'Only comparable' $\mathcal{T}$-transitive property and its closures for $\mathcal{IVFR}s$

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Abstract

In this paper a weaker kind of transitive property for interval-valued fuzzy relations ($\mathcal{IVFR}s$) is introduced. It is called 'only comparable' T-transitivity because it relaxes the need that all intervals must be comparable, by just the need of having T-transitive cycles only for comparable intervals.

This paper also defines a weak concept of closure, and it is proved that it exists just one $\mathcal{T}$-transitive and weak $\mathcal{T}$-transitive closure, it does not exists an 'only comparable' $\mathcal{T}$-transitive weak closure, but there exist many 'only comparable' $\mathcal{T}$-transitive weak closures of an $\mathcal{IVFR}$.

Keywords: Interval-valued Fuzzy Relations, Interval-valued Fuzzy Sets, T-transitivity, T-transitive closure, P weak closures, weak T-transitivity.

1. Introduction

Since Fuzzy sets, $\mathcal{FS}s$, were introduced by Zadeh in 1965 [19] many generalizations of fuzzy sets have been proposed to model the uncertainty and the vagueness in linguistic variables replacing the unit interval by another structure such as posets or lattices [4, 8, 16]. Interval-valued fuzzy sets ($\mathcal{IVFS}s$) were introduced in the 70s by Grattan-Guinness [12], Jahn [14], Sambuc [17] and Zadeh [20]. They are extensions of classical fuzzy sets where the membership value between 0 and 1 is replaced by an interval in [0,1]. They easily allow to model uncertainty and vagueness because sometimes it is easier for experts to give a "membership interval" than a membership degree to objects on a universe. Interval-valued fuzzy relations ($\mathcal{IVFR}s$) are fuzzy relations which experts express the relation degree between two objects by using interval numbers instead of numeric values. $\mathcal{IVFS}s$ are a special case of type-2 fuzzy sets [20, 21, 22] that simplifies the calculations while preserving their richness as well.

Transitivity is a fundamental notion in decision theory. It is universally assumed in disciplines of decision theory and accepted in a principle of rationality for some kind of relations. A first task for decision science is the resolution of intransitivities when the transitive property is violated [15]. The transitive closure is a usual way to generate a transitive relation from an intransitive relation. The T-transitive closure of fuzzy relations has been studied for $\mathcal{FR}s$ by De Baets and De Meyer [1]. They showed that it always exists and it is unique. Gonzalez-del-Campo and Garmendia proposed an algorithm to compute the transitive closure for an $\mathcal{IVFR}$ under a t-norm $\mathcal{T}$ [11]. However, the transitive property for interval-valued fuzzy relations [10] is a much stronger condition than for fuzzy relations because it needs that all intervals must be comparable in the inequality that defines $\mathcal{T}$-transitivity. Due to the fact that the set of intervals in [0,1] is a lattice, it is possible to relax the "classical" transitivity by satisfying the inequality just when the intervals are comparable. This new property will be called 'only comparable' $\mathcal{T}$-transitivity. In [3] it is possible to see some similar ideas about reflexive, symmetric and transitive relations for intuitionistic fuzzy relations. In [18] other weaker transitive property is defined for $\mathcal{IVFR}s$.

In this paper 'only comparable' $\mathcal{T}$-transitivity is defined and is compared with fuzzy $\mathcal{T}$-transitivity for $\mathcal{FR}s$ and $\mathcal{IVFR}s$. Sometimes imposing fuzzy $\mathcal{T}$-transitivity to $\mathcal{FR}s$ or $\mathcal{IVFR}s$ by computing the $\mathcal{T}$-transitive closure [10] gives in a completely different $\mathcal{IVFR}$, with much more higher interval degrees. So it is important to look for a weaker condition to impose coherence not in contradiction to $\mathcal{T}$-transitivity and resulting in a much closer closure to a given $\mathcal{IVFR}$.

The paper is organized as follow: in Section 3 'only comparable' $\mathcal{T}$-transitivity of an $\mathcal{IVFR}$ is defined. In Section 4 the weak closure of an $\mathcal{IVFR}$ under property $\mathcal{P}$ is defined. In Section 5 the relation between weak closure and closure of an $\mathcal{IVFR}$ under 'only comparable' $\mathcal{T}$-transitivity and $\mathcal{T}$-transitivity is studied. In Section 6 are obtained some particular results for t-representable t-norms. In Section 7 weak closures under 'only comparable' $\mathcal{T}$-transitivity and the closure under $\mathcal{T}$-transitivity for an $\mathcal{IVFR}$ are compared. Finally, in Section 8 an application is shown.

2. Preliminaries

Definition 2.1. *[5] Let $(L, \leq_L)$ be the lattice of intervals in [0,1] that satisfies:*

1. $L = \{[x_1, x_2] \in [0,1]^2$ *with* $x_1 \leq x_2\}$.
2. $[x_1, x_2] \leq_L [y_1, y_2]$ *if and only if* $x_1 \leq y_1$ *and* $x_2 \leq y_2$

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Also by definition:

$[x_1, x_2] <_L [y_1, y_2] \Leftrightarrow x_1 < y_1, x_2 \leq y_2$ or
$x_1 \leq y_1, x_2 < y_2$
$[x_1, x_2] =_L [y_1, y_2] \Leftrightarrow x_1 = y_1, x_2 = y_2$.

$0_L =_L [0, 0]$ and $1_L =_L [1, 1]$ are the smallest and the greatest elements in L respectively.

Definition 2.2. *[5] An interval-valued fuzzy set A on a universe X is a mapping* $A : X \to L$*:*

Definition 2.3. *[5] Let X be a universe and A and B two interval-valued fuzzy sets. The equality between A and B is defined as:* $A =_L B$ *if and only if* $A(a) =_L B(a)\ \forall a \in X$.

