Oscillation criteria for an advanced 2-D discrete system

Chunhua Yuan[a *], Liang Zhang[b] and Jian Liu[c]
School of Mathematical Science, University of Jinan, Jinan, 250022, China
[a]yuanchunhua72@163.com, [b]522599239@qq.com, [c]liujian1990@163.com

Keywords: 2-D discrete system, Characteristic equation, Oscillation.

Abstract. This paper is concerned with oscillatory behavior of an advanced 2-D discrete system. Some necessary and sufficient conditions for the oscillation of all solutions of the advanced 2-D discrete system are established. Numerical examples are provided to demonstrate the main results of this paper.

Introduction

2-D discrete systems arise in applications involving image processing, random walk problems, population dynamics with spatial migrations and finite difference schemes[1-4]. In recent years, lots of results for oscillatory behavior of 2-D discrete systems have been obtained (see [3- 11] and the references therein).

In this paper, we are concerned with the oscillatory behavior of the advanced 2-D discrete system:

$$u_{m+1,n}+u_{m,n+1}-pu_{m,n}+qu_{m+k,n+l}=0, \qquad n, m=0,1,2,\ldots, \tag{1}$$

where p, q are real numbers, k, l are nonnegative integers with $k^2+l^2\neq 0$. However, to the best of our knowledge, no research has been done on the oscillatory behavior of the advanced 2-D discrete system (1). In this paper, we establish some necessary and sufficient conditions for the oscillation of all solutions of the advanced 2-D discrete system (1).

Eq. 1 is said to be an advanced 2-D discrete system if k, l *are nonnegative integers with* $k^2+l^2\neq 0$.

A solution of Eq. 1 is a real double sequence $\{u_{m,n}\}$ *which is defined for* $m\geq 0, n\geq 0$ *and satisfies Eq. 1 for* $m\geq 0, n\geq 0$.

A solution $\{u_{m,n}\}$ *of Eq. 1 is said to be eventually positive (or negative) if* $u_{m,n}>0$ *(or* $u_{m,n}<0$*) for large numbers* $m\geq 0$ *and* $n\geq 0$. *It is said to be oscillatory if it is neither eventually positive nor eventually negative. Eq. 1 is called oscillatory if all of its nontrivial solutions are oscillatory.*

Preliminary

On basis of Corollary 2.9 in [4], we can easily obtain the following lemma, which is used in the proofs of the main results in the next section.

Lemma 1 The following statements are equivalent:

1) Every solution of Eq. 1 is oscillatory.

2) The characteristic equation $\lambda+\mu-p+q\lambda^k\mu^l=0$ *of Eq. 1 has no positive roots.*

Main Results

Theorem 1. Assume that $k>0$ *and* $l>0$. *Then every solution of Eq. 1 oscillates if and only if* $p\leq 0$ *and* $q\geq 0$.

Proof . When $k>0$ and $l>0$, then the characteristic equation of Eq. 1 is

$$\phi(p,q,\lambda,\mu)=\lambda+\mu-p+q\lambda^k\mu^l=0\,. \tag{2}$$

Sufficiency. Assume on the contrary that $\{u_{i,j}\}$ is an eventually positive solution of Eq. 1. In view of Eq. 1, we have

$$u_{i+1,j} + u_{i,j+1} = pu_{i,j} - qu_{i+k,j+l} . \tag{3}$$

Since $p \le 0$ and $q \ge 0$, we can see that the left side of Eq. 3 is strictly greater than 0 and the right sight of Eq. 3 is less than or equal to 0. This is a contradiction.

Necessity. Assume on the contrary that $p \le 0$ and $q \ge 0$ do not hold. We only need to consider the following three cases.

Case 1) $p > 0$ and $q < 0$. Letting $\mu = p + \delta$ in Eq. 2, we obtain

$$\phi(p, q, \lambda) = q(p+\delta)^l \lambda^k + \lambda + \delta = 0 . \tag{4}$$

When $k = 1$, it follows from Eq. 4 that $\lambda = -\delta/(1+q(p+\delta)^l)$. Choosing $\delta^* > \max\{0, \sqrt[l]{-1/q} - p\}$, we have $\lambda^* = -\delta^*/(1+q(p+\delta^*)^l) > 0$. It follows from $p > 0$ and $\delta^* > 0$ that $\mu^* = p + \delta^* > 0$. Hence, (λ^*, μ^*) is a positive root of Eq. 2. When $k > 1$, choosing $\delta = \delta^* > 0$ in Eq. 4, we have $\lim_{\lambda \to 0^+} \phi(p, q, \lambda) = \delta^* > 0$ and $\lim_{\lambda \to +\infty} \phi(p, q, \lambda) = -\infty$. Since Eq. 4 is continuous on $(0, +\infty)$, there exists $\lambda^* \in (0, +\infty)$ satisfying Eq. 4. Notice that $\mu^* = p + \delta^* > 0$ for $p > 0$ and $\delta^* > 0$. Therefore, (λ^*, μ^*) is a positive root of Eq. 2.

