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Square roots of matrices over a complete lattice

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Abstract

This paper deals with square roots of a matrix over a complete lattice, where the matrix composition is $\vee - U$ with U being an infinitely $\vee$-distributive isotonic operator. We give a general characterization for the existence of a square root of a matrix over a complete lattice. Furthermore, we give methods to construct a square root of a matrix while U is idempotent or a semi-uninorm.

Keywords: Square roots of a matrix; complete lattices; isotonic operator; infinitely $\vee$-distributive; semi-uninorm.

1. Introduction

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set, L be a linear lattice with universal bounds 0 and 1, $\underline{n} = \{1, 2, \cdots, n\}$, and $F(X \times X) = \{R : X \times X \to L\}$. Di Nola et al. [4] considered the decomposition problem: for a given fuzzy relation $Q \in F(X \times X)$, find a fuzzy relation $S \in F(X \times X)$ such that $Q = S \odot S$, where $\odot$ denotes the $\vee - \wedge$ composite operation, i.e.,

$$\bigvee_{k=1}^{n} (s_{ik} \wedge s_{kj}) = q_{ij}, i, j \in \underline{n}. \quad (1)$$

Actually, the decomposition problem can be viewed as a generalization of finding a square root of a Boolean matrix in [8]. Di Nola [4] gave a necessary and sufficient condition for judging whether there is a square root for a given matrix over linear lattices and presented a corresponding numerical algorithm. Wang [14] pointed out a mistake in the algorithm given in [4] and modified. An algebraic characterization of minimal and maximal elements of the square roots of a given matrix over linear lattices was given in [5]. In 2004, Martin Kutz [10] showed that finding roots of Boolean matrices is NP-complete, so is finding square roots of matrices over linear lattices. As a generalization, Sun [13] considered square roots of matrices over complete lattices. Since a T-norm (also t-norm or, unabbreviated, triangular norm) is a kind of binary operation (see [9]) that generalizes intersection in a lattice, in this paper we will consider square roots of a matrix over a complete lattice based on some generalized T-norms.

The rest of this paper is organized as follows. In Section 2, some definitions and preliminaries are given. Residual operators induced by an isotonic operator U and their properties are discussed in Section 3. In Section 4, a general characterization of the existences of square roots of a matrix over a complete lattice, and a theoretical algorithm to find such square roots are given when the matrix composition is $\vee - U$ with U being an infinitely $\vee$-distributive isotonic operator. In Section 5, we give further results for determining a square root of a matrix while U is idempotent or a semi-uninorm.

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2. Basic definitions and preliminaries

Here we recall some notions from lattice theory(see [1]). Let $(P, \leq)$ be a partially ordered set, i.e., poset, and S be a subset of P. An element $p \in P$ is a *join* (or *least upper bound*) of S if p is an upper bound of S, and $p \leq x$ for every upper bound x of S. Similarly, q is a *meet* (or *greatest lower bound*) of S if q is a lower bound of S, and $y \leq q$ for every lower bound y of S. A *lattice* $L = \langle L, \leq \rangle$ is a poset in which every pair of elements $p, q \in L$ has a join $p \vee q$ and a meet $p \wedge q$. A lattice L in which every subset possesses a meet and a join is *complete*. Throughout the paper, L is always assumed to be a complete lattice with universal bounds 0 and 1, unless otherwise specified. Denote $L^{n\times m} = \{B = (b_{ij})_{n\times m} : b_{ij} \in L, i \in \underline{n}, j \in \underline{m}\}$.

Definition 2.1 (Di Nola et al. [6]). *Let $Q = (q_{ij})_{n\times p} \in L^{n\times p}$ and $S = (s_{ij})_{p\times m} \in L^{p\times m}$. Define the max-min composition of Q and S to be $R = (r_{ij})_{n\times m} \in L^{n\times m}$, in symbols $R = Q \odot S$, given by*

$$r_{ij} = \bigvee_{k=1}^{p} (q_{ik} \wedge s_{kj})$$

for all $i \in \underline{n}, j \in \underline{m}$.

Definition 2.2 (Di Nola et al. [6]). *Let $Q_1 \in L^{n\times p}$ and $Q_2 \in L^{n\times p}$. Define $Q_1 \vee Q_2$*

as $(Q_1 \vee Q_2)_{ij} = (Q_1)_{ij} \vee (Q_2)_{ij}$, $Q_1 \wedge Q_2$ *as* $(Q_1 \vee Q_2)_{ij} = (Q_1)_{ij} \wedge (Q_2)_{ij}$, *and* $Q_1 \le Q_2$ *if and only if* $(Q_1)_{ij} \le (Q_2)_{ij}$ *for any* $i \in \underline{n}, j \in \underline{p}$.