Definition 2.4. *[5] Let X be a universe and A and B two interval-valued fuzzy sets. The inclusion of A in to B is defined as:* $A \subseteq_L B$ *if and only if* $A(a) \leq_L B(a)\ \forall a \in X$.

Definition 2.5. *[5] A t-norm* $\mathcal{T}$ *on L is a monotone increasing, symmetric and associative operator,* $\mathcal{T} : L^2 \to L$*, that satisfies:* $\mathcal{T}(1_L, [x_1, x_2]) =_L [x_1, x_2]$ *for all* $[x_1, x_2]$ *in L.*

Definition 2.6. *[5] A t-norm* $\mathcal{T}$ *on L is t-representable in L if there exist two t-norms:* T_1 *and* T_2 *(*T_1, T_2*, in ([0,1],≤)) that satisfy:*

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) =_L [T_1(x_1, y_1), T_2(x_2, y_2)]$$

where $T_1(v, w) \leq T_2(v, w)\ \forall v, w \in [0, 1]$.

Let $x =_L [x_1, x_2]$ and $y =_L [y_1, y_2]$ be two intervals on L:

Example 2.1. $\mathcal{T}([x_1, x_2], [y_1, y_2]) =_L [\min(x_1, y_1), \min(x_2, y_2)]$ *is t-representable in ([0,1],≤). Note that* min *is the highest t-norm.*

Example 2.2. *The following product t-norm* $\mathcal{T}$ *on L is t-representable:*

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) =_L [x_1 * y_1, x_2 * y_2]$$

Example 2.3. *Two generalizations of the Lukasiewicz t-norm [6] are the following:*

- $T_w([x_1, x_2], [y_1, y_2]) =_L [\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1)]$
- $T_W([x_1, x_2], [y_1, y_2]) =_L [\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$

Note that T_w *is t-representable but* T_W *is not t-representable.*

Definition 2.7. *[7] A t-norm operator* $\mathcal{T}$ *on L is pseudo-t-representable if there exists a t-norm T in ([0,1],≤) that satisfies:*

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) =_L [T(x_1, y_1), \max\{T(x_1, y_2), T(x_2, y_1)\}]$$

The t-norm T is called the representant of $\mathcal{T}$.

Example 2.4. *Some examples of pseudo-t-representable t-norms on L are shown:*

T	$\mathcal{T}$
$\min(x, y)$	$[\min(x_1, y_1), \max(\min(x_1, y_2), \min(x_2, y_1))]$
$x * y$	$[x_1 * y_1, \max(x_1 * y_2, x_2 * y_1)]$
$\max(0, x + y - 1)$	$[\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$

Definition 2.8. *[2] Let* X_1 *and* X_2 *be two universes of discourse. An interval-valued fuzzy relation* $R : X_1 \times X_2 \to L$ *is a mapping:*

$$R = \{((a, b), [x, y]) \mid a \in X_1, b \in X_2, [x, y] \in L\}$$

In the rest of the paper $X_1 = X_2$.
Let X be the universe $X = \{e_1, \ldots, e_n\}$.

Definition 2.9. *[10] Let* $\mathcal{T}$ *be a t-norm on L and let R interval-valued fuzzy relation on X. Then, R is* $\mathcal{T}$*-transitive if:*

$$\mathcal{T}(R(a, b), R(b, c)) \leq_L R(a, c)\ \forall a, b, c \in X$$

Definition 2.10. *[9] An interval-valued fuzzy relation* $R : X^2 \to L$ *is a generalized* $\mathcal{T}$*-indistinguishability if it is reflexive, symmetric and* $\mathcal{T}$*-transitive.*

Definition 2.11. *[9] Let P be a property of* $\mathcal{IVFR}$*s. Let* $R : X^2 \to L$ *be an interval-valued fuzzy relation on a finite universe X. The P closure of R is an* $\mathcal{IVFR}$ $R^{\mathcal{P}} : X^2 \to L$ *that satisfies:*

1. $R^{\mathcal{P}}$ *satisfies P.*
2. $R \subseteq_L R^{\mathcal{P}}$.
3. *If* $R \subseteq_L R'$ *and* R' *satisfies P then* $R^{\mathcal{P}} \subseteq_L R'$.

Lemma 2.1. *[9] Let R be an interval-valued fuzzy relation on a universe X and let* $\mathcal{T}$ *be an arbitrary t-norm on L. Then the* $\mathcal{T}$*-transitive closure of R always exists and it is unique.*

Let R be an interval-valued fuzzy relation on $X = \{e_1, \ldots, e_n\}$. For convenience, $R(e_i, e_j)$ can be written $[\underline{R}(e_i, e_j), \overline{R}(e_i, e_j)]$ or $[\underline{R}, \overline{R}]$.

Proposition 2.1. *[9] If* $\mathcal{T}$ *is t-representable with* T_1 *and* T_2 *(*$\mathcal{T} = [T_1, T_2]$*) then an interval-valued relation* $R : X^2 \to L$ *is* $\mathcal{T}$*-transitive if and only if* $\underline{R}$ *is* T_1*-transitive and* $\overline{R}$ *is* T_2*-transitive.*

Theorem 2.1. *[9] Let* $\mathcal{T}$ *be a t-representable t-norm (*$\mathcal{T} = [T_1, T_2]$*) and let* $R = [\underline{R}, \overline{R}]$ *be a interval-valued relation. Then, the* $\mathcal{T}$*-transitive closure interval-valued of R,* $R^{\mathcal{T}}$*, satisfies:*

$$R^{\mathcal{T}} = [\underline{R}^{T_1}, \overline{R}^{T_2}]$$

Definition 2.12. *[13] Let* A_X *the set of interval-valued fuzzy sets on* $X = \{e_1, \ldots, e_n\}$*. The Hamming distance d between M and N (*$M, N \in A_X$*) is defined by:*

$$d(M, N) = \sum \mid \overline{M}(e_i) - \overline{N}(e_i) \mid + \mid \underline{M}(e_i) - \underline{N}(e_i) \mid$$

for all e_i *in X.*

3. Only comparable $\mathcal{T}$-transitivity for interval-valued fuzzy relations

In this section the 'only comparable' $\mathcal{T}$-transitive property is defined. The relation between the 'only comparable' T-transitivity and the T-transitivity property for $\mathcal{FR}s$ and $\mathcal{IVFR}s$ is shown.

