfied by the single and product moments of dual generalized order statistics from a general class of distributions with p.d.f. $f(x)$ and c.d.f. $F(x)$ satisfying the characterizing differential equation:

$$\Big(\sum_{u=0}^{w}\alpha_u x^u\Big)f(x)=\Big(\sum_{v=0}^{z}\beta_v x^v\Big)F(x), \qquad -\infty<x<\infty, \tag{1.8}$$

where w and z are integers and $\alpha^{'s}$ and $\beta^{'s}$ are arbitrary real constants.

2. Recurrence Relations for Single Moments

In this section, we shall establish recurrence relations for single moments of dual generalized order statistics from a general class of distributions satisfying the relation given in equation (1.8).
Case I: $m_1=m_2=\ldots=m_{n-1}=m$.

Theorem 2.1. *For a positive integer $k\geq 1$, $n\in N$, $m\in Z$, $1\leq r\leq n$ and $i=0,1,2,\ldots$,*

$$\sum_{u=0}^{w}\alpha_u\mu_{(r,n,m,k)}^{(i+u)}=\gamma_r\sum_{v=0}^{z}\frac{\beta_v}{(i+v+1)}\Big[\mu_{(r,n,m,k)}^{(i+v+1)}-\mu_{(r-1,n,m,k)}^{(i+v+1)}\Big]. \tag{2.1}$$

Proof. For $1\leq r\leq n$ and $i=0,1,2,\ldots$, we have in view of (1.2),

$$\sum_{u=0}^{w}\alpha_u\mu_{(r,n,m,k)}^{(i+u)}=\frac{c_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}x^i[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]\Big\{\sum_{u=0}^{w}\alpha_u x^u f(x)\Big\}dx.$$

Utilizing Eq. (1.8), we get

$$\sum_{u=0}^{w}\alpha_u\mu_{(r,n,m,k)}^{(i+u)}=\frac{c_{r-1}}{(r-1)!}\sum_{v=0}^{z}\beta_v\int_{-\infty}^{\infty}x^{i+v}[F(x)]^{\gamma_r}g_m^{r-1}[F(x)]dx. \tag{2.2}$$

Integrating by parts of equation (2.2), treating x^{i+v} for integration and rest of the integrand for differentiation, we get

$$\sum_{u=0}^{w}\alpha_u\mu_{(r,n,m,k)}^{(i+u)}=\frac{c_{r-1}}{(r-1)!}\sum_{v=0}^{z}\frac{\beta_v}{(i+v+1)}\Big[\gamma_r\int_{-\infty}^{\infty}x^{i+v+1}[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx$$
$$-(r-1)\int_{-\infty}^{\infty}x^{i+v+1}[F(x)]^{\gamma_{r-1}-1}g_m^{r-2}[F(x)]f(x)dx\Big].$$

Simplifying above terms and rearranging them, we derive the relation in (2.1) on using (1.2). □

Remark 2.1. For $r=1$, the relation in (2.1) involves a term $\mu_{(0,n,m,k)}^{(i+v+1)}$ whose value will be taken as zero since we have defined $X'(0,n,m,k)=0$.

Remark 2.2. Under the assumptions of Theorem 2.1, with $k=1, m=0$ we shall obtain the recurrence relation for single moments of reversed order statistics from the general class of distributions (1.8).

Remark 2.3. Putting $k=0, m=-1$ in Theorem 2.1, we obtain the recurrence relation for single moments of lower record values from the general class of distributions (1.8).

Case II: $\gamma_i\neq\gamma_j,\ i\neq j,\ i,j=1,2,\ldots,n-1$.

Theorem 2.2. *For $n \in N,\ 1 \le r \le n,\ k \ge 1$ and $p = 0,1,2,\ldots,$*

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,\tilde{m},k)}^{(p+u)} = \gamma_r \sum_{v=0}^{z} \frac{\beta_v}{p+v+1} \left[\mu_{(r,n,\tilde{m},k)}^{(p+v+1)} - \mu_{(r-1,n,\tilde{m},k)}^{(p+v+1)} \right]. \tag{2.3}$$

Proof. For $1 \le r \le n$ and $p = 0,1,2,\ldots,$ we have from (1.2),

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,\tilde{m},k)}^{(p+u)} = c_{r-1} \int_{-\infty}^{\infty} x^p \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1} \Big\{ \sum_{u=0}^{w} \alpha_u x^u f(x) \Big\} dx,$$

which on using (1.8), gives

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,n,\tilde{m},k)}^{(p+u)} = c_{r-1} \sum_{v=0}^{z} \beta_v \int_{-\infty}^{\infty} x^{p+v} \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} dx.$$

Integrating by parts treating x^{p+v} for integration and rest of the integrand for differentiation, we get the required result (2.3).