Definition 2.3 (De Baets and Mesiar [3]). *A triangular norm is a binary operation* T *on* L *(i.e.,* $T : L \times L \to L$*) satisfying the following properties:*
(T1) *existence of neutral element 1:*
$T(x, 1) = x$, $\forall x \in L$;
(T2) *monotonicity:*
$T(x, z) \le T(y, w)$ *whenever* $x \le y$, $z \le w$;
(T3) *commutativity:*
$T(x, y) = T(y, x)$;
(T4) *associativity:*
$T(x, T(y, z)) = T(T(x, y), z)$.

As a generalization, a uninorm was introduced by Yager and Rybalov [16], we state it over L as follows.

Definition 2.4 (De Baets and Mesiar [3]). *A uninorm is a binary operation* U *on* L *satisfying the following properties:*
(U1) *existence of neutral element* $e \in L$*:*
$U(x, e) = x$, $\forall x \in L$;
(U2) *monotonicity:*
$U(x, z) \le U(y, w)$ *whenever* $x \le y$, $z \le w$;
(U3) *commutativity:*
$U(x, y) = U(y, x)$;
(U4) *associativity:*
$U(x, U(y, z)) = U(U(x, y), z)$.

For any uninorm U over the interval $[0, 1]$, one of the following two cases always holds ([7]):
(i) U is a conjunctive uninorm: $U(0, 1) = U(1, 0) = 0$;
(ii) U is a disjunctive uninorm: $U(0, 1) = U(1, 0) = 1$.
By omitting commutativity in uninorms, Sander [12] introduced pseudo-uninorms over the interval $[0, 1]$, which were further discussed over lattices by Wang and Fang [15].

Definition 2.5 (Wang and Fang [15]). *A binary operation* U *on* L *is called a pseudo-uninorm if it satisfies the following conditions:*
(PU1) *existence of neutral element* $e \in L$*:*
$U(x, e) = x = U(e, x)$, $\forall x \in L$;
(PU2) *monotonicity:*
$U(x, z) \le U(y, w)$ *whenever* $x \le y$, $z \le w$;
(PU3) *associativity:*
$U(x, U(y, z)) = U(U(x, y), z)$.

Further, a semi-uninorm was introduced by Liu [11] through removing associativity from pseudo-uninorms.

Definition 2.6 (Liu [11]). *A binary operation* U *on* L *is called a semi-uninorm if it satisfies the following conditions:*
(SU1) *existence of neutral element* $e \in L$*:*
$U(x, e) = x = U(e, x)$, $\forall x \in L$;
(SU2) *monotonicity:*
$U(x, z) \le U(y, w)$ *whenever* $x \le y$, $z \le w$.

Here we omit the neutral property of a semi-uninorm.

Definition 2.7. *A binary operation* U *on* L *is called an isotonic operator if it satisfies:*
(I1) *monotonicity:*
$U(x, z) \le U(y, w)$ *whenever* $x \le y$, $z \le w$.

Definition 2.8. *Let* $Q \in L^{n\times n}$ *and* U *be a binary operation on* L. *If there exists* $S \in L^{n\times n}$ *such that* $Q = S \otimes S$, *where* $\otimes$ *denotes the* $\vee - U$ *composite operation, i.e.,*

$$q_{ij} = \bigvee_{k=1}^{n} U(s_{ik}, s_{kj}), i, j \in \underline{n}, \tag{2}$$

then we call S *the square root of* Q *based on* $\vee - U$ *composition.*

Definition 2.9 (Wang and Fang [15]). *A binary operation* U *on* L *is called left (resp. right) infinitely* $\vee$*-distributive if for all* y, $x_j \in L$ *(*$j \in J$*, where* J *stands for any index set.)*

$$U(\bigvee_{j\in J} x_j, y) = \bigvee_{j\in J} U(x_j, y),$$

$$(\textit{resp.}\, U(y, \bigvee_{j\in J} x_j) = \bigvee_{j\in J} U(y, x_j)).$$

A binary operation U *on* L *is said to be infinitely* $\vee$*-distributive if* U *is both left infinitely* $\vee$*-distributive and right infinitely* $\vee$*-distributive.*

It is worth noting that for every left (resp. right) infinitely $\vee$-distributive operation U on L, $U(0, x) = 0$ (resp. $U(x, 0) = 0$) holds for all $x \in L$ since $\bigvee \emptyset = 0$.