Definition 3.1. *Let $[x_1, x_2]$ and $[y_1, y_2]$ be two intervals in $(L, \leq_L)$. Then $[x_1, x_2]$ is not greater than $[y_1, y_2]$ (denoted by $[x_1, x_2] \not>_L [y_1, y_2]$) if it is satisfied: $x_1 \leq y_1$ or $x_2 \leq y_2$.*

Lemma 3.1. *Let $[x_1, x_2]$ and $[y_1, y_2]$ be two intervals in $(L, \leq_L)$ such that $[x_1, x_2] \leq_L [y_1, y_2]$. Then $[x_1, x_2] \not>_L [y_1, y_2]$.*

Proof. Trivial □

Remark. $[x_1, x_2] \not>_L [y_1, y_2]$ does not imply $[x_1, x_2] \leq_L [y_1, y_2]$. Let us see an example. If $[x_1, x_2] = [0.3, 0.4]$ and $[y_1, y_2] = [0.2, 0.5]$ it is verified that $[0.3, 0.4] \not>_L [0.2, 0.5]$ but it is not verified that $[0.3, 0.4] \leq_L [0.2, 0.5]$.

Definition 3.2. *Let $\mathcal{T}$ be a t-norm on L and let R be an interval-valued fuzzy relation on X. R is 'only comparable' $\mathcal{T}$-transitive if:*

$$\mathcal{T}(R(a,b), R(b,c)) \not>_L R(a,c) \text{ for all } a, b, c \text{ in } X$$

In a similar way an 'only comparable' T-transitive property can be defined for $\mathcal{FR}s$ but in this case all relation degrees are comparable, so it is equivalent to T-transitivity property.

Definition 3.3. *Let T be a t-norm on L and let R be a fuzzy relation on X. R is 'only comparable' T-transitive if:*

$$T(R(a,b), R(b,c)) \not> R(a,c) \text{ for all } a, b, c \text{ in } X$$

Lemma 3.2. *Let $R : X^2 \to [0,1]$ be a fuzzy relation. Then R is T-transitive if and only if R is 'only comparable' T-transitive.*

Proof. Trivial due to the fact that $([0,1], \leq)$ is a totally ordered set, so in $([0,1], \leq)$ the boolean operator $\not>$ is equivalent to $\leq$ □

Lemma 3.3. *Let R be an $\mathcal{IVFR}s$. If R is $\mathcal{T}$-transitive then R is 'only comparable' $\mathcal{T}$-transitive.*

Proof. If R is $\mathcal{T}$-transitive then $\mathcal{T}(R(a,b), R(b,c)) \leq_L R(a,c)$ for all a, b, c in X, then by Lemma 3.1 $\mathcal{T}(R(a,b), R(b,c)) \not>_L R(a,c)$ for all a, b, c in X, so R is 'only comparable' $\mathcal{T}$-transitive □

Lemma 3.4. *Let R be an $\mathcal{IVFR}s$. If R is 'only comparable' $\mathcal{T}$-transitive then R may not be $\mathcal{T}$-transitive.*

Proof. Let $X = \{a_1, a_2, a_3\}$ be the universe. Let $R : X^2 \to L$ be the next relation:

$$R = \begin{pmatrix} [1,1] & [0.4, 0.6] & [0.4, 0.6] \\ [0.4, 0.6] & [1,1] & [0.5, 0.5] \\ [0.4, 0.6] & [0.5, 0.5] & [1,1] \end{pmatrix}$$

If $\mathcal{T} = (min, min)$ then R is not (min, min)-transitive because:

$$(min, min)(R(a_2, a_1), R(a_1, a_3)) = [0.4, 0.6] \not\leq_L R(a_2, a_3) = [0.5, 0.5]$$

but R is 'only comparable' (min, min)-transitive because:

$$(min, min)(R(a_i, a_k), R(a_k, a_j)) \not>_L R(a_i, a_j) \text{ for all } a_i, a_j, a_k \in X$$

□

Theorem 3.1. *Let $\mathcal{IVFR}s^T$ be the set of $\mathcal{T}$-transitive $\mathcal{IVFR}s$. Let $\mathcal{IVFR}s^{only\ comparable-T}$ be the set of 'only comparable' $\mathcal{T}$-transitive $\mathcal{IVFR}s$.*
Then:

$$\mathcal{IVFR}s^T \subseteq \mathcal{IVFR}s^{only\ comparable-T}$$

Proof. Trivial due to Lemmas 3.3,3.4 □

4. Weak closures of any property P for interval-valued fuzzy relations

Definition 4.1. *Let A and B be two interval-valued fuzzy sets. A is included in B ($A \subseteq_L B$) if and only if $A(e_i) \leq_L B(e_i)$ for all e_i in $X = \{e_1, \ldots, e_n\}$.*

In order be able to compare closures and weak closures for interval-valued fuzzy relations under a property $\mathcal{P}$ the inclusion between interval-valued fuzzy relations is defined.

Definition 4.2. *Let R and S be two interval-valued fuzzy relations on X. R is included in S $R \subseteq_L S$ if $R(e_i, e_j) \leq_L S(e_i, e_j)$ for all e_i, e_j in $X = \{e_1, \ldots, e_n\}$.*

Corollary 4.1. *Let R and S be two interval-valued fuzzy relations on X. R is not included in S ($R \not\subseteq_L S$) if there exist two elements e_p, e_q in $X = \{e_1, \ldots, e_n\}$ such that $R(e_p, e_q) \not\leq_L S(e_p, e_q)$.*

A weaker definition of P closure of a $\mathcal{IVFR}s$ is now defined relaxing the Axiom 3 of the Definition 2.11.

