A dual decomposition of the single-parameter Gini social evaluation functions

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Abstract

For each single-parameter Gini social evaluation function, and by using the dual decomposition of the OWA operators, we derive two contributing factors. The first one, the self-dual core that can be considered as a positional measure, similar to the mean. The second one, the anti-self dual remainder, that we will prove is an equality measure with balanced sensitivity to both tails.

Keywords: Income inequality; Social welfare; Aggregation functions; OWA operators; Dual decomposition.

1. Introduction

In the literature, there exist families of social evaluation functions which are defined in order to represent ethical orderings of alternative distributions of income, or some other social or economic variables, among individuals.

Following the Atkinson -Kolm -Sen (AKS) [2][13][17] approach, for each family of social evaluation functions it may be considered the corresponding family of inequality measures. In fact, each social evaluation function can be decomposed into two contributing factors, the mean of the distribution and the corresponding inequality measure.

Aristondo et al. [1] propose an alternative to the AKS decomposition of some particular welfare functions by using the dual decomposition of the OWA operators introduced by García-Lapresta and Marques Pereira [8]. Here, we do a similar exercise. We focus on the single-parameter Gini social evaluation functions. For each of these functions, and by using the dual decomposition of the OWA operators, we derive two contributing factors. The first one, the self-dual core that can be considered as a positional measure, similar to the mean. The second one, the anti-self-dual remainder, which we will prove is an equality measure with balanced sensitivity to both tails. In fact, this equality measure is consistent with two properties, the up-down positional transfer sensitivity and the symmetric positional transfer sensitivity principles.

The paper is organized as follows. Section 2 reviews the dual decomposition of an OWA operator due to Garca-Lapresta and Marques Pereira (2008). Section 3 introduces the single-parameter Gini social evaluation family and reviews its main properties according the traditional AKS [2][13][17] decomposition. In Section 4 we work out the dual decomposition of these particular social evaluation functions and we establish the main properties of the two contributing factors, and Section 5 concludes.

2. The dual decomposition of an OWA operator

We assume throughout that variables are drawn from an interval $[0, x^*]$ which is a compact subset of $\boldsymbol{R}$. Points in $[0, x^*]^n$ will be denoted by means of boldface characters: $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. For $x \in [0, x^*]$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$, by $\mathbf{x} \geq \mathbf{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$; by $\mathbf{x} > \mathbf{y}$ we mean $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$. Given $\mathbf{x} \in [0, x^*]^n$, with $(x_{(1)}, \ldots, x_{(n)})$ we denote the increasing ordered version of $\mathbf{x}$, i.e., $x_{(i)}$ is the i-th lowest number of $\{x_1, \ldots, x_n\}$. Moreover, $x_{(1)} = \min\{x_1, \ldots, x_n\}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\}$. Given a permutation on $\{1, \ldots, n\}$, i.e., a bijection $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, with $\mathbf{x}_\sigma$ we denote $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We begin by defining standard properties of real functions on $[0, x^*]^n$.[1]

Definition 1 *Let* $A : [0, x^*]^n \longrightarrow \boldsymbol{R}$ *be a function.*

1. *A is* idempotent *if for every* $x \in [0, x^*]$*:*

$$A(x \cdot \mathbf{1}) = x.$$

2. *A is* symmetric *if for every permutation* σ *on* $\{1, \ldots, n\}$ *and every* $\mathbf{x} \in [0, x^*]^n$*:*

$$A(\mathbf{x}_\sigma) = A(\mathbf{x}).$$

[1] For further details the interested reader is referred to Fodor and Roubens [7], Calvo *et al.* [4], Beliakov *et al.* [3], García-Lapresta and Marques Pereira [8] and Grabisch *et al.* [10].

3. A *is* monotonic *if for all* $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$*:*

$$\mathbf{x} \geq \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) \geq A(\mathbf{y}).$$

4. A *is* strictly monotonic *if for all* $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$*:*

$$\mathbf{x} > \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) > A(\mathbf{y}).$$

5. A *is* compensative *if for every* $\mathbf{x} \in [0, x^*]^n$*:*

$$x_{(1)} \leq A(\mathbf{x}) \leq x_{(n)}.$$

6. A *is* self-dual *if for every* $\mathbf{x} \in [0, x^*]^n$*:*

$$A(\, x^* \cdot \mathbf{1} - \mathbf{x}) = x^* - A(\mathbf{x}).$$

7. A *is* anti-self-dual *if for every* $\mathbf{x} \in [0, x^*]^n$*:*

$$A(\, x^* \cdot \mathbf{1} - \mathbf{x}) = A(\mathbf{x}).$$

8. A *is* invariant for translations *if for all* $t \in \boldsymbol{R}$ *and* $\mathbf{x} \in [0, x^*]^n$*:*

$$A(\mathbf{x} + t \cdot \mathbf{1}) = A(\mathbf{x})$$

whenever $\mathbf{x} + t \cdot \mathbf{1} \in [0, x^*]^n$.

