

Residual implications derived from uninorms satisfying Modus Ponens

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Abstract

Modus Ponens is a key property for fuzzy implication functions that are going to be used in fuzzy inference processes. In this paper it is investigated when fuzzy implication functions derived from uninorms via residuation, usually called *RU*-implications, satisfy the modus ponens with respect to a continuous t-norm T , or equivalently, when they are T -conditionals. For *RU*-implications it is proved that T -conditionality only depends on the underlying t-norm T_U of the uninorm U used to derive the residual implication and this fact leads to a lot of new solutions of the Modus Ponens property. Along the paper the particular cases when the uninorm lies in any of the most usual classes of uninorms are considered.

Keywords: Fuzzy implication function, residual implication, Modus Ponens, uninorm

1. Introduction

Fuzzy implication functions play a fundamental role in fuzzy logic and approximate reasoning. This kind of logical operations are essential in modelling all fuzzy conditionals and also in the inference process. Moreover, they are also useful in many application fields not only derived from the proper approximate reasoning, but also in other aspects as fuzzy subset-hood measures, fuzzy relational equations, fuzzy mathematical morphology, and computing with words among others. For this reason, investigations on fuzzy implication functions have been extensively developed in last decades even from the pure theoretical point of view, as it can be seen in the survey [18] and in the books [3, 4], entirely devoted to this kind of logical operations.

One of the main topics in this theoretical study consists on the investigation of additional properties of implication functions, properties that usually come from the concrete applications where implications functions are going to be applied. The study of each one of these additional properties usually leads to solve a functional equation (or inequality) involving fuzzy implication functions (see for instance Chapter 7 in [4] and the references therein).

One of these additional properties, that in this case comes from approximate reasoning, is known as the (generalized) *Modus Ponens*. In fact, forward inference schemes in approximate reasoning are usually based on the Modus Ponens that is carried out through the well known *Compositional Rule of Inference* (CRI) of Zadeh, based on the sup- T composition, where T is a t-norm (see for instance, Section 8.3 in [4]). Thus, if I is a fuzzy implication function and T is a t-norm, the Modus Ponens property for I with respect to T becomes the functional inequality:

$$T(x, I(x, y)) \leq y \quad \text{for all } x, y \in [0, 1],$$

property that is also known as T -conditionality.

The Modus Ponens has been extensively studied in the literature by some authors (namely [2, 4, 16, 24, 25, 26, 27]). However, all these studies involve only the main classes of implication functions:

1. *R*-implications derived from (left-continuous) t-norms, $I_T(x, y) =$

$$\sup\{z \in [0, 1] \mid T(x, z) \leq y, x, y \in [0, 1]\},$$

2. (S, N) -implications derived from a t-conorm S and a fuzzy negation N ,

$$I_{S, N}(x, y) = S(N(x), y), x, y \in [0, 1],$$

3. *QL*-implications derived from a t-norm T , a t-conorm S and a fuzzy negation N ,

$$I_{S, N, T}(x, y) = S(N(x), T(x, y)), x, y \in [0, 1].$$

On the other hand, note that there exist other kinds of implication functions like *D*-implications and Yager's implications. Moreover, some generalizations of *R*, (S, N) , and *QL*-implications have been introduced, by substituting the t-norm and the t-conorm by more general aggregation functions (for more details see [4] and also [19] with the references therein). One of these generalizations is based on uninorms obtaining the so-called *RU*-implications ([7]), (U, N) -implications ([5]), and even *QL* and *D*-implications derived from conjunctive and disjunctive uninorms ([15]).

For these kinds of implications the Modus Ponens has not been studied yet and this is the idea of the present paper. In particular we want to deal with the case of RU -implications, leaving the other cases for a future work. Specifically, we want to study the T -conditionality with respect to any continuous t -norm T for the case of RU -implications. We will prove that there are a lot of them that satisfy the Modus Ponens with respect to any t -norm T , and we will characterize the special case when the t -norm T and the underlying operations of the uninorm are continuous.