Definition 4.3. *Let P be a property of $\mathcal{IVFR}s$. Let $R : X^2 \to L$ be an interval-valued fuzzy relation on a finite universe X. The P weak closure of R is a fuzzy relation $R^{\sim\mathcal{P}} : X^2 \to L$ that satisfies:*

1. *$R^{\sim\mathcal{P}}$ satisfies P.*
2. *$R \subseteq_L R^{\sim\mathcal{P}}$.*
3. *It does not exist any R' satisfying P such that $R \subseteq_L R' \subset_L R^{\sim\mathcal{P}}$.*

Note that if R satisfies P, then: $R =_L R^{\sim\mathcal{P}} =_L R^{\mathcal{P}}$.

Lemma 4.1. *Let $R : X^2 \to L$ be an interval-valued fuzzy relation on a finite universe X. If $R^{\mathcal{P}}$ exists then $R^{\sim\mathcal{P}}$ exists.*

Proof. Axiom 3 in Definition 2.11 implies Axiom 3 in the Definition 4.3 □

Lemma 4.2. *Let $R : X^2 \to L$ be an interval-valued fuzzy relation on a finite universe X. If $R^{\mathcal{P}}$ exists, then $R^{\sim\mathcal{P}}$ exists, it is unique and it is verified that:*

$$R^{\sim\mathcal{P}} =_L R^{\mathcal{P}}$$

Proof. It is followed from Definition 2.11 and 4.3:

- By Axiom 3 of Definition 2.11: $R^{\mathcal{P}} \subseteq_L R^{\sim\mathcal{P}}$
- By Axiom 3 of Definition 4.3: $R^{\mathcal{P}} \not\subset_L R^{\sim\mathcal{P}}$

Hence $R^{\mathcal{P}} =_L R^{\sim\mathcal{P}}$ □

5. Closures and weak closures of $\mathcal{T}$-transitivity and 'only comparable' $\mathcal{T}$-transitivity for $\mathcal{IVFR}s$

The following sections study the closures and weak closures of the $\mathcal{T}$-transitive and 'only comparable' $\mathcal{T}$-transitive relations of $\mathcal{IVFR}s$.

5.1. $\mathcal{T}$-transitive closures of $\mathcal{IVFR}s$

As well as the T-transitive closure of a $\mathcal{FR}$ exists [1], and it is unique, also the $\mathcal{T}$-transitive closures of $\mathcal{IVFR}$ always exists and it is unique. The $\mathcal{T}$-transitive closure of $\mathcal{FR}$ have been widely studied. There exist many optimal algorithms to compute it in the literature, specially for the minimum t-norm.

Lemma 5.1. *[10] Let R be an interval-valued fuzzy relation on a universe X and let $\mathcal{T}$ be an arbitrary t-norm on L. Then the $\mathcal{T}$-transitive closure of R, $R^{\mathcal{T}}$ always exists.*

Theorem 5.1. *[10] Let $\mathcal{T}$ be a t-representable t-norm ($\mathcal{T} = [T_1, T_2]$) and let $R = [\underline{R}, \overline{R}]$ be an interval-valued fuzzy relation. Then $R^{\mathcal{T}} = [\underline{R}^{T_1}, \overline{R}^{T_2}]$ where $\underline{R}^{T_1}$ is the T_1-transitive closure of $\underline{R}$ and $\overline{R}^{T_2}$ is the T_2-transitive closure of $\overline{R}$.*

5.2. $\mathcal{T}$-transitive weak closures of $\mathcal{IVFR}s$

As well as for the $\mathcal{T}$-transitive closure of $\mathcal{IVFR}$, the $\mathcal{T}$-transitive weak closure of $\mathcal{IVFR}$ also exists, it is unique, and it is equal to the $\mathcal{T}$-transitive closure of an $\mathcal{IVFR}$.

Lemma 5.2. *Let $R : X^2 \to L$ be an interval-valued fuzzy relation on a finite universe X. The $\mathcal{T}$-transitive weak closure of R exists, it is unique and $R^{\sim\mathcal{T}} = R^{\mathcal{T}}$.*

Proof. Trivial by Lemma 5.1 and Lemma 4.2 □

Notation. An 'only comparable' $\mathcal{T}$-transitive weak closure of R is denoted by $R^{\sim only\ comparable-\mathcal{T}}$.

5.3. 'Only comparable' $\mathcal{T}$-transitive closures of $\mathcal{IVFR}s$

Lemma 5.3. *The 'only comparable' $\mathcal{T}$-transitive closure of R may not exist.*

Proof. Let R be an $\mathcal{IVFR}$. It is necessary to find two non comparable 'only comparable' $\mathcal{T}$-transitive $\mathcal{IVFR}s$ S_1, S_2 such that:

- $R \subseteq_L S_1$,
- $R \subseteq_L S_2$ and
- There does not exist any $\mathcal{IVFR}$ S such that $R \subseteq_L S \subseteq_L S_1$ and $R \subseteq_L S \subseteq_L S_2$

Let $\mathcal{T} = [T_1, T_2]$ be a t-norm on L and let R be an $\mathcal{IVFR}$ such that $\underline{R}^{T_1} \subseteq \overline{R}$. If $S_1 = [\underline{R}^{T_1}, \overline{R}]$ and $S_2 = [\underline{R}, \overline{R}^{T_2}]$, then:

- S_1 and S_2 are not comparable because $[\underline{R}^{T_1}, \overline{R}] \not\subseteq_L [\underline{R}, \overline{R}^{T_2}]$ and $[\underline{R}, \overline{R}^{T_2}] \not\subseteq_L [\underline{R}^{T_1}, \overline{R}]$
- S_1 and S_2 are 'only comparable' $\mathcal{T}$-transitive from Lemma 5.4 and 5.5
- There does not exist any $\mathcal{IVFR}$ S such that $R \subseteq_L S \subseteq_L S_1$ and $R \subseteq_L S \subseteq_L S_2$ from Lemma 5.4 and 5.5