9. A *is* stable for translations *if for all* $t \in \boldsymbol{R}$ *and* $\mathbf{x} \in [0, x^*]^n$*:*

$$A(\mathbf{x} + t \cdot \mathbf{1}) = A(\mathbf{x}) + t$$

whenever $\mathbf{x} + t \cdot \mathbf{1} \in [0, x^*]^n$.

Definition 2 *Consider the binary relation* $\succcurlyeq$ *on* $[0, \infty)^n$ *defined as*

$$\mathbf{x} \succcurlyeq \mathbf{y} \;\Leftrightarrow\; \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \;\; \text{and}$$
$$\sum_{i=1}^{k} x_{(i)} \leq \sum_{i=1}^{k} y_{(i)} \;\; \text{for every} \;\; k \in \{1, \ldots, n-1\}$$

The relations $\sim$ *and* $\succ$ *are derived from* $\succeq$ *in the usual way.*

1. A *is* strictly S-convex *if for all* $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$*:*

$$\mathbf{x} \succ \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) > A(\mathbf{y}).$$

2. A *is* strictly S-concave *if for all* $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$*:*

$$\mathbf{x} \succ \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) < A(\mathbf{y}).$$

3. A *is* S-convex *if for all* $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$*:*

$$\mathbf{x} \succcurlyeq \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) \geq A(\mathbf{y}).$$

4. A *is* S-concave *if for all* $\mathbf{x}, \mathbf{y} \in [0, x^*]^n$*:*

$$\mathbf{x} \succcurlyeq \mathbf{y} \;\Rightarrow\; A(\mathbf{x}) \leq A(\mathbf{y}).$$

In this paper we will use also the following definition

Definition 3 *A function* $A : [0, x^*]^n \longrightarrow [0, x^*]$ *is called a* n-ary aggregation function *in* $[0, x^*]^n$ *if it is monotonic and satisfies* $A(\mathbf{0}) = 0$ *and* $A(x^* \cdot \mathbf{1}) = x^*$*. An n-ary aggregation function is said to be* strict *if it is strictly monotonic.*

For the sake of simplicity, the n-arity is omitted whenever it is clear from the context.

The following definition will play an important role in our paper.

Definition 4 *Let* $A : [0, x^*]^n \longrightarrow [0, x^*]$ *be an aggregation function. The aggregation function* $A^* : [0, x^*]^n \longrightarrow [0, x^*]$ *defined as*

$$A^*(\mathbf{x}) = x^* - A(\, x^* \cdot \mathbf{1} - \mathbf{x})$$

is called the dual *of the aggregation function* A*.*

Clearly, an aggregation function A is self-dual if and only if $A^* = A$.

By taking into account García-Lapresta and Marques Pereira [8], and García-Lapresta et al. [9], the following result is straightforward.

Proposition 1 *Let* $A : [0, x^*]^n \longrightarrow [0, x^*]$ *be an aggregation function. The dual* A^* *inherits from the aggregation function* A *the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, self-duality, and stability for translations, whenever* A *has these properties. In addition,* A^* *is* S*-convex (resp.* S*-concave) whenever* A *is* S*-concave (resp.* S*-convex).*

We follow the proposal of the dual decomposition of an aggregation function into its self-dual core and associated anti-self-dual remainder, due to García-Lapresta and Marques Pereira [8], in order to propose a similar decomposition for an aggregation function in our context. For this, first we need a previous definition of the so-called self-dual core and of the anti-self-dual remainder of an aggregation function A.