Lemma 2.1. *Let* Q_1, $Q_2 \in L^{n\times p}$, $S \in L^{p\times m}$, $R \in L^{q\times n}$ *and* U *be an isotonic operator. If* $Q_1 \le Q_2$ *then* $Q_1 \otimes S \le Q_2 \otimes S$ *and* $R \otimes Q_1 \le R \otimes Q_2$.

Proof. Let $Q_1 \le Q_2$. From the definition of $\otimes$ and Definition 2.7 for any $i \in \underline{n}$, $j \in \underline{m}$,

$$\begin{aligned}(Q_1 \otimes S)_{ij} &= \bigvee_{k=1}^{p} U((Q_1)_{ik}, S_{kj}) \\ &\le \bigvee_{k=1}^{p} U((Q_2)_{ik}, S_{kj}) \\ &= (Q_2 \otimes S)_{ij}.\end{aligned}$$

Therefore $Q_1 \otimes S \le Q_2 \otimes S$. Similarly, we can prove $R \otimes Q_1 \le R \otimes Q_2$. □

Lemma 2.2. *Let I, J be any index sets, $Q_i \in L^{n\times p}$ for any $i \in I$, $S_j \in L^{p\times m}$ for any $j \in J$, and U be an infinitely $\vee$-distributive operation. Then $(\bigvee_{i\in I} Q_i)\otimes(\bigvee_{j\in J} S_j) = \bigvee_{i\in I, j\in J}(Q_i \otimes S_j)$.*

Proof. Suppose that U is infinitely $\vee$-distributive. Then for any $s \in \underline{n}$, $t \in \underline{m}$,

$$\begin{aligned}((\bigvee_{i\in I} Q_i)\otimes(\bigvee_{j\in J} S_j))_{st} &= \bigvee_{k=1}^{p} U((\bigvee_{i\in I} Q_i)_{sk}, (\bigvee_{j\in J} S_j)_{kt}) \\ &= \bigvee_{k=1}^{p} \bigvee_{i\in I, j\in J} U((Q_i)_{sk}, (S_j)_{kt}) \\ &= \bigvee_{i\in I, j\in J} \bigvee_{k=1}^{p} U((Q_i)_{sk}, (S_j)_{kt}) \\ &= \bigvee_{i\in I, j\in J} (Q_i \otimes S_j)_{st} \\ &= (\bigvee_{i\in I, j\in J} Q_i \otimes S_j)_{st}\end{aligned}$$

Therefore $(\bigvee_{i\in I} Q_i)\otimes(\bigvee_{j\in J} S_j) = \bigvee_{i\in I, j\in J}(Q_i \otimes S_j)$. □

3. Residual operators induced by an isotonic operator U

We first introduce residual operators induced by an isotonic operator U.

Definition 3.1. *Let U be an isotonic operator. The binary operations R_U^l and R_U^r on L, defined as: for any $x, z \in L$,*

$$R_U^l(x,z) = \bigvee\{t \in L : U(t,x) \le z\},$$

$$R_U^r(x,z) = \bigvee\{t \in L : U(x,t) \le z\},$$

are called left residual operator and right residual operator induced by U.

Theorem 3.1. *If $U(0,1) = 0$, then the following assertions are equivalent:*
(i) U is left infinitely $\vee$-distributive.
(ii) $U(y,x) \le z$ if and only if $y \le R_U^l(x,z)$ for all $x, z \in L$.
(iii) $U(R_U^l(x,z), x) \le z$ for all $x, z \in L$.
(iv) $R_U^l(x,z) = \max\{t \in L : U(t,x) \le z\}$ for all $x, z \in L$.

Proof. (i)⇒(ii): For all $x, z \in L$, if $U(y,x) \le z$, then $y \le R_U^l(x,z)$ holds by the definition of $R_U^l(x,z)$. Conversely, if $y \le R_U^l(x,z)$, by isotonicity and left infinitely $\vee$-distributivity of U, we have $U(y,x) \le U(R_U^l(x,z),x) = U(\bigvee\{t \in L : U(t,x) \le z\}, x) = U(\bigvee_{t\in L, U(t,x)\le z} t, x) = \bigvee_{t\in L, U(t,x)\le z} U(t,x) = z$.

(ii)⇒(iii): Let $y = R_U^l(x,z)$. Then $U(R_U^l(x,z),x) = U(y,x) \le z$ since (ii) holds.

(iii)⇒(iv): $R_U^l(x,z) = \max\{t \in L : U(t,x) \le z\}$ is obvious since $U(R_U^l(x,z),x) \le z$.