□

Lemma 5.4. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm on L. Let $R : X^2 \to L$ be a non 'only comparable' $\mathcal{T}$-transitive interval-valued fuzzy relation on a finite universe X. There does not exist any 'only comparable' $\mathcal{T}$-transitive relation S such that $S \subseteq_L [\underline{R}^{T_1}, \overline{R}]$ if $\underline{R}^{T_1} \subseteq \overline{R}$.*

Proof. Suppose that there is an 'only comparable' $\mathcal{T}$-transitive relation S such that $[\underline{R}, \overline{R}] \subseteq_L [\underline{S}, \overline{S}] \subset_L [\underline{R}^{T_1}, \overline{R}]$ but then $\overline{R} \subseteq S \subset \overline{R}$ which is not possible □

Lemma 5.5. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm on L. Let $R : X^2 \to L$ be a non 'only comparable' $\mathcal{T}$-transitive interval-valued fuzzy relation on a finite universe X. There does not exist any 'only comparable' $\mathcal{T}$-transitive relation S such that $S \subseteq_L [\underline{R}, \overline{R}^{T_2}]$.*

Proof. Suppose that there exists an 'only comparable' $\mathcal{T}$-transitive relation S such that $[\underline{R}, \overline{R}] \subseteq_L [\underline{S}, \overline{S}] \subset_L [\underline{R}, \overline{R}^{T_2}]$ but then $\underline{R} \subseteq S \subset \underline{R}$ which is not possible □

5.4. 'Only comparable' $\mathcal{T}$-transitive weak closures of $\mathcal{IVFR}s$

Nevertheless, there may exist several 'only comparable' $\mathcal{T}$-transitive weak closures of $\mathcal{IVFR}s$.

Lemma 5.6. *Let R be an interval-valued fuzzy relation on a universe X and let $\mathcal{T}$ be an arbitrary generalized t-norm. Then, they may exist several 'only comparable' $\mathcal{T}$-transitive weak closures of R.*

Proof. Let $\mathcal{T} = [T_1, T_2]$ be a t-norm on L. Let R be an $\mathcal{IVFR}$ such that $\underline{R}^{T_1} \subseteq \overline{R}$. Then $[\underline{R}^{T_1}, \overline{R}]$ and $[\underline{R}, \overline{R}^{T_2}]$ are 'only comparable' $\mathcal{T}$-transitive weak closures of R according to Lemmas 5.4 and 5.5 □

An algorithm to compute the $\mathcal{T}$-transitive closure of any $\mathcal{IVFR}$ for any t-norm $\mathcal{T}$ on L is given in [10].

Example 5.1. *Let $X = \{a_1, a_2, a_3\}$ be a universe and $R : X^2 \to L$ an interval-valued fuzzy relation:*

$$R = \begin{pmatrix} [1,1] & [1,1] & [0,0.9] \\ [1,1] & [1,1] & [0.4,0.6] \\ [0,0.9] & [0.4,0.6] & [0,0] \end{pmatrix}$$

Let $\mathcal{T}$ be the following t-norm on L:

$$\mathcal{T}([x_1,x_2],[y_1,y_2]) = [min(x_1,y_1), min(x_2,y_2)]$$

R is not 'only comparable' $\mathcal{T}$-transitive $\mathcal{T}(R(a_3,a_2), R(a_2,a_3)) = [0.4, 0.6] >_L R(a_3,a_3) = [0,0]$.

Note that there exist several not comparable 'only comparable' $\mathcal{T}$-transitive approximations containing R, for instance:

$$R_1^{\sim only\ comparable-\mathcal{T}} = \begin{pmatrix} [1,1] & [1,1] & [0,0.9] \\ [1,1] & [1,1] & [0.4,0.6] \\ [0,0.9] & [0.4,0.6] & [0,0.9] \end{pmatrix}$$

$$R_2^{\sim only\ comparable-\mathcal{T}} = \begin{pmatrix} [1,1] & [1,1] & [0,0.9] \\ [1,1] & [1,1] & [0.4,0.6] \\ [0,0.9] & [0.4,0.6] & [0.4,0.6] \end{pmatrix}$$

In fact, there exist infinite 'only comparable' $\mathcal{T}$-transitive upper approximations. Let $\{S_k : X^2 \to L\}$ be the set of interval-valued fuzzy relations defined as follows:

$$S_k(a_i,a_j) = \begin{cases} [z_{k_1}, z_{k_2}], & \text{if } a_i = a_3 \wedge a_j = a_3; \\ R(a_i,a_j), & \text{otherwise.} \end{cases}$$

where $[z_{k_1}, z_{k_2}]$ is incomparable with [0,0.9] and [0.4,0.6], i.e. it is false that $[z_{k_1}, z_{k_2}] >_L [0,0.9]$ or $[z_{k_1}, z_{k_2}] <_L [0,0.9]$ (and similar for [0.4,0.6]). Then, it is easy to prove that S_k is 'only comparable' $\mathcal{T}$-transitive for all k. Moreover, there does not exist any 'only comparable' $\mathcal{T}$-transitive interval-valued fuzzy relation S_{min} such that $S_{min} \subseteq_L S_k$ for all k.