Definition 5 *Let* $A : [0, x^*]^n \longrightarrow [0, x^*]$ *be an aggregation function. The function* $\widehat{A} : [0, x^*]^n \longrightarrow [0, x^*]$ *defined as*

$$\widehat{A}(\mathbf{x}) = \frac{A(\mathbf{x}) + A^*(\mathbf{x})}{2} = \frac{A(\mathbf{x}) - A(\, x^* \cdot \mathbf{1} - \mathbf{x}) + x^*}{2}$$

is called the core *of the aggregation function* A*.*

Definition 6 *Let* $A : [0, x^*]^n \longrightarrow [0, x^*]$ *be an aggregation function. The function* $\widetilde{A} : [0, x^*]^n \longrightarrow \boldsymbol{R}$ *defined as* $\widetilde{A}(\mathbf{x}) = A(\mathbf{x}) - \widehat{A}(\mathbf{x})$*, that is*

$$\widetilde{A}(\mathbf{x}) = \frac{A(\mathbf{x}) - A^*(\mathbf{x})}{2} = \frac{A(\mathbf{x}) + A(\, x^* \cdot \mathbf{1} - \mathbf{x}) - x^*}{2},$$

is called the remainder *of the aggregation function* A*.*

From these definitions, clearly, every aggregation function A decomposes additively $A = \widehat{A} + \widetilde{A}$ into two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation function. Notice that the anti-self-dual remainder of an aggregation function A is not an aggregation function. Clearly $\widetilde{A}(\mathbf{0}) = \widetilde{A}(x^* \cdot \mathbf{1}) = 0$,which violates the boundary conditions and implies that $\widetilde{A}$ is either non monotonic or everywhere null.

The following two propositions state the properties inherited respectively by the self-dual core and the anti-self-dual remainder of an aggregation function A. The results are established in García-Lapresta and Marques Pereira [8], and García-Lapresta et al. [9].

Proposition 2 *The self-dual core $\widehat{A}$ is a self-dual aggregation function which inherits from the aggregation function A the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, and stability for translations, whenever A has these properties.*

Proposition 3 *The anti-self-dual remainder $\widetilde{A}$ is anti-self-dual and inherits from the aggregation function A the properties of continuity, symmetry, plus also S-convexity and S-concavity, whenever A has these properties.*

The next proposition is related to two more properties of the anti-self-dual remainder based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core (see García-Lapresta and Marques Pereira[8]).

Proposition 4 *Let $A : [0, x^*]^n \longrightarrow [0, x^*]$ be an aggregation function.*

1. *If A is idempotent, then $\widetilde{A}(x \cdot \mathbf{1}) = 0$ for every $x \in [0, x^*]$.*
2. *If A is stable for translations, then $\widetilde{A}$ is invariant for translations.*

We now turn to examine the self-dual decomposition of an important class of continuous aggregation operators, the OWA operators introduced by Yager [18]. For this class of aggregation operators, the aggregated value is obtained as a weighted average of the ordered $\mathbf{x}$ coordinate values.

Definition 7 *Given a weighting vector $\mathbf{w} = (w_1, ..., w_n) \in [0,1]^n$ satisfying $\sum_{i=1}^n w_i = 1$,* the OWA operator associated with $\mathbf{w}$ *is the aggregation function $A_w : [0, x^*]^n \to [0, x^*]$ defined as*

$$A_w(\mathbf{x}) = \sum_{i=1}^n w_i x_{\sigma(i)}$$

where σ is a permutation of $\{1, \ldots, n\}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$.

Proposition 5 *For every $\mathbf{w} = (w_1, ..., w_n) \in [0,1]^n$ satisfying $\sum_{i=1}^n w_i = 1$, , the OWA operator A_w is idempotent (hence, compensative), symmetric, monotonic, and stable for translations. In addition, A_w is S-convex whenever $w_1 \leq \cdots \leq w_n$ and S-concave whenever $w_1 \geq \cdots \geq w_n$.*

Proposition 6 *For every $\mathbf{w} = (w_1, ..., w_n) \in [0,1]^n$ satisfying $\sum_{i=1}^n w_i = 1$, , the dual of the OWA operator A_w is the aggregation function A_w^* given by*

$$A_w^*(\mathbf{x}) = \sum_{i=1}^n w_{n-i+1} x_{\sigma(i)}$$

Notice that A_w^* is also an OWA operator fulfilling $A_w^*(\mathbf{x}) = A_{w*}(\mathbf{x})$ where $w_i^* = w_{n-i+1}$.