(iv)⇒(i): Let J be any index set and $y_j \in L$ for any $j \in J$. If $J \ne \emptyset$, then $U(y_{j_0}, x) \le U(\bigvee_{j\in J} y_j, x)$ for any $j_0 \in J$ by the isotonicity of U. Therefore $\bigvee_{j\in J} U(y_j,x) \le U(\bigvee_{j\in J} y_j, x)$. Let $\bigvee_{j\in J} U(y_j,x) = z$. Then $U(y_j,x) \le z$ for any $j \in J$, i.e., $y_j \in \{t \in L : U(t,x) \le z\}$ for any $j \in J$. Since $R_U^l(x,z) = \max\{t \in L : U(t,x) \le z\}$, we have $y_j \le R_U^l(x,z)$ for any $j \in J$. Thus $\bigvee_{j\in J} y_j \le R_U^l(x,z)$. From the isotonicity of U again, it follows that $U(\bigvee_{j\in J} y_j, x) \le U(R_U^l(x,z),x) \le z = \bigvee_{j\in J} U(y_j,x)$. Then $\bigvee_{j\in J} U(y_j,x) = U(\bigvee_{j\in J} y_j, x)$ since $\bigvee_{j\in J} U(y_j,x) \le U(\bigvee_{j\in J} y_j, x)$.

If $J = \emptyset$, then $\bigvee_{j\in J} U(y_j,x) = 0$ and $U(\bigvee_{j\in J} y_j, x) = U(0,x) \le U(0,1) = 0$. Therefore $\bigvee_{j\in J} U(y_j,x) = U(\bigvee_{j\in J} y_j, x)$ holds for any index set J, i.e., U is left infinitely $\vee$-distributive. □

Similarly, we have:

Theorem 3.2. *If $U(1,0) = 0$, then the following assertions are equivalent:*
(i) U is right infinitely $\vee$-distributive.
(ii) $U(x,y) \le z$ if and only if $y \le R_U^r(x,z)$ for all $x, z \in L$.
(iii) $U(x, R_U^r(x,z)) \le z$ for all $x, z \in L$.
(iv) $R_U^r(x,z) = \max\{t \in L : U(x,t) \le z\}$ for all $x, z \in L$.

Let $A \in L^{n\times p}$, $B \in L^{n\times m}$. Define the $\circledast^l$ composition of A and B, $A \circledast^l B \in L^{p\times m}$, and the $\circledast^r$ composition of A and B, $A \circledast^r B \in L^{p\times m}$, respectively by

$$(A \circledast^l B)_{ij} = \bigwedge_{k=1}^{n} R_U^l(A_{ki}, B_{kj}), i \in \underline{p}, j \in \underline{m}, \quad (3)$$

and

$$(A \circledast^r B)_{ij} = \bigwedge_{k=1}^{n} R_U^r(A_{ki}, B_{kj}), i \in \underline{p}, j \in \underline{m}. \quad (4)$$

Theorem 3.3. *Let $A \in L^{n\times p}$, $B \in L^{n\times m}$ and U be a right infinitely $\vee$-distributive isotonic operator. Then $A \circledast^r B = \max\{X \in L^{p\times m} : A \otimes X \le B\}$.*

Proof. We first prove $A \circledast^r B \in \{X \in L^{p\times m} : A \otimes X \le B\}$. By Theorem 3.2, for any $i \in \underline{n}$,

$j \in m$,

$$\begin{aligned}(A \otimes (A \circledast^r B))_{ij} &= \bigvee_{k=1}^{p} U(A_{ik}, (A \circledast^r B)_{kj}) \\ &= \bigvee_{k=1}^{p} U(A_{ik}, \bigwedge_{t=1}^{n} R_U^r(A_{tk}, B_{tj})) \\ &\le \bigvee_{k=1}^{p} U(A_{ik}, R_U^r(A_{ik}, B_{ij})) \\ &\le B_{ij}.\end{aligned}$$

To complete the proof, we need to show for any $X \in \{X \in L^{p\times m} : A \otimes X \le B\}$, $X \le A \circledast^r B$. Let $X \in L^{p\times m}$ and $A \otimes X \le B$. Then $U(A_{ik}, X_{kj}) \le \bigvee_{k=1}^{p} U(A_{ik}, X_{kj}) \le B_{ij}$ for any $i \in \underline{n}$, $j \in \underline{m}$, $k \in \underline{p}$. Therefore $X_{kj} \le R_U^r(A_{ik}, B_{ij})$ for any $i \in \underline{n}$. Then $X_{kj} \le \bigwedge_{i=1}^{n} R_U^r(A_{ik}, B_{ij}) = (A \circledast^r B)_{kj}$, and hence $A \circledast^r B = \max\{X \in L^{p\times m} : A \otimes X \le B\}$. □

