On the stability of Jensen functional equation in Felbin's type fuzzy normed linear spaces

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Abstract. In this paper,we investigate the generalized Hyers-Ulam-Rassias stability of Jensen functional equation in Felbin's type fuzzy normed linear spaces .

1.Introduction

In 1940, Ulam[1] proposed the general Ulam stability problem.Next year, Hyers[2] solved this problem.In 1978, Rassias[3] took account of the unbounded Cauchy difference in Hyers' theorem and obtained the results for linear mappings. The stability problems of several functional equations have been extensively investigated by a number of authors (see [4,5] and references therein).In 1989,Kominek[6] proved the stability of Jensen functional equation on a restricted domain.In1998, Jung[7] proved the Hyers-Ulam-Rassias stability of Jensen functional equation. In 2014, Eskandani and Rassias[8] investigated the stability of a general cubic functional equation in Felbin's type fuzzy normed linear spaces. In this paper,we investigate the generalized Hyers-Ulam-Rassias stability of Jensen functional equation in Felbin's type fuzzy normed linear spaces .

We consider some basic concepts concerning in the theory of fuzzy real numbers. Let β be a fuzzy subset on R, i.e., a mapping $\beta: R \to [0,1]$ associating with each real number *t* its grade of membership $\beta(t)$.

Definition1.1[9] A fuzzy subset β on R is called a fuzzy real number,whose α-level set is denoted by $[\beta]_\alpha$,i.e.,$[\beta]_\alpha = \{t: \beta(t) \geq \alpha\}$,if it satisfies two axioms:

(1)There exists $t_0 \in R$ such that$\beta(t_0) = 1$.

(2)For each $\alpha \in (0,1]$; $[\beta]_\alpha = [\beta_\alpha^-, \beta_\alpha^+]$ where$-\infty < \beta_\alpha^- \leq \beta_\alpha^+ < +\infty$.

The set of all fuzzy real numbers denoted by $F(R)$.If $\beta \in F(R)$ and $\beta(t) = 0$ whenever $t < 0$,then β is called a nonnegative fuzzy real number and $F^*(R)$ denotes the set of all non-negative fuzzy real numbers.

Definition1.2[9] Let X be a real linear space, L and R (respectively, left norm and right norm) be symmetric and non-decreasing mappings in both arguments from $[0,1]\times[0,1]$ into $[0,1]$ satisfying $L(0,0)=0$ and $R(1,1)=1$.The mapping $\|.\|$ from X into $F^*(R)$ is called a fuzzy norm if for $x\in X$ and $\alpha\in(0,1]$:

(1)$\|x\|=\bar{0}$ if and only if $x=0$,

(2)$\|rx\|=|r|\|x\|$ for all $x\in X$ and$r\in(-\infty,+\infty)$;

(3)For all $x,y\in X$,

(a)if $s\le\|x\|_1^-$, $t\le\|y\|_1^-$ and $s+t\le\|x+y\|_1^-$ then $\|x+y\|(s+t)\ge L(\|x\|(s),\|y\|(t))$,

(b)if $s\le\|x\|_1^-$, $t\le\|y\|_1^-$ and $s+t\ge\|x+y\|_1^-$ then $\|x+y\|(s+t)\le R(\|x\|(s),\|y\|(t))$.

The quaternary (X,$\|.\|$,L,R) is called a fuzzy normed linear space.

Definition1.3[9] Let (X,$\|.\|$,L,R) be a fuzzy normed linear space and $\lim_{a\to0^+}R(a,a)=0$.A sequence $\{x_n\}$ in X is said to converge to $x\in X$,denoted by $\lim_{n\to\infty}x_n=x$,if $\lim_{n\to\infty}\|x_n-x\|_\alpha^+=0$ for every $\alpha\in(0,1]$ and is called a Cauchy sequence if $\lim_{n,m\to\infty}\|x_n-x_m\|_\alpha^+=0$ for every $\alpha\in(0,1]$.A subset A in X is said to be complete if every Cauchy sequence in A converges in A.The fuzzy normed space (X,$\|.\|$,L,R) is said to be a fuzzy Banach space if it is complete.

Theorem1.4[10] Let (X,$\|.\|$,L,R) be a fuzzy normed linear space, if $R(a,b)\le\max(a,b)$,then for any $\alpha\in(0,1]$,$\|x+y\|_\alpha^+\le\|x\|_\alpha^++\|y\|_\alpha^+$ for all $x,y\in X$.