Proposition 7 *For every $\mathbf{w} = (w_1, ..., w_n) \in [0,1]^n$ satisfying $\sum_{i=1}^n w_i = 1$, , the dual A_w^* is idempotent (hence, compensative), symmetric, monotonic, and stable for translations. In addition, A_w^* is S-concave whenever $w_1 \leq \cdots \leq w_n$ and S-convex whenever $w_1 \geq \cdots \geq w_n$.*

García-Lapresta and Marques Pereira [8] apply the dual decomposition to these type of aggregation operators, and analyze some properties inherited by the self-dual core $\widehat{A}_w$ and the anti-self-dual remainder $\widetilde{A}_w$. By following García-Lapresta et al. [9] we have that $\widetilde{A}_w$, inherits S-convexity (or respectively S-concavity) from A_w,whenever A_w has this property. Since S-convexity (or respectively S-concavity) has to do with a natural property for an inequality (or respectively equality) measure, we will show in the next section that, for a particular class of OWA operators, $\widetilde{A}_w$ can be considered as a form of equality measure.

3. The single-parameter Gini social evaluation functions

An income distribution for a population consisting of n identical individuals ($n \geq 2$) is a list $\mathbf{x} = (x_1, ..., x_n)$, where x_i is the income of individual i. We assume throughout that incomes are drawn from an subset D of $\boldsymbol{R}$. Let $\mu(\mathbf{x})$ be the mean of $\mathbf{x} \in \mathbf{D}^n \subseteq \boldsymbol{R}^n$, that is $\mu(\mathbf{x}) = \sum_{i=1}^n x_i/n$.

The object of welfare comparisons between two such distributions is to be able to say that one attains more or less social welfare than the other. More specifically, we wish to define a social evaluation function $W_n : \mathbf{D}^n \to \boldsymbol{R}$ which associates to every distribution a real number $W_n(\mathbf{x})$ that represents the social welfare attained in income distribution $\mathbf{x} \in \mathbf{D}^n$. When $W_n(\mathbf{x}) \geq W_n(\mathbf{y})$, then we will say that distribution $\mathbf{x}$ is at least as good as distribution $\mathbf{y}$.

To evaluate social welfare, obviously, we have to take into consideration not only the level of income but also the inequality in the income distribution. Because inequality and the income level enter as separate arguments into judgments of social well-being, it is reasonable for a welfare function to be decomposable into both arguments.

For this, it would be helpful to define an inequality index $I_n : \mathbf{D}^n \to \boldsymbol{R}$ which associates to every distribution a real number $I_n(\mathbf{x})$ that represents the inequality in the income distribution $\mathbf{x} \in \mathbf{D}^n$. When $I_n(\mathbf{x}) \geq I_n(\mathbf{y})$, then we will say that distribution $\mathbf{x}$ is at least as unequal as distribution $\mathbf{y}$.

Some properties will be proposed for these functions.

A simple and intuitive justification for a particular property in this field is that it is reasonable to regard as more equal, and hence more socially desirable, a distribution which can be obtained from another by a richer person giving a part of his income to a poorer one, without changing the income position of individuals in the society. Such a type of transfer is referred to as a rank preserving progressive transfer.

Hardy et al.[11] prove that the necessary and sufficient condition of $\mathbf{x} \prec \mathbf{y}$, according to definition 2, is that income distribution $\mathbf{x}$ can be derived from income distribution $\mathbf{y}$ by a sequence of progressive transfers which are rank preserving (see also Fields and Fei, [6]). Consequently and according to Definition 2, it would be reasonable for a social evaluation function W_n to be strictly S-concave, that is, W_n increases under a rank preserving progressive transfer. This property is known as the transfer principle. Whenever S-concavity is assumed instead of strictly S-concavity, W_n satisfies symmetry along with a weaker version of the transfer principle. This version states that the level of social evaluation function does not decrease under a rank preserving progressive transfer. In turn, for an inequality measure I_n it would be reasonable to be strictly S-convex or respectively S-convex. In this paper we will assume the weaker version of the transfer principle.

The following two definitions formally state when a function is considered as a measure of social evaluation and respectively of inequality.

Definition 8 *A function* $W_n : \mathbf{D}^n \longrightarrow \boldsymbol{R}$ *is called a* social evaluation function *if it is monotonic and S-concave. Moreover,* W_n *is called* unit-translatable *if is stable for translations.*

Definition 9 *A function* $I_n : \mathbf{D}^n \longrightarrow \boldsymbol{R}$ *is called an* inequality measure *if it is S-convex and satisfies* $I_n(x \cdot \mathbf{1}) = c$ *for every* $x \in \mathbf{D}$ *and a constant* $c \in \boldsymbol{R}$ *Moreover,* I_n *is said to be* absolute *if it is invariant for translations.*