□

Remark 2.4. For $r = 1$, from the relation in eq. (2.3) we get the term $\mu_{(0,n,\tilde{m},k)}^{(i+v+1)}$, the value of which will be zero as we have defined $X'(0,n,\tilde{m},k) = 0$.

Remark 2.5. Putting $m_i = m_j = m$ in eq. (1.4) and using eq. (1.6), the recurrence relation for single moments of dual generalized order statistics from the general class of distributions (1.8) for Case I, i.e., when $m_1 = m_2 = \ldots = m_{n-1} = m$, as obtained in Theorem 2.1, can be deduced from Theorem 2.2 with $\tilde{m}$ replaced by m.

3. Recurrence Relations for Product Moments

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

Theorem 3.1. *For a positive integer $k \ge 1,\ n \in N,\ m \in Z,\ 1 \le r \le s-2 < n$ and $i, j = 0,1,2,\ldots,$*

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,s,n,m,k)}^{(i,j+u)} = \gamma_s \sum_{v=0}^{z} \frac{\beta_v}{(j+v+1)} \left[\mu_{(r,s,n,m,k)}^{(i,j+v+1)} - \mu_{(r,s-1,n,m,k)}^{(i,j+v+1)} \right], \tag{3.1}$$

and for $s = r+1$,

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,r+1,n,m,k)}^{(i,j+u)} = \gamma_{r+1} \sum_{v=0}^{z} \frac{\beta_v}{(j+v+1)} \left[\mu_{(r,r+1,n,m,k)}^{(i,j+v+1)} - \mu_{(r,n,m,k)}^{(i+j+v+1)} \right]. \tag{3.2}$$

Proof. On employing (1.3) we have, for $1 \le r \le s < n$ and $i, j = 0,1,2,\ldots$

$$\sum_{u=0}^{w} \alpha_u \mu_{(r,s,n,m,k)}^{(i,j+u)} = \frac{c_{s-1}}{(r-1)!(s-r-1)!} \int_{-\infty}^{\infty} x^i [F(x)]^m f(x) g_m^{r-1}[F(x)] I(x) dx, \tag{3.3}$$

where

$$I(x) = \int_{x}^{\infty} y^j \Big[h_m(F(y)) - h_m(F(x)) \Big]^{s-r-1} [F(y)]^{\gamma_s - 1} \Big\{ \sum_{u=0}^{w} \alpha_u y^u f(y) \Big\} dy.$$

Utilizing Eq. (1.8), we get

$$I(x)=\sum_{v=0}^{z}\beta_v\int_x^{\infty}y^{j+v}\Big[h_m(F(y))-h_m(F(x))\Big]^{s-r-1}[F(y)]^{\gamma_s}dy. \tag{3.4}$$

Integrating by parts treating y^{j+v} for integration and rest of the integrand for differentiation and substituting the value of $I(x)$ in (3.3) and simplifying, we get the required result (3.1). In a similar manner, (3.2) can be easily established. □

Remark 3.1. Under the assumptions of Theorem 3.1, with $k=1, m=0$ we shall obtain the recurrence relations for product moments of reversed order statistics from the general class of distributions (1.8).

Remark 3.2. Putting $k=0, m=-1$ in Theorem 3.1, we obtain the recurrence relations for product moments of lower record values from the general class of distributions (1.8).

Case II: $\gamma_i\neq\gamma_j,\ i\neq j,\ i,j=1,2,\ldots,n-1$.

Theorem 3.2. *For* $n\in N,\ 1\le r<s\le n,\ k\ge 1$ *and* $p,q=0,1,2,\ldots,$

$$\sum_{u=0}^{w}\alpha_u\mu_{(r,s,n,\tilde{m},k)}^{(p,q+u)}=\gamma_s\sum_{v=0}^{z}\frac{\beta_v}{(q+v+1)}\Big[\mu_{(r,s,n,\tilde{m},k)}^{(p,q+v+1)}-\mu_{(r,s-1,n,\tilde{m},k)}^{(p,q+v+1)}\Big]. \tag{3.5}$$

Proof. For $1\le r<s\le n$ and $p,q=0,1,2,\ldots,$ we have from (1.5),

$$\sum_{u=0}^{w}\alpha_u\mu_{(r,s,n,\tilde{m},k)}^{(p,q+u)}=c_{s-1}\int_{-\infty}^{\infty}x^p\Big\{\sum_{i=1}^{r}a_i(r)\Big(F(x)\Big)^{\gamma_i}\Big\}\frac{f(x)}{F(x)}I(x)dx, \tag{3.6}$$

where

$$I(x)=\int_x^{\infty}y^q\Big\{\sum_{i=r+1}^{s}a_i^{(r)}(s)\Big(\frac{F(y)}{F(x)}\Big)^{\gamma_i}\Big\}\Big\{\sum_{u=0}^{w}\alpha_u y^u f(y)\Big\}\frac{1}{F(y)}dy.$$