2. Preliminaries

We will suppose the reader to be familiar with the theory of t -norms, t -conorms and fuzzy negations (all necessary results and notations can be found in [11]). We also suppose that some basic facts on uninorms are known (see for instance [9]) as well as their most usual classes, that is, uninorms in \mathcal{U}_{\min} and \mathcal{U}_{\max} ([9]), representable uninorms ([9]), idempotent uninorms ([6, 14, 23]) and uninorms continuous in the open unit square ([10]).

We recall here only some facts on implications and uninorms in order to establish the necessary notation that we will use along the paper.

Definition 1 A binary operator $I : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a fuzzy implication function, or an implication, if it satisfies:

(I1) $I(x, z) \geq I(y, z)$ when $x \leq y$, for all $z \in [0, 1]$.

(I2) $I(x, y) \leq I(x, z)$ when $y \leq z$, for all $x \in [0, 1]$.

(I3) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

Note that, from the definition, it follows that $I(0, x) = 1$ and $I(x, 1) = 1$ for all $x \in [0, 1]$ whereas the symmetrical values $I(x, 0)$ and $I(1, x)$ are not derived from the definition.

Definition 2 A uninorm is a two-place function $U : [0, 1]^2 \rightarrow [0, 1]$ which is associative, commutative, increasing in each place and such that there exists some element $e \in [0, 1]$, called neutral element, such that $U(e, x) = x$ for all $x \in [0, 1]$.

Evidently, a uninorm with neutral element $e = 1$ is a t -norm and a uninorm with neutral element $e = 0$ is a t -conorm. For any other value $e \in]0, 1[$ the operation works as a t -norm in the $[0, e]^2$ square, as a t -conorm in $[e, 1]^2$ and its values are between minimum and maximum in the set of points $A(e)$ given by

$$A(e) = [0, e[\times]e, 1] \cup]e, 1] \times [0, e[.$$

We will usually denote a uninorm with neutral element e and underlying t -norm and t -conorm, T and S , by $U \equiv \langle T, e, S \rangle$. For any uninorm it is satisfied that $U(0, 1) \in \{0, 1\}$ and a uninorm U

is called *conjunctive* if $U(1, 0) = 0$ and *disjunctive* when $U(1, 0) = 1$. On the other hand, let us recall the most studied classes of uninorms in the literature.

Theorem 1 ([9]) Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm with neutral element $e \in]0, 1[$.

(a) If $U(0, 1) = 0$, then the section $x \mapsto U(x, 1)$ is continuous except in $x = e$ if and only if U is given by $U(x, y) =$

$$\begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\ e + (1 - e)S\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in A(e), \end{cases}$$

where T is a t -norm, and S is a t -conorm.

(b) If $U(0, 1) = 1$, then the section $x \mapsto U(x, 0)$ is continuous except in $x = e$ if and only if U is given by the same structure as above, changing minimum by maximum in $A(e)$.

The set of uninorms as in case (a) will be denoted by \mathcal{U}_{\min} and the set of uninorms as in case (b) by \mathcal{U}_{\max} . We will denote a uninorm in \mathcal{U}_{\min} with underlying t -norm T , underlying t -conorm S and neutral element e as $U \equiv \langle T, e, S \rangle_{\min}$ and in a similar way, a uninorm in \mathcal{U}_{\max} as $U \equiv \langle T, e, S \rangle_{\max}$.

Idempotent uninorms were characterized first in [6] for those with a lateral continuity and in [14] for the general case. An improvement of this last result was done in [23] as follows.

Theorem 2 ([23]) U is an idempotent uninorm with neutral element $e \in [0, 1]$ if and only if there exists a non increasing function $g : [0, 1] \rightarrow [0, 1]$, symmetric with respect to the identity function, with $g(e) = e$, such that $U(x, y) =$

$$\begin{cases} \min(x, y) & \text{if } y < g(x) \text{ or} \\ & (y = g(x) \text{ and } x < g^2(x)), \\ \max(x, y) & \text{if } y > g(x) \text{ or} \\ & (y = g(x) \text{ and } x > g^2(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x), \end{cases}$$

being commutative in the points (x, y) such that $y = g(x)$ with $x = g^2(x)$.