Theorem 3.4. *Let $A \in L^{p\times m}$, $B \in L^{n\times m}$ and U be a left infinitely $\vee$-distributive isotonic operator. Then $A \circledast^l B = \max\{X \in L^{n\times p} : X \otimes A \le B\}$.*

Proof. Similar to the proof of Theorem 3.3. □

Corollary 3.1. *Let $A \in L^{n\times p}$, $B \in L^{n\times m}$ and U be a right infinitely $\vee$-distributive isotonic operator. For any $X \in L^{p\times m}$, if $X \le A \circledast^r B$, then $A \otimes X \le B$.*

Proof. From Lemma 2.1 and Theorem 3.3, we have $A \otimes X \le A \otimes (A \circledast^r B) \le B$. □

Corollary 3.2. *Let $A \in L^{p\times m}$, $B \in L^{n\times m}$ and U be a left infinitely $\vee$-distributive isotonic operator. For any $X \in L^{n\times p}$, if $X \le A \circledast^l B$, then $X \otimes A \le B$.*

Proof. Similar to the proof of Corollary 3.1. □

Corollary 3.3. *Let $A, B \in L^{n\times n}$ and U be an infinitely $\vee$-distributive isotonic operator. Then $A \le A \circledast^l B$ if and only if $A \le A \circledast^r B$.*

Proof. If $A \le A \circledast^l B$, then by Corollary 3.2 $A \otimes A \le B$, therefore $A \le A \circledast^r B$ by Theorem 3.3. Vice versa. □

If U is commutative, then $R_U^l(x, y) = R_U^r(x, y)$ for all $x, y \in L$. When U is commutative, we denote $R_U = R_U^l = R_U^r$ and $A \circledast B = A \circledast^l B = A \circledast^r B$.

4. General results on finding square roots of a matrix over L when U is an infinitely $\vee$-distributive isotonic operator

In this section, U is assumed to be an infinitely $\vee$-distributive isotonic operator. We shall give general results on the existence of square roots of a matrix over complete lattices and show a theoretical way to find the square roots. Let $Q \in L^{n\times n}$. For any s, $t \in \underline{n}$, define an $n \times n$ matrix $RQ^{(s,t)}$ with $(RQ^{(s,t)})_{ij} = 0$ whenever $s \ne i$ and $t \ne j$. Denote $\mathcal{RQ}^{(s,t)} = \{RQ^{(s,t)} : (RQ^{(s,t)} \otimes RQ^{(s,t)})_{st} = Q_{st}, RQ^{(s,t)} \otimes RQ^{(s,t)} \le Q\}$. Pick one $RQ^{(s,t)}$ in each $\mathcal{RQ}^{(s,t)}$, s, $t \in \underline{n}$, to form $\mathcal{RQ}$, i.e., $\mathcal{RQ} = \{RQ^{(1,1)}, RQ^{(1,2)}, \cdots, RQ^{(n,n)}\}$, such that $A \otimes B \le Q$ for any A, $B \in \mathcal{RQ}$. Denote $\mathfrak{RQ}$ the set of all such $\mathcal{RQ}$s.

We now give a characterization for the existence for a square root of a given matrix over a complete lattice.

Theorem 4.1. *A matrix $Q \in L^{n\times n}$ has square roots if and only if $\mathfrak{RQ} \ne \emptyset$. Furthermore, if $\mathfrak{RQ} \ne \emptyset$, then for any $\mathcal{RQ} \in \mathfrak{RQ}$, $(\bigvee \mathcal{RQ}) \otimes (\bigvee \mathcal{RQ}) = Q$, and all square roots of Q can be represented by $\bigvee \mathcal{RQ}$.*

Proof. Suppose that S is a square root of $Q \in L^{n\times n}$. For any s, $t \in \underline{n}$, define $RQ^{(s,t)}$ as

$$(RQ^{(s,t)})_{ij} = \begin{cases} 0, & s \ne i \text{ and } t \ne j, \\ S_{ij}, & \text{otherwise.} \end{cases}$$

It is obvious that $RQ^{(s,t)} \le S$ and $(RQ^{(s,t)} \otimes RQ^{(s,t)})_{st} = \bigvee_{k=1}^{n} U(S_{sk}, S_{kt}) = Q_{st}$. Then by Lemma 2.1 $RQ^{(s,t)} \otimes RQ^{(i,j)} \le S \otimes S = Q$. Therefore $\mathcal{RQ} = \{RQ^{(s,t)}, s, t \in \underline{n}\} \in \mathfrak{RQ}$, i.e., $\mathfrak{RQ} \ne \emptyset$.