Utilizing Eq. (1.8), we get

$$I(x)=\sum_{v=0}^{z}\beta_v\int_x^{\infty}y^{q+v}\Big\{\sum_{i=r+1}^{s}a_i^{(r)}(s)\Big(\frac{F(y)}{F(x)}\Big)^{\gamma_i}\Big\}dy. \tag{3.7}$$

Integrating by parts treating y^{q+v} for integration and rest of the integrand for differentiation, we get

$$I(x)=\sum_{j=0}^{q}\frac{\beta_j}{(q+v+1)}\int_x^{\infty}y^{q+v+1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\gamma_i\Big(\frac{F(y)}{F(x)}\Big)^{\gamma_i-1}f(y)dy.$$

Substituting the above expression for $I(x)$ in (3.6) and simplifying, we derive the relation in (3.5). □

Remark 3.3. Putting $m_i=m_j=m$ in (1.5) and using (1.7), the recurrence relation for product moments of generalized order statistics from the general class of distributions (1.8) for Case I, i.e., when $m_1=m_2=\ldots=m_{n-1}=m$ as obtained in Theorem 3.1 can be deduced from Theorem 3.2 with $\tilde{m}$ replaced by m.

Remark 3.4.

$$\text{Setting}\alpha_u \;=\begin{cases}\frac{1}{p}, & \text{if} \quad u=1\\ 0, & \text{if} \quad u\neq 1\end{cases} \tag{3.8}$$

and

$$\beta_v=\begin{cases}1, & \text{if} \quad v=1\\ 0, & \text{if} \quad v\neq 1,\end{cases} \tag{3.9}$$

we observe that (1.8) reduces to

$$\frac{x}{p}f(x)=F(x), \tag{3.10}$$

which is the characterizing differential equation for power function distribution with p.d.f. in the form

$$f(x)=p\;\nu^{-p}x^{p-1} \quad , \quad 0<x\leq \nu,\;\; \nu>0. \tag{3.11}$$

One can deduce the recurrence relations for single and product moments of dual generalized order statistics from power function distribution for the particular values of parameters involved.

Remark 3.5.

$$\text{Setting}\alpha_u \;=\begin{cases}\frac{1}{p\;\theta^p}, & \text{if} \quad u=p+1\\ 0, & \text{if} \quad u\neq p+1\end{cases} \tag{3.12}$$

and

$$\beta_v=\begin{cases}1, & \text{if} \quad v=1\\ 0, & \text{if} \quad v\neq 1,\end{cases} \tag{3.13}$$

we observe that (1.8) reduces to

$$\frac{f(x)}{p\theta^p}x^{p+1}=F(x), \tag{3.14}$$

which is the characterizing differential equation for inverse Weibull distribution with p.d.f. in the form

$$f(x)=p\;\theta^p x^{-(p+1)}e^{-\left(\frac{\theta}{x}\right)^p} , \quad x>0,\; p,\theta>0. \tag{3.15}$$

One can deduce the recurrence relations for single and product moments of dual generalized order statistics from inverse Weibull distribution for the particular values of parameters.

Remark 3.6.

$$\text{Setting}\alpha_u = \begin{cases} \frac{\lambda^\theta}{\theta}, & \text{if} \quad u = 0 \\ 0, & \text{if} \quad u \neq 0 \end{cases} \tag{3.16}$$

and

$$\beta_v = \begin{cases} 1, & \text{if} \quad v = 1 \\ 0, & \text{if} \quad v \neq 1, \end{cases} \tag{3.17}$$

we observe that (1.8) reduces to

$$\frac{\lambda^\theta}{\theta} f(x) = x^{-(\theta-1)} F(x), \tag{3.18}$$

which is the characterizing differential equation for two parameter Frechet distribution with p.d.f. in the form

$$f(x) = \left(\frac{x}{\lambda}\right)^{-\theta-1} \left(\frac{\theta}{\lambda}\right) \exp\left\{-\left(\frac{x}{\lambda}\right)^{-\theta}\right\}, \quad x > 0, \ \theta > 0, \ \lambda > 0. \tag{3.19}$$

One can deduce the recurrence relations of single and product moments of dual generalized order statistics from Frechet distribution for the particular values of parameters.

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