Any idempotent uninorm U with neutral element e and associated function g , will be denoted by $U \equiv \langle g, e \rangle_{\text{ide}}$ and the class of idempotent uninorms will be denoted by \mathcal{U}_{ide} . Obviously, for any of these uninorms the underlying t -norm T is the minimum and the underlying t -conorm S is the maximum.

Definition 3 ([9]) Let e be in $]0, 1[$. A binary operation $U : [0, 1]^2 \rightarrow [0, 1]$ is a representable uninorm if and only if there exists a strictly increasing function $h : [0, 1] \rightarrow [-\infty, +\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that

$$U(x, y) = h^{-1}(h(x) + h(y))$$

for all $(x, y) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$ and $U(0, 1) = U(1, 0) \in \{0, 1\}$. The function h is usually called an additive generator of U .

Remark 1 Recall that there are no continuous uninorms with neutral element $e \in]0, 1[$. In fact, representable uninorms were characterized as those uninorms that are continuous in $[0, 1]^2 \setminus \{(1, 0), (0, 1)\}$ (see [21]) as well as those that are strictly increasing in the open unit square (see [8]).

Any representable uninorm U with neutral element e and additive generator h , will be denoted by $U \equiv \langle h, e \rangle_{\text{rep}}$ and the class of representable uninorms will be denoted by \mathcal{U}_{rep} . For any of these uninorms the underlying t -norm T is always strict and the underlying t -conorm S is strict as well.

A more general class containing representable uninorms is given by those uninorms that are continuous in the open unit square $]0, 1[^2$. This class was characterized in [10] as follows.

Theorem 3 ([10] and [21] for the current version) Suppose U is a uninorm continuous in $]0, 1[^2$ with neutral element $e \in]0, 1[$. Then either one of the following cases is satisfied:

(a) There exist $u \in [0, e]$, $\lambda \in [0, u]$, two continuous t -norms T_1 and T_2 and a representable uninorm R such that U can be represented as $U(x, y) =$

$$\begin{cases} \lambda T_1\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) & \text{if } x, y \in [0, \lambda], \\ \lambda + (u - \lambda) T_2\left(\frac{x - \lambda}{u - \lambda}, \frac{y - \lambda}{u - \lambda}\right) & \text{if } x, y \in [\lambda, u], \\ u + (1 - u) R\left(\frac{x - u}{1 - u}, \frac{y - u}{1 - u}\right) & \text{if } x, y \in]u, 1[, \\ 1 & \text{if } \min(x, y) \in]\lambda, 1] \\ & \text{and } \max(x, y) = 1, \\ \lambda \text{ or } 1 & \text{if } (x, y) = (\lambda, 1) \\ & \text{or } (x, y) = (1, \lambda), \\ \min(x, y) & \text{elsewhere.} \end{cases} \quad (1)$$

(b) There exist $v \in]e, 1]$, $\omega \in [v, 1]$, two continuous t -conorms S_1 and S_2 and a representable uninorm R such that U can be represented as $U(x, y) =$

$$\begin{cases} v + (\omega - v) S_1\left(\frac{x - v}{\omega - v}, \frac{y - v}{\omega - v}\right) & \text{if } x, y \in [v, \omega], \\ \omega + (1 - \omega) S_2\left(\frac{x - \omega}{1 - \omega}, \frac{y - \omega}{1 - \omega}\right) & \text{if } x, y \in [\omega, 1], \\ v R\left(\frac{x}{v}, \frac{y}{v}\right) & \text{if } x, y \in]0, v[, \\ 0 & \text{if } \max(x, y) \in [0, \omega[\\ & \text{and } \min(x, y) = 0, \\ \omega \text{ or } 0 & \text{if } (x, y) = (0, \omega) \\ & \text{or } (x, y) = (\omega, 0), \\ \max(x, y) & \text{elsewhere.} \end{cases} \quad (2)$$