Because equality in the distribution is directly related to inequality, sometimes instead of an inequality index, it would seem reasonable to define an equality index $E_n : \mathbf{D}^n \to \boldsymbol{R}$, which associates to every distribution a real number $E(\mathbf{x})$ that represents the equality in the income distribution $\mathbf{x} \in \mathbf{D}^n$. Obviously, the properties that characterized an equality index are the same as those in Definition 9 but for S-convexity, which in this case would be S-concavity. Formally,

Definition 10 *A function* $E_n : \mathbf{D}^n \longrightarrow \boldsymbol{R}$ *is called an* equality measure *if it is S-concave and satisfies* $E_n(x \cdot \mathbf{1}) = c$ *for every* $x \in \mathbf{D}$ *and a constant* $c \in \boldsymbol{R}$ *Moreover,* E_n *is said to be* absolute *if it is invariant for translations.*

An index of inequality is called ethical if it implies, and is implied by, a social evaluation function. Particularly, for each family of unit-translatable social evaluation functions we may consider the corresponding family of absolute inequality measures. Following the AKS [2][13][17] approach, the absolute index of inequality for a social evaluation W_n, is given by

$$I_n(\mathbf{x}) = \mu(\mathbf{x}) - W_n(\mathbf{x}). \tag{1}$$

If we define the corresponding equality measure as $E_n(\mathbf{x}) = -I_n(\mathbf{x})$, it holds that $W_n(\mathbf{x}) = \mu(\mathbf{x}) + E_n(\mathbf{x})$

In this section, we will focus on the family of the single-parameter Gini social evaluation functions, the S-Gini family, (see Donaldson and Weymark [5], and Yitzhaki[19]), defined for every $\mathbf{x} \in \mathbf{D}^n$ as

$$W_\theta(\mathbf{x}) = \sum_{i=1}^{n} \left(\left(\frac{n-i+1}{n} \right)^\theta - \left(\frac{n-i}{n} \right)^\theta \right) x_{\sigma(i)} \tag{2}$$

with $\theta \geq 1$, where σ is a permutation of $\{1, \ldots, n\}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$.[2]

This social evaluation function on $\mathbf{D}^n$, W_θ, treats individuals symmetrically. More precisely, if σ is any permutation of $\{1, \ldots, n\}$ such that $x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)}$ then income distribution $\mathbf{y}$ that results from $\mathbf{x}$, under this permutation, has the same level of social welfare.

Let Y_n be the correponding rank-ordered income distributions subset of $\mathbf{D}^n$. Suppose that W_θ is first defined on Y_n. Then the assumption that W_θ treats individuals symmetrically, allows us to extend it uniquely to the entire subset $\mathbf{D}^n$. For convenience, we shall restrict our attention to the subset Y_n.

In the traditional decomposition of this family, and according to Eq. (1), the corresponding absolute index of inequality is given by $I_\theta(\mathbf{x}) = \mu(\mathbf{x}) - W_\theta(\mathbf{x})$.

[2] In Donaldson and Weymark [5] $\mathbf{D}^n$ coincides with $\boldsymbol{R}^n$ and incomes aredecreasingly ordered, that is $W_\theta(\mathbf{x}) = \sum_{i=1}^{n} \left(\left(\frac{i+1}{n} \right)^\theta - \left(\frac{i}{n} \right)^\theta \right) x_{\sigma'(i)}$, where σ' is a permutation of $\{1, \ldots, n\}$ such that $x_{\sigma'(1)} \geq \cdots \geq x_{\sigma'(n)}$. They prove that the S-Gini functions are unchanged when the population is replicated, income by income. This property allows us to rank all the income distributions, independently of the size of the population. Hence, it can be considered that the domain of every W_θ in Eq. (2) is $Y = \bigcup_{n \geq 2} [0, x^*]^n$, that is, the set of all income distributions for every population of size $n \geq 2$.

If $\theta = 1$, W_1 is the utilitarian rule with $W_1(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i = \mu(\mathbf{x})$. The corresponding absolute index of inequality is $I_1(\mathbf{x}) = 0$. If $\theta = 2$, we have the Gini social evaluation function, W_2, and its corresponding absolute inequality index is given by $I_2(\mathbf{x}) = \mu(\mathbf{x}) - W_2(\mathbf{x})$. If $\theta = \infty$, $W_\infty(\mathbf{x}) = \lim_{\theta\to\infty} W_\theta(\mathbf{x}) = x_1 = \min_i \{x_i\}$, the maximin rule. The corresponding absolute index of inequality is $I_\infty(\mathbf{x}) = \mu(\mathbf{x}) - \min_i \{x_i\}$.