Conversely, let $\mathcal{RQ} = \{RQ^{(s,t)}, s, t \in \underline{n}\} \in \mathfrak{RQ}$. Since $(RQ^{(s,t)} \otimes RQ^{(s,t)})_{st} = Q_{st}$ and $RQ^{(s,t)} \otimes RQ^{(i,j)} \le Q$ for any $i, j, s, t \in \underline{n}$, from Lemma 2.2 it follows that

$$\begin{aligned}&(\bigvee \mathcal{RQ}) \otimes (\bigvee \mathcal{RQ}) \\ &= (\bigvee_{RQ^{(s,t)} \in \mathcal{RQ}} RQ^{(s,t)}) \otimes (\bigvee_{RQ^{(i,j)} \in \mathcal{RQ}} RQ^{(i,j)}) \\ &= \bigvee_{RQ^{(s,t)}, RQ^{(i,j)} \in \mathcal{RQ}} (RQ^{(s,t)} \otimes RQ^{(i,j)}) \\ &= Q.\end{aligned}$$

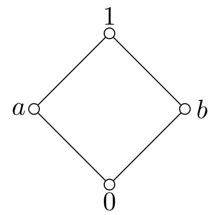
Moreover, the assertion that all square roots of Q can be represented by $\bigvee \mathcal{RQ}$ has been implied in the first part of the proof. □

From the proof of Theorem 4.1, we can get the following theoretical way to find the square roots.

Algorithm 4.1. *Finding square roots of* $Q \in L^{n\times n}$.
Step 1. Find $RQ^{(s,t)}$ *such that* $(RQ^{(s,t)} \otimes RQ^{(s,t)})_{st} = Q_{st}$.
Step 2. Determine $\mathcal{RQ}^{(s,t)}$.
Step 3. Construct $\mathcal{RQ}$ *and* $\mathfrak{RQ}$.
Step 4. If $\mathfrak{RQ} = \emptyset$*, then* Q *has no square roots, otherwise* $\bigvee \mathcal{RQ}$ *is a square root for any* $\mathcal{RQ} \in \mathfrak{RQ}$.

At the end of this section, we give an example to illustrate Algorithm 4.1.

Example 4.1. Let L be a complete lattice as follows.



Define a binary operation U, shown below, over L.

U	0	a	b	1
0	0	0	0	0
a	0	1	a	1
b	0	a	b	1
1	0	1	1	1

We can check U is an infinitely $\vee$-distributive conjunctive uninorm. Let $Q = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Then $\mathcal{RQ}^{(1,1)} = \mathcal{RQ}^{(2,2)} = \{\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\}$, $\mathcal{RQ}^{(1,2)} = \{\begin{pmatrix} b & b \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix}\}$, $\mathcal{RQ}^{(2,1)} = \{\begin{pmatrix} b & 0 \\ b & b \end{pmatrix}, \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b & b \end{pmatrix}\}$.

Since $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} b & 0 \\ b & b \end{pmatrix} = \begin{pmatrix} b & b \\ a & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ a & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ b & b \end{pmatrix} = \begin{pmatrix} b & b \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \otimes \begin{pmatrix} b & b \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & b \end{pmatrix}$, $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \otimes \begin{pmatrix} b & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & b \end{pmatrix}$, $\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & b \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$, we have $\mathfrak{RQ} = \emptyset$. Therefore Q has no square roots.

Theorem 4.1 gives a general characterization of the existence of a square root for a given matrix over a complete lattice when U is an infinitely $\vee$-distributive isotonic operator. However, determining $\mathfrak{RQ}$ and $\mathcal{RQ}$ is not easy. In the following section we try to determine $\mathcal{RQ}$.

5. Determine $\mathcal{RQ}$ while U is idempotent or a semi-uninorm

Theorem 4.1 and Algorithm 4.1 reveal that finding square roots of $Q \in L^{n\times n}$ is equivalent to determine $\mathcal{RQ}$ when U is an infinitely $\vee$-distributive isotonic operator. In this section, we further assume U to be idempotent (i.e., $U(x,x) = x$ for all $x \in L$, see [2]) or be a semi-uninorm, then determine $\mathcal{RQ}$ under such assumptions.

For an idempotent infinitely $\vee$-distributive isotonic operator U, for any $s,t \in \underline{n}$, define $\mathcal{RQ}_{=}^{(s,t)} = \{RQ_{=}^{(s,k,t)} : k \in \underline{n}\}$, where $RQ_{=}^{(s,k,t)}$ is defined as: for any $i,j \in \underline{n}$,

$$(RQ_{=}^{(s,k,t)})_{ij} = \begin{cases} Q_{st}, & (s,k) = (i,j), \\ Q_{st}, & (k,t) = (i,j), \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 5.1. *For any* $s,t \in \underline{n}$*, for any* $M \in \mathcal{RQ}_{=}^{(s,t)}$*, if* $M \le M \circledast^l Q$*, then* $M \in \mathcal{RQ}^{(s,t)}$.