Note that all the shown 'only comparable' $\mathcal{T}$-transitive upper approximations are contained in the the $\mathcal{T}$-transitive closure [10] of R:

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [1,1] & [0,0.9] \\ [1,1] & [1,1] & [0.4,0.9] \\ [0,0.9] & [0.4,0.9] & [0.4,0.9] \end{pmatrix}$$

6. Weak $\mathcal{T}$-transitive weak closure for t-representable t-norms

Lemma 6.1. *Let $\mathcal{T}$ be a t-representable t-norm on L such that $\mathcal{T} = [T_1, T_2]$. Let $R : X^2 \to L$ be an interval-valued fuzzy relation on a finite universe X. If $\underline{R}$ is T_1-transitive or $\overline{R}$ is T_2-transitive then R is 'only comparable' $\mathcal{T}$-transitive.*

Proof. If $\underline{R}$ is T_1-transitive it is verified:

$$T_1(\underline{R(a_i,a_k)}, \underline{R(a_k,a_j)}) \leq \underline{R(a_i,a_j)}$$

for all i, j, k.

By Definition 3.1:

$$T_1(\underline{R(a_i,a_k)}, \underline{R(a_k,a_j)}) \leq \underline{R(a_i,a_j)} \text{ or } T_2(\overline{R(a_i,a_k)}, \overline{R(a_k,a_j)}) \leq \overline{R(a_i,a_j)}$$

is equivalent to

$$\mathcal{T}(R(a_i,a_k), R(a_k,a_j)) \not>_L R(a_k,a_j)$$

In a similar way it is possible to show that R is 'only comparable' $\mathcal{T}$-transitive if $\overline{R}$ is T_2-transitive □

Theorem 6.1. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm on L. Let $R : X^2 \to L$ be a non 'only comparable' $\mathcal{T}$-transitive interval-valued fuzzy relation on a finite universe X. Let $R^{\mathcal{T}}_{down}$ be defined as $[\underline{R}^{T_1}, \overline{R}]$. If $\underline{R}^{T_1} \subseteq \overline{R}$ then $R^{\mathcal{T}}_{down}$ is a $R^{\sim only\ comparable-\mathcal{T}}$.*

Proof. Axioms of weak closure under 'only comparable' $\mathcal{T}$-transitivity are satisfied:

- Axiom 1: $R^{\mathcal{T}}_{down} = [\underline{R}^{T_1}, \overline{R}]$ is 'only comparable' $\mathcal{T}$-transitive:

  Trivial because $\underline{R}^{T_1}$ is T_1-transitive by Lemma 6.1.
- Axiom 2: $R \subseteq_L R^{\mathcal{T}}_{down}$: Trivial due to $[\underline{R}, \overline{R}] \subseteq_L [\underline{R}^{T_1}, \overline{R}]$.
- Axiom 3: Trivial by Lemma 5.4

□

Corollary 6.1. *Let $R^{\mathcal{T}}_{down}$ be defined as $[\underline{R}^{T_1}, \overline{R}]$. If $\underline{R}^{T_1} \subseteq \overline{R}$ then $R^{\mathcal{T}}_{down} \subseteq_L R^{\mathcal{T}}$*

Proof. Trivial from Theorem 2.1 □

Theorem 6.2. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm on L. Let $R : X^2 \to L$ be a non 'only comparable' $\mathcal{T}$-transitive relation on a finite universe X. Let $R^{\mathcal{T}}_{up}$ be the interval-valued fuzzy relation defined as $[\underline{R}, \overline{R}^{T_2}]$. Then $R^{\mathcal{T}}_{up}$ is a $R^{\sim only\ comparable-\mathcal{T}}$.*

Proof. Axioms of weak closure under 'only comparable' $\mathcal{T}$-transitivity are satisfied:

- Axiom 1: $R^{\mathcal{T}}_{up} = [\underline{R}, \overline{R}^{T_2}]$ is 'only comparable' $\mathcal{T}$-transitive:

  Trivial due to the fact $\overline{R}^{T_2}$ is T_2-transitive and Lemma 6.1.
- Axiom 2: $R \subseteq_L R^{\mathcal{T}}_{up}$: Trivial due to $[\underline{R}, \overline{R}] \subseteq_L [\underline{R}, \overline{R}^{T_2}]$.
- Axiom 3: Trivial by Lemma 5.5

□

Corollary 6.2. *Let $R^{\mathcal{T}}_{up}$ be defined as $[\underline{R}, \overline{R}^{T_2}]$. Then $R^{\mathcal{T}}_{up} \subseteq_L R^{\mathcal{T}}$*

Proof. Trivial from Theorem 2.1 □

The next section includes some Theorems and can be usefull to generate some 'only comparable' $\mathcal{T}$-transitive weak closures.

7. Comparing $\mathcal{T}$-transitive closures and 'only comparable' $\mathcal{T}$-transitive weak closures of $\mathcal{IVFR}s$

Theorem 7.1. *Let $R : X^2 \to L$ be an interval-valued fuzzy relation on a finite universe X. Then*

$$R^{\sim only\ comparable-\mathcal{T}} \not\supset_L R^{\mathcal{T}}$$

Proof. If $R^{\mathcal{T}}$ is $\mathcal{T}$-transitive then $R^{\mathcal{T}}$ is an 'only comparable' $\mathcal{T}$-transitive relation. It is not possible that $R^{\sim only\ comparable-\mathcal{T}} \supset_L R^{\mathcal{T}}$ due to the Axiom 3 of definition of 'only comparable' $\mathcal{T}$-transitive weak closure of R in Definition 4.3 □

Lemma 7.1. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm on L. Let $R : X^2 \to L$ be a non 'only comparable' $\mathcal{T}$-transitive interval-valued fuzzy relation on a finite universe X. If $R^{\mathcal{T}}_{down}$ exists then it is satisfied:*

$$R^{\mathcal{T}}_{down} \subseteq_L R^{\mathcal{T}}$$

Proof.

$$R^{\mathcal{T}}_{down} =_L [\underline{R}^{T_1}, \overline{R}] \subseteq_L [\underline{R}^{T_1}, \overline{R}^{T_2}] =_L R^{\mathcal{T}}$$