The class of all uninorms continuous in $]0, 1[^2$ will be denoted by \mathcal{U}_{cos} . A uninorm as in (1) will be denoted by $U \equiv \langle T_1, \lambda, T_2, u, (R, e) \rangle_{\text{cos}, \min}$ and the class of all uninorms continuous in the open unit

square of this form will be denoted by $\mathcal{U}_{\text{cos}, \min}$. Analogously, a uninorm as in (2) will be denoted by $U \equiv \langle (R, e), v, S_1, \omega, S_2 \rangle_{\text{cos}, \max}$ and the class of all uninorms continuous in the open unit square of this form will be denoted by $\mathcal{U}_{\text{cos}, \max}$. For any uninorm $U \equiv \langle T_1, \lambda, T_2, u, (R, e) \rangle_{\text{cos}, \min}$, the underlying t -norm of U is given by an ordinal sum of three t -norms, T_1, T_2 and a strict t -norm, whereas the underlying t -conorm is always strict. Similarly, for any uninorm $U \equiv \langle (R, e), v, S_1, \omega, S_2 \rangle_{\text{cos}, \max}$, the underlying t -norm of T is always strict, whereas the underlying t -conorm is given by an ordinal sum of three t -conorms, a strict t -conorm, S_1 , and S_2 .

On the other hand, different classes of implications derived from uninorms have been studied. We recall here RU -implications.

Definition 4 Let U be a uninorm. The residual operation derived from U is the binary operation given by $I_U(x, y) =$

$$\sup\{z \in [0, 1] \mid U(x, z) \leq y\} \text{ for all } x, y \in [0, 1].$$

Proposition 4 ([7]) Let U be a uninorm and I_U its residual operation. Then I_U is an implication if and only if the following condition holds

$$U(x, 0) = 0 \quad \text{for all } x < 1. \quad (3)$$

In this case I_U is called an RU -implication.

This includes all conjunctive uninorms but also many disjunctive ones, for instance in the classes of representable uninorms (see [7]), idempotent uninorms (see [20]), and uninorms continuous in the open unit square (see [22]). However, when we deal with left-continuous uninorms U we clearly have that U satisfies condition (3) if and only if it is conjunctive.

Some properties of RU -implications have been studied involving the main classes of uninorms, those previously stated: uninorms in \mathcal{U}_{\min} , representable uninorms, idempotent uninorms and uninorms continuous in the open unit square (for more details see [1, 4, 7, 17, 20, 22]). However, although the strong interest of the Modus Ponens property, its study is not among the properties investigated for implications derived from uninorms. Let us recall the definition of the Modus Ponens in the framework of fuzzy logic.

Definition 5 Let I be an implication function and T a t -norm. It is said that I satisfies the Modus Ponens property with respect to T , or that I is a T -conditional if

$$T(x, I(x, y)) \leq y \quad \text{for all } x, y \in [0, 1]. \quad (4)$$

A well known general result on T -conditionality was proved in [24].

Proposition 5 Let I be an implication function and T a left-continuous t -norm. Then I is a T -conditional if and only if $I \leq I_T$, where I_T denotes the residual implication derived from T .

3. RU -implications that are T -conditionals

In this section we want to deal with the case of RU -implications. Thus, the main goal of this section is to characterize when an RU -implication derived from a uninorm U is a T -conditional for a t -norm T , specially when T is continuous. All along this section it will be understood that any considered uninorm U satisfies $U(x, 0) = 0$ for all $x < 1$, in order to ensure that the corresponding residual I_U is an RU -implication, according to Proposition 4.

Proposition 6 *Let U be a uninorm with neutral element $e \in]0, 1[$ and underlying t -norm T_U , and let I_U be the corresponding RU -implication. Then the following items are equivalent:*

- i) I_U is a T -conditional.
- ii) I_U satisfies Equation (4) for all $y < x < e$.
- iii) The inequality

$$T\left(x, eI_{T_U}\left(\frac{x}{e}, \frac{y}{e}\right)\right) \leq y$$

holds for all x, y such that $y < x < e$, where I_{T_U} denotes the residual implication derived from the t -norm T_U .

This result proves that the underlying t -conorm S_U of the uninorm U and the values of U in the region $A(e)$ are not relevant in order I_U to be a T -conditional. Only the underlying t -norm T_U is relevant and the inequality corresponding to T -conditionality only needs to be checked in the region

$$R_e = \{(x, y) \in [0, 1]^2 \mid y < x < e\}.$$

The region R_e is pictured in Figure 3.