Proof. By the definition of $\mathcal{RQ}^{(s,t)}$, we need to prove $(M \otimes M)_{st} = Q_{st}$ and $M \otimes M \le Q$. For any $M \in \mathcal{RQ}_{=}^{(s,t)}$, there is some $k_0 \in \underline{n}$ such that $M = RQ_{=}^{(s,k_0,t)}$, i.e., $M_{sk_0} = M_{k_0 t} = Q_{st}$. Since U is infinitely $\vee$-distributive, we have $U(0,x) = U(x,0) = 0$ for any $x \in L$. Therefore $(M \otimes M)_{st} = \bigvee_{k=1}^{n} U(M_{sk}, M_{kt}) = U(M_{sk_0}, M_{k_0 t}) = U(Q_{st}, Q_{st}) = Q_{st}$. From $M \le M \circledast^l Q$, $M \otimes M \le Q$ holds by Corollary 3.2. Thus $M \in \mathcal{RQ}^{(s,t)}$. $\square$

Take each $RQ_{=}^{(s,k_{st},t)}$, where $k_{st} \in \underline{n}$, in $\mathcal{RQ}_{=}^{(s,t)}$ to form $\mathcal{RQ}_{=}$, i.e., $\mathcal{RQ}_{=} = \{RQ_{=}^{(1,k_{11},1)}, RQ_{=}^{(1,k_{12},2)}, \cdots, RQ_{=}^{(n,k_{nn},n)}\}$.

Theorem 5.2. *If* $\bigvee \mathcal{RQ}_{=} \le (\bigvee \mathcal{RQ}_{=}) \circledast^l Q$*, then* $(\bigvee \mathcal{RQ}_{=}) \otimes (\bigvee \mathcal{RQ}_{=}) = Q$.

Proof. If $\bigvee \mathcal{RQ}_{=} \le (\bigvee \mathcal{RQ}_{=}) \circledast^l Q$, then Corollary 3.2 ensures that $(\bigvee \mathcal{RQ}_{=}) \otimes (\bigvee \mathcal{RQ}_{=}) \le Q$. Now, we shall prove $(\bigvee \mathcal{RQ}_{=}) \otimes (\bigvee \mathcal{RQ}_{=}) \ge Q$. From the proof of Theorem 5.1, we know for any $RQ_{=}^{(s,k_{st},t)} \in \mathcal{RQ}_{=}$, $(RQ_{=}^{(s,k_{st},t)} \otimes RQ_{=}^{(s,k_{st},t)})_{st} = Q_{st}$. Then for any $i,j \in \underline{n}$, $((\bigvee \mathcal{RQ}_{=}) \otimes (\bigvee \mathcal{RQ}_{=}))_{ij} = ((\bigvee_{RQ_{=}^{(s,k_{st},t)} \in \mathcal{RQ}_{=}} RQ_{=}^{(s,k_{st},t)}) \otimes (\bigvee_{RQ_{=}^{(s,k_{st},t)} \in \mathcal{RQ}_{=}} RQ_{=}^{(s,k_{st},t)}))_{ij} \ge (RQ_{=}^{(i,k_{ij},j)} \otimes RQ_{=}^{(i,k_{ij},j)})_{ij} = Q_{ij}$. Therefore $(\bigvee \mathcal{RQ}_{=}) \otimes (\bigvee \mathcal{RQ}_{=}) = Q$. $\square$

For an infinitely $\vee$-distributive semi-uninorm U (e is the neutral element), for any $s,t \in \underline{n}$, define $\mathcal{RQ}_e^{(s,t)} = \{RQ_e^{(s,k,t)} : k \in \underline{n}\}$ as

$$(RQ_e^{(s,k,t)})_{ij} = \begin{cases} Q_{st}, & (s,k) = (i,j), \\ e, & (k,t) = (i,j), \\ 0, & \text{otherwise,} \end{cases}$$

or

$$(RQ_e^{(s,k,t)})_{ij} = \begin{cases} e, & (s,k) = (i,j), \\ Q_{st}, & (k,t) = (i,j), \\ 0, & \text{otherwise,} \end{cases}$$

where $|\{s,k,t\}| \neq 1$ or $|\{s,k,t\}| = 1$ with $Q_{ss} = e$.