Hence, it can be observed that the distributional sensitivity of a social evaluation function W_θ increases as θ increases from 1 to plus infinity. For every $\theta \geq 1$, W_θ is S-concave and therefore satisfies the transfers principle. As mentioned before, this principle states that any rank preserving progressive transfer increases social welfare (and decreases inequality). The next question we can ask is whether the size of this positive impact depends on the location where this transfer takes place. If the answer is yes, it may be the case that the lower this transfer is applied, the better it is. This is the idea behind positional transfer sensitivity principle, introduced and analyzed by Mehran [15] and Kakwani [12][3]. Formally, this principle is depicted in the next definition.

For a function $f : Y_n \longrightarrow R$ and $\mathbf{x} \in Y_n$, let $\Delta f_t(\delta,\rho,\mathbf{x})$ represents the change in f resulting from a transfer δ from the individual $t+\rho$ to the individual t which leaves all individual-ranks in the distribution of $\mathbf{x}$ unchanged.

Definition 11 *A social evaluation function $W : Y_n \longrightarrow \boldsymbol{R}$ (respectively and inequality function I) satisfies the positional transfer sensitivity principle, if for any $\mathbf{x} \in Y_n$, $\delta,\rho > 0$, and any pair of individuals i,j such that $i < j$, $\Delta W_i(\delta,\rho,\mathbf{x}) \geq \Delta W_j(\delta,\rho,\mathbf{x})$ ($\Delta I_i(\delta,\rho,\mathbf{x}) \leq \Delta I_j(\delta,\rho,\mathbf{x})$).*

In Mehran [15] it can be seen that the positional transfer sensitivity principle is satisfied by the S-Gini family if and only if $\theta \geq 2$.

4. The dual decomposition of the single-parameter Gini social evaluation functions

In this section we assume that incomes are drawn from a compact subset $[0,x^*]$ of $\boldsymbol{R}$, but we can also apply this analysis to any bounded variable such as literacy, health status or nutritional intake. Notice that in this context the set $\mathbf{D}^n$ defined in the above section coincides with $[0,x^*]^n$ and Y_n is the corresponding set of increasingly rank-ordered incomes distributions. Moreover, from Definition 7, any social evaluation function W_θ in Eq.(2) is an OWA operator.

We can apply the dual decomposition of an OWA operator analyzed in García-Lapresta and Marques Pereira[8] to every single-parameter Gini social evaluation function W_θ in Eq.(2).[4]

Denoting

$$
\begin{aligned}
w_i^\theta &= \left(\frac{n-i+1}{n}\right)^\theta - \left(\frac{n-i}{n}\right)^\theta, \\
\widehat{w}_i^\theta &= \frac{w_i^\theta + w_{n-i+1}^\theta}{2} \text{ and } \widetilde{w}_i^\theta = \frac{w_i^\theta - w_{n-i+1}^\theta}{2}
\end{aligned}
$$

we obtain that the corresponding self-dual core $\widehat{W}_\theta$ and the anti-self-dual remainder $\widetilde{W}_\theta$ for every $\mathbf{x} \in Y_n$, can be respectively written as

$$\widehat{W}_\theta(\mathbf{x}) = \sum_{i=1}^{n} \widehat{w}_i^\theta x_i \text{ and } \widetilde{W}_\theta(\mathbf{x}) = \sum_{i=1}^{n} \widetilde{w}_i^\theta x_i. \quad (3)$$

Notice that $\widehat{W}_\theta$ is an OWA operator fulfilling that $\widehat{W}_\theta = A_{\widehat{w}}$ for every $i \in \{1,\ldots,n\}$. However, the anti-self-dual remainder $\widetilde{W}_\theta$ verifies that $\widetilde{W}_\theta(\mathbf{0}) = \widetilde{W}_\theta(x^*\mathbf{1}) = 0$ which implies that $\widetilde{W}_\theta$ is not an aggregation operator.

The following two propositions state the properties inherited respectively by self-dual core $\widehat{W}_\theta$ and anti-self-dual remainder $\widetilde{W}_\theta$, and they will allow us to interpret both functions as two different contributing factors of the social evaluation function of a particular society.