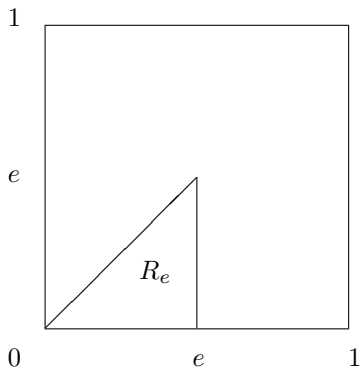


Figure 1: Region R_e .

Thus, we continue our study depending on how the underlying t -norm T_U is. Note that if $T(e, e) = 0$, then condition iii) in Proposition 6 is always satisfied. Then, any t -norm T_U (continuous or not) will work in this case. From now on, we will restrict ourselves to the case when T_U is continuous. Taking into account the classification of continuous t -norms (see for instance [11]), we will divide our study in

three steps respectively devoted to the cases when $T_U = \min$, T_U is Archimedean or T_U is given by an ordinal sum.

Let us begin with the case when $T_U = \min$.

Proposition 7 *Let U be a uninorm with neutral element $e \in]0, 1[$ and underlying t -norm T_U given by the minimum. Then the RU -implication I_U is a T -conditional for any t -norm T . In particular, this is the case for any idempotent uninorm U .*

Example 1 *Let N be a strong negation. Among the class of idempotent uninorms, an important example is given by uninorms whenever $g = N$ (see [20]), that is, they have the form*

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } y < N(x), \\ \max(x, y) & \text{if } y > N(x), \\ \min(x, y) \text{ or } \max(x, y) & \text{otherwise,} \end{cases}$$

being commutative in the points (x, y) such that $y = N(x)$. In these cases the corresponding RU -implication is given by

$$I_U(x, y) = \begin{cases} \min(N(x), y) & \text{if } y < x, \\ \max(N(x), y) & \text{if } y \geq x. \end{cases}$$

From the proposition above, all these implications are T -conditionals for any t -norm T . Figure 3 shows the idempotent uninorm $U \equiv \langle N, \frac{1}{2} \rangle_{ide}$ and its RU -implication given in this example when the considered negation is the classical one $N_c(x) = 1 - x$.

Let us now deal with the case when the underlying t -norm T_U is Archimedean and the t -norm T is continuous. It is well known that when a t -norm is Archimedean it is represented by a decreasing additive generator $\varphi : [0, 1] \rightarrow [0, +\infty]$, with $\varphi(0) = +\infty$ if the t -norm is strict, and with $\varphi(0) = 1$ when the t -norm is nilpotent. Moreover, in this last case the function $N(x) = \varphi^{-1}(1 - \varphi(x))$ for all $x \in [0, 1]$ is a strong negation usually called the *associated negation* of the nilpotent t -norm T . Let us consider both cases separately.

Proposition 8 *Let U be a uninorm with neutral element $e \in]0, 1[$ and underlying t -norm T_U strict. Let I_U be the RU -implication derived from U and T a continuous t -norm.*

- i) *If I_U is a T -conditional, then there exists $a \geq e$ such that T is an ordinal sum of the form $T = (\langle 0, a, T_a \rangle, \langle a, 1, T_1 \rangle)$, where T_a is Archimedean and T_1 continuous.*
- ii) *Let φ and φ_a be the additive generators of T_U and T_a , respectively. Then I_U is a T -conditional if and only if the function g defined from $[0, +\infty]$ to $[\varphi_a(\frac{e}{a}), \varphi_a(0)]$ given by the expression $g(u) = \varphi_a(\frac{e}{a}\varphi^{-1}(u))$ is sub-additive.*

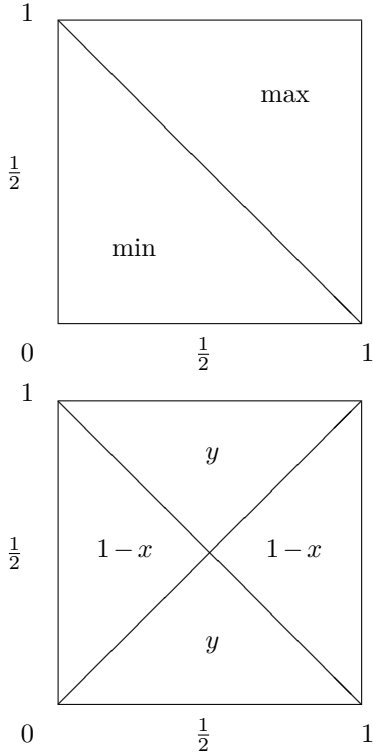


Figure 2: Structure of $U \equiv \langle N, \frac{1}{2} \rangle_{\text{ide}}$ (top) and I_U (bottom) with $N_c(x) = 1 - x$.