Theorem 5.3. *For any $s,t \in \underline{n}$, for any $M \in \mathcal{RQ}_e^{(s,t)}$, if $M \leq M \circledast^l Q$, then $M \in \mathcal{RQ}^{(s,t)}$.*

Proof. Similar to the proof of Theorem 5.1 since $U(Q_{st}, e) = U(e, Q_{st}) = Q_{st}$. □

Pick one $RQ_e^{(s,k_{st},t)}$ ($k_{st} \in \underline{n}$) in each $\mathcal{RQ}_e^{(s,t)}$ to form $\mathcal{RQ}_e$, i.e., $\mathcal{RQ}_e = \{RQ_e^{(1,k_{11},1)}, RQ_e^{(1,k_{12},2)}, \cdots, RQ_e^{(n,k_{nn},n)}\}$.

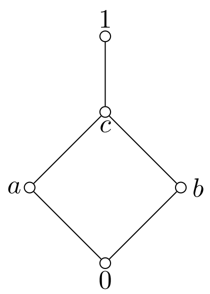
Theorem 5.4. *If $\bigvee \mathcal{RQ}_e \leq (\bigvee \mathcal{RQ}_e) \circledast^l Q$, then $(\bigvee \mathcal{RQ}_e) \otimes (\bigvee \mathcal{RQ}_e) = Q$.*

Proof. Similar to the proof of Theorem 5.2. □

We point out here that $\circledast^l$ in Theorems 5.1, 5.2, 5.3 and 5.4 can be replaced by $\circledast^r$ with the help of Corollary 3.3.

In the following, we give examples to help understanding Theorems 5.2 and 5.4.

Example 5.1. Let L be a complete lattice as follows.



Consider the following binary operation U over L.

U	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	1
1	0	a	b	1	1

We can check U is infinitely $\vee$-distributive, isotonic, idempotent, and has neutral element c.

Let $Q = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$. Consider $\mathcal{RQ}_{=} = \{RQ_{=}^{(1,1,1)}, RQ_{=}^{(1,1,2)}, RQ_{=}^{(2,2,1)}, RQ_{=}^{(2,2,2)}\}$, where $RQ_{=}^{(1,1,1)} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $RQ_{=}^{(1,1,2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $RQ_{=}^{(2,2,1)} = \begin{pmatrix} 0 & 0 \\ b & b \end{pmatrix}$, $RQ_{=}^{(2,2,2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, $\bigvee \mathcal{RQ}_{=} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, $(\bigvee \mathcal{RQ}_{=}) \circledast^l Q = \begin{pmatrix} 1 \wedge 1 & b \wedge 1 \\ 1 \wedge b & 1 \wedge 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix} \geq \bigvee \mathcal{RQ}_{=}$. From Theorem 5.2, $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ is a square root of $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$. Moreover, it is the unique square root.

Let $Q = \begin{pmatrix} c & a \\ b & c \end{pmatrix}$. Consider $\mathcal{RQ}_e = \{RQ_e^{(1,1,1)}, RQ_e^{(1,2,2)}, RQ_e^{(2,2,1)}, RQ_e^{(2,2,2)}\}$, where $RQ_e^{(1,1,1)} = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$, $RQ_e^{(1,2,2)} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$, $RQ_e^{(2,2,1)} = \begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix}$, $RQ_e^{(2,2,2)} = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$. Obviously, $\bigvee \mathcal{RQ}_e = \begin{pmatrix} c & a \\ b & c \end{pmatrix}$, $(\bigvee \mathcal{RQ}_e) \circledast^l Q = \begin{pmatrix} c \wedge 1 & a \wedge 1 \\ 1 \wedge b & 1 \wedge c \end{pmatrix} = \begin{pmatrix} c & a \\ b & c \end{pmatrix} \geq \bigvee \mathcal{RQ}_e$. From Theorem 5.4, $\begin{pmatrix} c & a \\ b & c \end{pmatrix}$ is a square root of $\begin{pmatrix} c & a \\ b & c \end{pmatrix}$. Moreover, it is the unique square root.

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Consider $\mathcal{RQ}_e = \{RQ_e^{(1,1,1)}, RQ_e^{(1,1,2)}, RQ_e^{(2,1,1)}, RQ_e^{(2,2,2)}\}$, where $RQ_e^{(1,1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $RQ_e^{(1,1,2)} = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}$, $RQ_e^{(2,1,1)} = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$, $RQ_e^{(2,2,2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, $\bigvee \mathcal{RQ}_e = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$, $(\bigvee \mathcal{RQ}_e) \circledast^l Q = \begin{pmatrix} 1 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 1 & 1 \wedge 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \geq \bigvee \mathcal{RQ}_e$. From Theorem 5.4, $\begin{pmatrix} c & 1 \\ 1 & c \end{pmatrix}$ is a square root of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

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