Case 2) $p > 0$ and $q \ge 0$. Letting $\mu = \lambda$ in Eq. 2, we obtain

$$\phi(p, q, \lambda) = q\lambda^{k+l} + 2\lambda - p = 0 . \tag{5}$$

From Eq. 5, we have $\lim_{\lambda \to 0^+} \phi(p, q, \lambda) = -p < 0$, $\lim_{\lambda \to +\infty} \phi(p, q, \lambda) = +\infty$. Since Eq. 5 is continuous on $(0, +\infty)$, there exists $\lambda^* \in (0, +\infty)$ satisfying Eq. 5. Notice that $\mu^* = \lambda^* > 0$. Therefore, (λ^*, μ^*) is a positive root of Eq. 2.

Case 3) $p \le 0$ and $q < 0$. Letting $\mu = \lambda$ in Eq. 2, we have Eq. 5.

When $p = 0$, it follows from Eq. 5 that $\lambda^* = \sqrt[k+l-1]{-2/q} > 0$. Notice that $\mu^* = \lambda^* > 0$. Hence, (λ^*, μ^*) is a positive root of Eq. 2. When $p < 0$, in view of Eq. 5, we have $\lim_{\lambda \to 0^+} \phi(p, q, \lambda) = -p > 0$ and $\lim_{\lambda \to +\infty} \phi(p, q, \lambda) = -\infty$. Since Eq. 5 is continuous on $(0, +\infty)$, there exists $\lambda^* \in (0, +\infty)$ satisfying Eq. 5. Notice that $\mu^* = \lambda^* > 0$. Therefore, (λ^*, μ^*) is a positive root of Eq. 2.

In all the cases above, Eq. 2 always has positive solutions. This contradicts Lemma 1. The proof is thus completed.

Theorem 2. Assume that $k > 1$ and $l = 0$ (or $k = 0$ and $l > 1$). Then every solution of Eq. 1 oscillates if and only if $p \le 0$ and $q \ge 0$.

Proof. When $k > 1$ and $l = 0$, then the characteristic equation of Eq. 1 is

$$\phi(p, q, \lambda, \mu) = \lambda + \mu - p + q\lambda^k = 0 . \tag{6}$$

Sufficiency. Suppose to the contrary that $\{u_{i,j}\}$ is an eventually positive solution of Eq. 1. Then from Eq. 1, we have

$$u_{i+1,j} + u_{i,j+1} = pu_{i,j} - qu_{i+k,j} . \tag{7}$$

Since $p \le 0$ and $q \ge 0$, we can see that the left side of Eq. 7 is strictly greater than 0 and the right sight of Eq. 7 is less than or equal to 0. This is a contradiction.

Necessity. Suppose that every solution of Eq. 1 is oscillatory. From Lemma 1, one can obtain that Eq. 6 has no positive roots. Substituting $\mu = c\lambda$ into Eq. 6, we have $c = (-q\lambda^k - \lambda + p)/\lambda$. It is obvious that Eq. 6 has no positive roots if and only if there does not exist $c \in (0, +\infty)$ such that $c = (-q\lambda^k - \lambda + p)/\lambda$ for any $\lambda \in (0, +\infty)$. That is, for any $\lambda \in (0, +\infty)$, $q\lambda^k + \lambda - p \geq 0$ always holds. Let $f(\lambda) = q\lambda^k + \lambda - p$. Then, we have $f'(\lambda) = kq\lambda^{k-1} + 1$.

i) When $q \geq 0$, notice that $f'(\lambda) > 0$ for $\lambda > 0$. Therefore, $f(\lambda)$ is strictly increasing on $(0, +\infty)$. For any $\lambda \in (0, +\infty)$, to ensure $f(\lambda) \geq 0$, it only needs to satisfy the relation $f(\lambda) \geq f(0) = -p \geq 0$, that is, $p \leq 0$.

ii) When $q < 0$, since $f(\lambda)$ is continuous on $(0, +\infty)$ and $\lim_{\lambda \to +\infty} f(\lambda) = -\infty$, it is obvious that $f(\lambda) \geq 0$ does not always hold for any $\lambda \in (0, +\infty)$.

Combining i) and ii) implies that necessity is true. The proof is thus completed. The proof for the case $k = 0$ and $l > 1$ is similar to the case for $k > 1$ and $l = 0$ and is not repeated here.

Theorem 3. Assume that $k = 1$ and $l = 0$ (or $k = 0$ and $l = 1$). Then every solution of Eq. 1 oscillates if and only if $p \leq 0$ and $q \geq -1$.

Proof. When $k = 1$ and $l = 0$, the characteristic equation of Eq.1 is

$$(1+q)\lambda + \mu - p = 0. \tag{8}$$

It is clear that Eq. 8 does not have any positive roots if and only if $p \leq 0$ and $q \geq -1$. Hence, Lemma 1 implies the statement of this theorem. This completes the proof. The proof for the case $k = 0$ and $l = 1$ is similar to the case for $k = 1$ and $l = 0$ and is not repeated here.