□

Lemma 7.2. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm on L. Let $R : X^2 \to L$ be a non 'only comparable' $\mathcal{T}$-transitive interval-valued fuzzy relation on a finite universe X. Then, it is satisfied:*

$$R^{\mathcal{T}}_{up} \subseteq_L R^{\mathcal{T}}$$

Proof.

$$R^{\mathcal{T}}_{up} =_L [\underline{R}, \overline{R}^{T_2}] \subseteq_L [\underline{R}^{T_1}, \overline{R}^{T_2}] =_L R^{\mathcal{T}}$$

□

Lemma 7.3. *Let R be an $\mathcal{IVFR}$. Let S be an $\mathcal{IVFR}$ defined as $S(a_i, a_j) = [\underline{R(a_i, a_j)}, R'(a_i, a_j)]$ such that $\underline{R(a_i, a_j)} \le R'(a_i, a_j)$. If S is 'only comparable' $\mathcal{T}$-transitive, then S is an 'only comparable' $\mathcal{T}$-transitive weak closure of R.*

Proof. S is an 'only comparable' $\mathcal{T}$-transitive weak closure of R because S is 'only comparable' $\mathcal{T}$-transitive and there does not exist any $\mathcal{IVFR}$ contained in S □

Lemma 7.4. *Let R be an $\mathcal{IVFR}$. There may exist an 'only comparable' $\mathcal{T}$-transitive weak closure of R that is not contained in the $\mathcal{T}$-transitive weak closure of R.*

Proof. A counterexample is provided.

Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm. Let S be an $\mathcal{IVFR}$ defined as $S = [\underline{R}, \overline{R}^{T_2}]$. Let i_0, j_0 be two integers such that $1 \le i_0, j_0 \le n$. Let ϵ be an arbitrary small real number. Let Q be an $\mathcal{IVFR}$ defined as follows:

$$Q(a_i, a_j) = \begin{cases} \overline{R}^{T_2}(a_i, a_j) + \epsilon, & i = i_0, j = j_0; \\ \overline{R}^{T_2}(a_i, a_j), & otherwise. \end{cases}$$

Q is a T_2-transitive relation. By Lemma 7.3 it is proved that $S' = [\underline{R}, Q]$ is an 'only comparable' $\mathcal{T}$-transitive weak closure of R. However, S' and $R^{\mathcal{T}}$ are not comparable □

According to the Theorem 7.1 the 'only comparable' $\mathcal{T}$-transitive weak closure of an interval-valued fuzzy relation can not be greater than its $\mathcal{T}$-transitive closure. Moreover, in many cases the 'only comparable' $\mathcal{T}$-transitive weak closure of an interval-valued fuzzy relation is contained in its $\mathcal{T}$-transitive closure but Lemma 7.4 shows that is not always true.

In order to compute the distance between interval-valued fuzzy relations a measure of distance based on the Hamming distance is defined.

Definition 7.1. *Let R_X the set of interval-valued fuzzy relations on $X = \{e_1, \ldots, e_n\}$. The distance d between R and S ($R, S \in R_X$) is defined by:*

$$d(R, S) = \sum_{\forall i,j} | \overline{R}(e_i, e_j) - \overline{S}(e_i, e_j) |$$
$$+ \sum_{\forall i,j} | \underline{R}(e_i, e_j) - \underline{S}(e_i, e_j) |$$

Proposition 7.1. *The distance defined in Definition 7.1 is a classical measure of distance.*

Proof. Let R,S and Q be interval-valued fuzzy relations. Then:

- $d(R, R) = 0$: trivial.
- $d(R, S) = d(S, R)$: trivial.
- $d(R, S) \le d(R, Q) + d(Q, S)$:

  We denote $R(e_i, e_j)$ by $R_{i,j}$ (and for the rest of interval-valued fuzzy relations) for convenience. Then for all i, j it is satisfied:

  $$| \overline{R}_{i,j} - \overline{S}_{i,j} | \le | \overline{R}_{i,j} - \overline{Q}_{i,j} | + | \overline{Q}_{i,j} - \overline{S}_{i,j} |$$

  and

  $$| \underline{R}_{i,j} - \underline{S}_{i,j} | \le | \underline{R}_{i,j} - \underline{Q}_{i,j} | + | \underline{Q}_{i,j} - \underline{S}_{i,j} |$$

  due to the triangle inequality. Thus $d(R, S) \le d(R, Q) + d(Q, S)$

□

Lemma 7.5. *Let $\mathcal{T} = [T_1, T_2]$ be a t-representable t-norm. For an interval-valued fuzzy relation R the distance between R and $R^{\mathcal{T}}$ is:*

$$d(R, R^{\mathcal{T}}) = \sum_{\forall i,j} \mid \overline{R}^{T_2}(e_i, e_j) - \overline{R}(e_i, e_j) \mid$$

$$+ \sum_{\forall i,j} \mid \underline{R}^{T_1}(e_i, e_j) - \underline{R}(e_i, e_j) \mid$$

Lemma 7.6. *Let $R^{\mathcal{T}}_{down}$ be the 'only comparable' $\mathcal{T}$-transitive weak closure of R given in Theorem 6.1. The distance between R and $R^{\mathcal{T}}_{down}$ is:*

$$d(R, R^{\mathcal{T}}_{down}) = \sum_{\forall i,j} \mid \underline{R}^{T_1}(e_i, e_j) - \underline{R}(e_i, e_j) \mid$$

Proof. Trivial □

Lemma 7.7. *Let $R^{\mathcal{T}}_{down}$ be the 'only comparable' $\mathcal{T}$-transitive weak closure of R given in Theorem 6.1. Then:*

$$d(R, R^{\mathcal{T}}_{down}) \leq d(R, R^{\mathcal{T}})$$

Proof. Trivial from Lemmas 7.5 and 7.6 □

Lemma 7.8. *Let $R^{\mathcal{T}}_{up}$ be the 'only comparable' $\mathcal{T}$-transitive weak closure of R given in Theorem 6.2. The distance between R and $R^{\mathcal{T}}_{up}$ is:*

$$d(R, R^{\mathcal{T}}_{up}) = \sum_{\forall i,j} \mid \overline{R}^{T_2}(e_i, e_j) - \overline{R}(e_i, e_j) \mid$$