Proposition 8 *For every single-parameter Gini social evaluation function W_θ defined as in Eq.(2) with $\theta \geq 1$, the self-dual core of W_θ, $\widehat{W}_\theta$, is idempotent (hence, compensative), symmetric, monotonic, and stable for translations.*

Hence, for every $\mathbf{x} \in Y_n$, self-dual core $\widehat{W}_\theta(\mathbf{x})$ depends of the overall average of the coordinates of $\mathbf{x}$ and it is independent of the specific distribution of these coordinates, it can be considered as a positional measure of the income distribution $\mathbf{x}$.

Proposition 9 *For every single-parameter Gini social evaluation function W_θ defined as in Eq.(2) with $\theta \geq 1$, the anti-self-dual remainder of W_θ, $\widetilde{W}_\theta$, is invariant for translations. Moreover,*

1. *$\widetilde{W}_\theta(x\cdot\mathbf{1}) = 0$ for every $x \in [0,x^*]$,*
2. *$\widetilde{W}_\theta(\mathbf{x}) \leq 0$ for every $\mathbf{x} \in Y_n$ and $\widetilde{W}_\theta$ is S-concave.*

Therefore, in this case, from Definition 10, we have that anti-self-dual remainder $\widetilde{W}_\theta$ can be considered as an equality measure. Moreover, in this case, equality is measured from an absolute point of view and remains invariant if the incomes of all individuals are increased by the same amount.

Hence, we have that W_θ can be decomposed as $W(\mathbf{x}) = \widehat{W}_\theta(\mathbf{x}) + \widetilde{W}_\theta(\mathbf{x})$, where $\widehat{W}_\theta$ is a positional measure and $\widetilde{W}_\theta$ is an equality measure. It

[3]Positional transfer sensitivity property is the positional version of the diminishing transfer sensitivity property in Kolm [14].

[4]Aristondo et al. [1] do a similar exercise for three particular welfare functions.

would be interesting to know something more about both contributing factors.

First of all, we have to say that, if we consider $\theta = 1$, the dual decomposition coincides with the traditional one. It is straightforward to see that $\widehat{W_1}(\mathbf{x}) = \mu(\mathbf{x})$ and $\widetilde{W_1}(\mathbf{x}) = E_1(\mathbf{x}) = -I_1(\mathbf{x}) = 0$.

A similar result is obtained when $\theta = 2$. In this case, we have that W_2 coincides with the Gini social evaluation function. Aristondo et al. [1] prove that for this function both the traditional decomposition and that obtained by using the dual decomposition coincide. That is $\widehat{W_2}(\mathbf{x}) = \mu(\mathbf{x})$, and $\widetilde{W_2}(\mathbf{x}) = E_2(\mathbf{x}) = -I_2(\mathbf{x})$. Moreover, in the limiting case, when $\theta \to \infty$, we have that

$$\begin{aligned}\widehat{W_\infty}(\mathbf{x}) &= \lim_{\theta\to\infty} \widehat{W_\theta}(\mathbf{x}) = \frac{1}{2}(x_1 + x_n) = \\ &\frac{1}{2}\left(\min_i \{x_i\} + \max_i \{x_i\}\right)\end{aligned}$$

and

$$\begin{aligned}\widetilde{W_\infty}(\mathbf{x}) &= \lim_{\theta\to\infty} \widetilde{W_\theta}(\mathbf{x}) = \frac{1}{2}(x_1 - x_n) = \\ &\frac{1}{2}\left(\min_i \{x_i\} - \max_i \{x_i\}\right)\end{aligned}$$

when the weights of the incomes tend to zero except for the individuals at the extremes of the tails of the distribution. Therefore, the measure only considers transfers either from the richest individual, or to the poorest one.

In the following we will show that the remainder of any single-parameter Gini social evaluation function satisfies two principles related to a particular perception of inequality sensitivity. From the definitions of these principles, it holds that the remainder is an equallyty measure with balanced sensitivity to both tails.

Since the concern with inequality stems from the injustice of extremely low incomes, it seems appropriate to choose social evaluation functions sensitive to what happens to the poorest. However, the intuitive appeal of the positional diminishing transfer sensitivity principle can be questioned when we consider only transfers between "rich" people. It is easy to imagine people arguing that an equalizing transfer between persons who are both "rich" (in an absolute sense) can be more inequality reducing the higher up it occurs in the distribution. Therefore for transfers between "rich" people, it would seem right to ask just the opposite of that required by the positional diminishing transfer sensitivity principle. Consequently, Puerta and Urrutia [16], considering two income classes, the "poor" and the "rich" people, below and above the median, introduce a new principle, the up-down positional transfer sensitivity principle which depicts this idea. For this, the population is split into two groups according to the median. Whenever n is even, the population is taken as $\left\{1, ..., \frac{n}{2}\right\} \cup \left\{\frac{n}{2}+1, ..., n\right\}$ and whenever n is odd as $\left\{1, ..., \left\lfloor\frac{n}{2}\right\rfloor\right\} \cup \left\{\left\lfloor\frac{n}{2}\right\rfloor + 1, ..., n\right\}$, where $\left\lfloor\frac{n}{2}\right\rfloor$ is the largest integer not greater than $\frac{n}{2}$. For the sake of simplicity, furthermore, we will use the former classification, although the results also apply to the latter one.