- Remark 2**
- i) Note that when $a = e$, the result above can be related to the results about T -conditionality for residual implications derived from continuous t -norms given in [24]. Specifically, we can derive that I_U is a T -conditional if and only if $T = (\langle 0, e, T_e \rangle, \langle e, 1, T_1 \rangle)$, where T_e is Archimedean, T_1 continuous and I_{T_U} is a T_e -conditional.
 - ii) Of course that one can take $a = 1$ in the proposition above obtaining RU -implications that are T -conditional for Archimedean t -norms T .
 - iii) Note that uninorms with T_U strict include, but are not limited to, all representable uninorms as well as those uninorms continuous in $]0, 1[^2$ lying in $\mathcal{U}_{\text{cos}, \text{max}}$. In addition, the subset of all uninorms with T_U and S_U strict have been recently characterized in [13].

Example 2 Let us take, for instance, the conjunctive uninorm given by $U(x, y) =$

$$\begin{cases} 0 & \text{if } (x, y) \in \{(1, 0), (0, 1)\}, \\ \frac{xy}{xy + (1-x)(1-y)} & \text{otherwise,} \end{cases}$$

whose residual implication I_U is given by

$$I_U(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{(0, 0), (1, 1)\}, \\ \frac{(1-x)y}{x+y-2xy} & \text{otherwise,} \end{cases}$$

and take also $T = T_{\mathbf{P}}$ the product t -norm. It is easy to see that in this case I_U is a T -conditional.

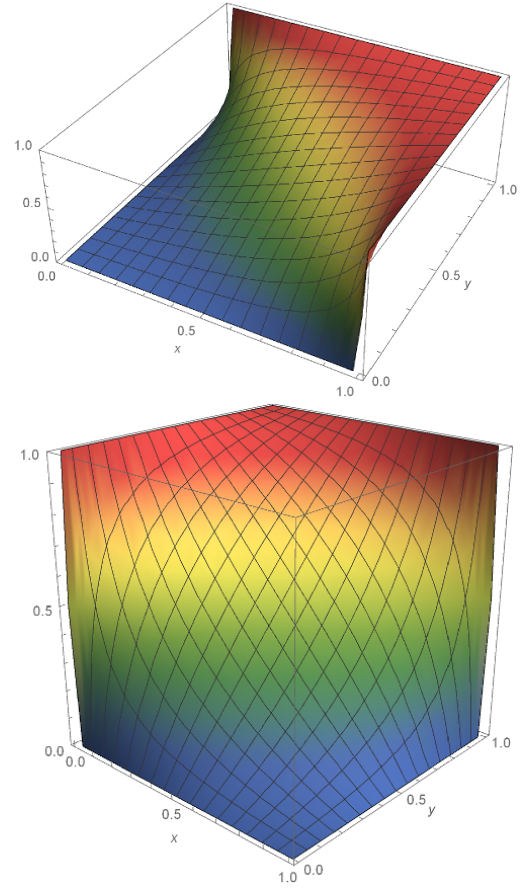


Figure 3: Representable uninorm U with additive generator $h(x) = \log\left(\frac{x}{1-x}\right)$ (top) and its residual implication I_U (bottom).

Namely, it is well known that U is a representable uninorm with neutral element $e = \frac{1}{2}$ and additive generator $h(x) = \log\left(\frac{x}{1-x}\right)$. In this case, the underlying t -norm T_U is strict with additive generator given by $\varphi(x) = \log\left(\frac{2-x}{x}\right)$ (see [9] or Section 10.2 in [11]). Moreover, this situation corresponds to take $a = 1$ and $\varphi_1(x) = -\log(x)$ in Proposition 8 and consequently the corresponding function g is given by

$$\begin{aligned} g(x) &= \varphi_1\left(\frac{1}{2}\varphi^{-1}(x)\right) = -\log\left(\frac{1}{2}\frac{2}{1+e^x}\right) \\ &= \log(1+e^x), \end{aligned}$$

which is clearly sub-additive. In Figure 3 uninorm U and its residual implication I_U have been depicted.