Illustrative examples

Example 1. Consider the advanced 2-D discrete system

$$u_{m+1,n} + u_{m,n+1} + 1.05u_{m,n} + u_{m+1,n+1} = 0. \tag{9}$$

From Eq. 9, we have $k = 1, l = 1, p = -1.05, q = 1$. Since $p = -1.05 < 0$ and $q = 1 > 0$, by Theorem 1, every solution of Eq. 9 is oscillatory. The oscillatory behavior of Eq. 9 is demonstrated by Fig. 1.

Example 2. Consider the advanced 2-D discrete system

$$u_{m+1,n} + u_{m,n+1} + 0.6u_{m,n} + u_{m+2,n+1} = 0. \tag{10}$$

From Eq. 10, we have $k = 2, l = 0, p = -0.6, q = 1$. Since $p = -0.6 < 0$ and $q = 1 > 0$, by Theorem 2, every solution of Eq. 10 is oscillatory. The oscillatory behavior of Eq. 10 is demonstrated by Fig. 2.

Example 3. Consider the advanced 2-D discrete system

$$u_{m+1,n} + 1.02u_{m,n+1} + 0.1u_{m,n} = 0. \tag{11}$$

From Eq. 11, we have $k = 0, l = 1, p = -0.1, 1 + q = 1.02$. Since $p = -0.1 < 0$ and $1 + q = 1.02 > 0$, in view of Theorem 3, every solution of Eq. 11 is oscillatory. The oscillatory behavior of Eq. 11 is demonstrated by Fig. 3.

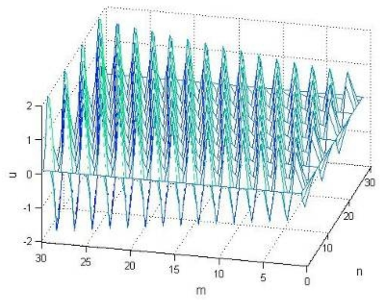

Fig. 1 Oscillatory behavior of Eq. 9

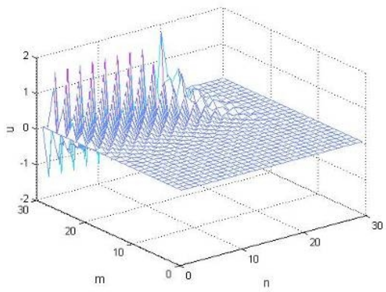

Fig. 2 Oscillatory behavior of Eq. 10

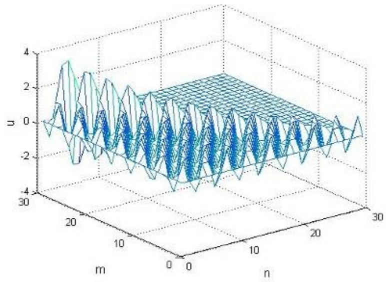

Fig. 3 Oscillatory behavior of Eq. 11

Conclusions

In this paper, we derived effective criteria to determine oscillations of an advanced 2-D discrete system. Oscillation criteria for advanced 2-D discrete systems are different from delay 2-D discrete systems. Numerical examples are given to illustrate the results presented in this paper.

Acknowledgements

The research work was supported by Natural Science Foundation of Shandong Province under Grant No. ZR2014FL015 and Natural Science Foundation of Shandong Province under Grant No. ZR2014AP011.

References

[1] B.E. Shi, L.O. Chua: IEEE Trans. Circuits Syst. I vol. 39 (1992), p. 531

[2] R.J. LeVeque: *Numerical Methods for Conservation Laws* (Birkhauser-Verlag, GER 1990).

[3] S.S. Cheng: *Partial Difference Equations, vol. 3 of Advances in Discrete Mathematics and Applications* (Taylor & Francis, UK 2003).

[4] B.G. Zhang, Y. Zhou: *Qualitative Analysis of Delay Partial Difference Equations* (Hindawi Publishing Corporation, USA 2007).

[5] B.G. Zhang, Q.J. Xing: J. Math. Anal. Appl. vol. 329 (2007), p. 567

[6] B.G. Zhang, R.P. Agarwal: Comput. Math. Appl. vol. 45 (2003), p.1253

[7] R.P. Agarwal, Y. Zhou: Math. Comput. Model. vol. 31 (2000), p. 17

[8] B.G. Zhang, J.S. Yu: Comput. Math. Appl. Vol. 35 (1998), p.111

[9] P.J.Y. Wong, R.P. Agarwal: Comput. Math. Appl. vol. 32 (1996), p. 57

[10] B.G. Zhang, S.T. Liu: Discrete Dynamics in Nature and Society vol. 1 (1998), p. 265

[11] J.F. Cheng, Y.M. Chu: Bull. Inst. Math. Acad. Sinica vol. 1 (2006), p. 507