Definition 12 *A social evaluation function $W : Y_n \longrightarrow \boldsymbol{R}$ (respectively and inequality function I) satisfies the up-down positional transfer sensitivity principle, if for any $\mathbf{x} \in Y_n$, $\delta, \rho > 0$, and any pair of individuals i, j such that $i < j < \frac{n}{2} + \rho$, $\Delta W_i(\delta, \rho, \mathbf{x}) \geq \Delta W_j(\delta, \rho, \mathbf{x})$, $(\Delta I_i(\delta, \rho, \mathbf{x}) \leq \Delta I_j(\delta, \rho, \mathbf{x}))$ and $\Delta W_{n-(i+\rho)+1}(\delta, \rho, \mathbf{x}) \geq \Delta W_{n-(j+\rho)+1}(\delta, \rho, \mathbf{x})$, $(\Delta I_{n-(i+\rho)+1}(\delta, \rho, \mathbf{x}) \leq \Delta I_{n-(j+\rho)+1}(\delta, \rho, \mathbf{x}))$.*

The following result states a condition that guarantees this principle to be fulfilled by the remainder of every member of the S-Gini family of social evaluation functions.

Proposition 10 *For every single-parameter Gini social evaluation function W_θ defined as in Eq.(2) with $\theta \geq 3$, the anti-self-dual remainder of W_θ, $\widetilde{W_\theta}$, satisfies the up-down positional transfer sensitivity principle.*

The next proposition shows that the anti-self-dual remainder of every single-parameter Gini social evaluation function with $\theta \geq 3$ verifies a second property related to the inequality sensitivity. It states that two progressive transfers, one below the median and another above it, have the same equalizing effect whenever the individuals involved in both transfers are at the same positional distance to the median position.

Definition 13 *A function $f : Y_n \longrightarrow R$ satisfies the symmetric positional transfer sensitivity principle, if for any $\mathbf{x} \in Y_n$, $\delta, \rho > 0$, and any pair of individuals $i + \rho \leq \frac{n}{2}$ and $n - (i+\rho) + 1 \geq \frac{n}{2} + 1$, $\Delta f_i(\delta, \rho, \mathbf{x}) = \Delta f_{n-(i+\rho)+1}(\delta, \rho, \mathbf{x})$.*

Notice that in this definition, all individuals involved in the transfers are by pairs, at the same positional distance to the median positions. That is, $\left|i + \rho - \frac{n}{2}\right| = \left|(n - (i+\rho) + 1) - \left(\frac{n}{2} + 1\right)\right|$ and $\left|i - \frac{n}{2}\right| = \left|(n - i + 1) - \left(\frac{n}{2} + 1\right)\right|$.

Proposition 11 *For every single-parameter Gini social evaluation function W_θ defined as in Eq.(2) with $\theta \geq 3$, the anti-self-dual remainder of W_θ, $\widetilde{W_\theta}$, satisfies the symmetric positional transfer sensitivity principle.*

5. Conclusions

Every single parameter social evaluation function can be decomposed into the mean income and an absolute equality measure, by following the AKS [2][13][17] approach. By following the dual decomposition of an OWA operator in García-Lapresta and Marques Pereira[8] and working in a different compact domain, we propose the dual decomposition for every single parameter social evaluation function, in order to obtain two contributing factors; the self-dual core, what we prove is a positional measure, and the anti-self-dual remainder, what we prove can be considered as an equality measure.

Whenever a single parameter social evaluation function satisfies the k degree positional diminishing transfer principle, the equality measure obtained in the traditional decomposition by following the AKS [2][13][17] approach satisfies the same principle. However, in this paper it is proved that the equality measure obtained in the dual decomposition satisfies a principle with more balanced sensitivity to both tails of the distribution, the $k-1$ degree up-down positional transfer principle. Moreover, this equality measure satisfies another new principle, the symmetric positional transfer sensitivity principle.

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