A mapping $f:X\to Y$ is called a Jensen function if f satisfies the functional equation

$$2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$$

for $x,y\in X$.For a given mapping $f:X\to Y$,we define the difference operator

$$Df(x,y)=f(x)+f(y)-2f\left(\frac{x+y}{2}\right)$$

for $x,y\in X$.Then f is a Jensen function if $Df(x,y)=0$ for all $x,y\in X$.

2. Stability of Jensen functional equation using direct method.

Theorem2.1 Let X be a real liner space and $(Y,\|.\|,L,R)$ be a fuzzy Banach space satisfying $R(a,b) \leq \max(a,b)$ .Let $f: X \to Y$ be a mapping for which there exists a function $\varphi: X \times X \to F^*(R)$ such that

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y)_\alpha^+ = 0, \qquad (2.1)$$

$$\tilde{\varphi}_\alpha(x) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \varphi\left(0, 2^{i+1}x\right)_\alpha^+ < \infty, \qquad (2.2)$$

$$\|Df(x,y)\|_\alpha^+ \leq \varphi(x,y)_\alpha^+ \qquad (2.3)$$

for all $x, y \in X$, $\alpha \in (0,1]$.Then there exists a Jensen function $J: X \to Y$ such that

$$\|f(x) - J(x)\|_\alpha^+ \leq \tilde{\varphi}_\alpha(x) \qquad (2.4)$$

for all $x \in X$, $\alpha \in (0,1]$.

Proof. Define $g: X \to Y$ by $g(x) = f(x) - f(0)$ for all $x \in X$. Letting $x = 0$ in (2.3), we get

$$\left\|f(0) + f(y) - 2f(\tfrac{y}{2})\right\|_\alpha^+ = \left\|f(y) - f(0) - 2(f\left(\tfrac{y}{2}\right) - f(0))\right\|_\alpha^+$$

$$= \left\|g(y) - 2g(\tfrac{y}{2})\right\|_\alpha^+ \leq \varphi(0,y)_\alpha^+ \qquad (2.5)$$

for all $y \in X$, $\alpha \in (0,1]$. Replacing y by $2^{n+1}x$ in (2.5) and dividing both sides by 2^{n+1}, we get

$$\left\|\frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^n} g(2^n x)\right\|_\alpha^+ \leq \frac{1}{2^{n+1}} \varphi(0, 2^{n+1}x)_\alpha^+. \qquad (2.6)$$

By theorem1.4 and inequality (2.6), we get

$$\left\|\frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^m} g(2^m x)\right\|_\alpha^+ \leq \sum_{i=m}^{n} \frac{1}{2^{i+1}} \varphi\left(0, 2^{i+1}x\right)_\alpha^+ \qquad (2.7)$$

for all $x \in X$, $\alpha \in (0,1]$ and all non-negative integers m and n with $n \geq m$. Passing the limit $m, n \to \infty$ in (2.7),we have

$$\lim_{m,n\to\infty} \left\|\frac{1}{2^{n+1}} g(2^{n+1}x) - \frac{1}{2^m} g(2^m x)\right\|_\alpha^+ = 0.$$

Therefore the sequence $\left\{\frac{1}{2^n}g(2^n x)\right\}$ is a Cauchy sequence in Y for all $x \in X$.Since Y is complete, the sequence $\left\{\frac{1}{2^n}g(2^n x)\right\}$ converges for all $x \in X$.So we can define the mapping $A: X \to Y$ by $A(x) = \lim_{n\to\infty}\frac{1}{2^n}g(2^n x)$.

If we define a function $J: X \to Y$ by $J(x) = A(x) + f(0)$,and let $m = 0$ and $n \to \infty$ in (2.7) , then $\|J(x) - f(x)\|_\alpha^+ = \|A(x) - f(x) + f(0)\|_\alpha^+ = \|A(x) - g(x)\|_\alpha^+ \le \tilde{\varphi}_\alpha(x)$.We get (2.4). Now, we show that J is a Jensen function.

$$\|DJ(x,y)\|_\alpha^+ = \left\|D\big(A(x,y) + f(0)\big)\right\|_\alpha^+ = \left\|\lim_{n\to\infty}\frac{1}{2^n}Df(2^n x, 2^n y)\right\|_\alpha^+$$

$$\le \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x, 2^n y)_\alpha^+ = 0.$$

So J is a Jensen function.

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