Proof. Trivial □

Lemma 7.9. *Let $R^{\mathcal{T}}_{up}$ be the 'only comparable' $\mathcal{T}$-transitive weak closure of R given in Theorem 6.2. Then:*

$$d(R, R^{\mathcal{T}}_{up}) \leq d(R, R^{\mathcal{T}})$$

Proof. Trivial from Lemmas 7.5 and 7.8 □

8. Example

A decision maker (for example: a potential buyer) intends to buy a car. He has four alternatives (cars in this case) to choose $X = \{c_1, c_2, c_3, c_4\}$. Due to the features of this kind of decision the decision maker chooses the $\mathcal{T}$-norm $\mathcal{T} = [prod, min]$. Taking into consideration various factors (car features in this case) the decision maker constructs the next interval-valued fuzzy relation:

$$R = \begin{pmatrix} [1,1] & [0.4,0.9] & [0.3,0.6] & [0.2,0.5] \\ [0.1,0.6] & [1,1] & [0.2,0.8] & [0.5,0.9] \\ [0.4,0.7] & [0.2,0.8] & [1,1] & [0.2,0.6] \\ [0.5,0.8] & [0.1,0.5] & [0.4,0.8] & [1,1] \end{pmatrix}$$

where $R[1,2] =_L [0.4, 0.9]$ means he prefers the cars number 1 over the car number 2 in a degree between 0.4 and 0.9.

This relation is no transitive under $\mathcal{T} = [prod, min]$. For example: $\mathcal{T}(R[1,2], R[2,3]) =_L [0.08, 0.8] \not\leq_L R[1,3] =_L [0.3, 0.6]$.

Probably, the decision maker thinks a non $\mathcal{T}$-transitive relation of preference is not rational. However, he can accept some small changes in order to compute it in a $\mathcal{T}$-transitive relation. Then he has two options. First, he can compute the $\mathcal{T}$-transitive closure of R. And second, he can compute the 'only comparable' $\mathcal{T}$-transitive weak closure of R.

Applying the algorithm given by Gonzalez-del-Campo and Garmendia [11] it is obtained the next relation $R^{\mathcal{T}}$:

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.4,0.9] & [0.4,0.8] & [0.4,0.9] \\ [0.5,0.8] & [1,1] & [0.4,0.8] & [0.5,0.9] \\ [0.4,0.8] & [0.4,0.8] & [1,1] & [0.4,0.8] \\ [0.5,0.8] & [0.4,0.8] & [0.4,0.8] & [1,1] \end{pmatrix}$$

For the second option he can compute the 'only comparable' $\mathcal{T}$-transitive weak closure of R using Theorems 6.1 and 7.2 with $\mathcal{T} = [T_1, T_2] = [prod, min]$:

$$R^{\mathcal{T}}_{down} = \begin{pmatrix} [1,1] & [0.4,0.9] & [0.4,0.6] & [0.4,0.5] \\ [0.5,0.6] & [1,1] & [0.4,0.8] & [0.5,0.9] \\ [0.4,0.7] & [0.4,0.8] & [1,1] & [0.4,0.6] \\ [0.5,0.8] & [0.4,0.5] & [0.4,0.8] & [1,1] \end{pmatrix}$$

$$R^{\mathcal{T}}_{up} = \begin{pmatrix} [1,1] & [0.4,0.9] & [0.3,0.8] & [0.2,0.9] \\ [0.1,0.8] & [1,1] & [0.2,0.8] & [0.5,0.9] \\ [0.4,0.8] & [0.2,0.8] & [1,1] & [0.2,0.8] \\ [0.5,0.8] & [0.1,0.8] & [0.4,0.8] & [1,1] \end{pmatrix}$$

Using Lemmas 7.5, 7.6 and 7.8 it is possible to compute the distances between R and $R^{\mathcal{T}}$, $R^{\mathcal{T}}_{down}$ and $R^{\mathcal{T}}_{up}$:

- $d(R, R^{\mathcal{T}}) = 3$
- $d(R, R^{\mathcal{T}}_{down}) = 1.6$
- $d(R, R^{\mathcal{T}}_{up}) = 1.4$

We can see that $R^{\mathcal{T}}_{down}$ and $R^{\mathcal{T}}_{up}$ are closer to R than $R^{\mathcal{T}}$.

9. Conclusions

Transitive property is a fundamental notion in decision theory. It is universally assumed in disciplines of decision theory and accepted in a principle of rationality in some relations. However, the transitive property for interval-valued fuzzy relations is a much stronger condition than for fuzzy relations because it needs that all intervals must be comparable in the inequality that defines $\mathcal{T}$-transitivity.

In this paper, it is defined the 'only comparable' $\mathcal{T}$-transitivity property of $\mathcal{IVFR}s$ relaxing the T-transitivity for $\mathcal{FR}s$ by satisfying the inequality just when the intervals are comparable. It is also defined the weak closure for a interval-valued fuzzy relation under a property $\mathcal{P}$. In particular, it is studied the weak closure for a interval-valued fuzzy relation under the 'only comparable' $\mathcal{T}$-transitive property. It is proved that the closure for a interval-valued fuzzy relation under 'only comparable' $\mathcal{T}$-transitivity does not exist and there may exist several weak closure for a interval-valued fuzzy relation under 'only comparable' $\mathcal{T}$-transitivity.

Finally, it is proposed the weak closure for a interval-valued fuzzy relation under 'only comparable' $\mathcal{T}$-transitivity as a method to compute an approximation of a non $\mathcal{T}$-transitive fuzzy relations and it is shown that it is closer than the $\mathcal{T}$-transitive closure for a interval-valued fuzzy relation. Some examples are provided.

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