Proposition 9 Let U be a uninorm with neutral element $e \in]0, 1[$ and underlying t -norm T_U nilpotent with associated negation N_U . Let I_U be the RU -implication derived from U and T a continuous t -norm.

- i) If I_U is a T -conditional with respect to T , then there exists $a \geq e$ such that T is an ordinal sum

of the form $T = (\langle 0, a, T_a \rangle, \langle a, 1, T_1 \rangle)$, where T_a is nilpotent with associated negation N_a such that N_a covers N_U , i.e., $eN_U(\frac{x}{e}) \leq aN_a(\frac{x}{a})$ for all $x \in [0, e]$, and T_1 is continuous.

- ii) Let φ and φ_a be the additive generators of T_U and T_a , respectively. Then I_U is a T -conditional with respect to T if and only if the function $g : [0, 1] \rightarrow [\varphi_a(\frac{e}{a}), 1]$ given by $g(u) = \varphi_a(\frac{e}{a}\varphi^{-1}(u))$ is sub-additive.

Remark 3 i) Note that in the previous proposition, the fact that g is subadditive already ensures that the corresponding negations N_a and N_U are such that N_a covers N_U . Namely, we have that N_a covers N_U if and only if

$$e\varphi^{-1}\left(1 - \varphi\left(\frac{x}{e}\right)\right) \leq a\varphi_a^{-1}\left(1 - \varphi_a\left(\frac{x}{a}\right)\right),$$

but taking the change $x = e\varphi^{-1}(z)$, this is equivalent to

$$g(1 - z) \geq 1 - g(z),$$

which is clearly satisfied when g is sub-additive.

- ii) Again note that when $a = e$, the result above can be stated as follows: I_U is a T -conditional if and only if $T = (\langle 0, e, T_e \rangle, \langle e, 1, T_1 \rangle)$, where T_e is Archimedean, T_1 continuous and I_{T_U} is a T_e -conditional. Of course $a = 1$ can be taken in the proposition above obtaining RU -implications that are T -conditional for nilpotent t -norms T .
- iii) Recall that uninorms U with T_U nilpotent and S_U Archimedean were characterized in [12] and [13].

Example 3 Let U be a uninorm in \mathcal{U}_{\min} with neutral element $e = \frac{1}{2}$ and underlying t -norm $T_U = T_{\mathbf{L}}$ the Łukasiewicz t -norm, that is, U is given by the expression

$$U(x, y) = \begin{cases} \max(0, x + y - \frac{1}{2}) & \text{if } (x, y) \in [0, \frac{1}{2}]^2, \\ \frac{1 + S_U(2x - 1, 2y - 1)}{2} & \text{if } (x, y) \in [\frac{1}{2}, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

where S_U can be any t -conorm. In this case I_U is given by (see [7]):

$$I_U(x, y) = \begin{cases} 1 & \text{if } x < \frac{1}{2} \text{ and } x \leq y, \\ \frac{1}{2} - x + y & \text{if } x < \frac{1}{2} \text{ and } x > y, \\ y & \text{if } y \leq \frac{1}{2} \leq x, \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq y < x, \\ \frac{1 + R_{S_U}(2x - 1, 2y - 1)}{2} & \text{if } \frac{1}{2} \leq x \leq y, \end{cases}$$

where $R_{S_U}(x, y) = \sup\{z \in [0, 1] | S_U(x, z) \leq y\}$. Then I_U is always a $T_{\mathbf{L}}$ -conditional because, using Proposition 9, we have $a = 1$ and $\varphi(x) = \varphi_1(x) = 1 - x$, obtaining $g(x) = \frac{x+1}{2}$ which is clearly sub-additive. In Figure 4 we can see the structure of this general uninorm in \mathcal{U}_{\min} as in this example and the corresponding RU -implication.

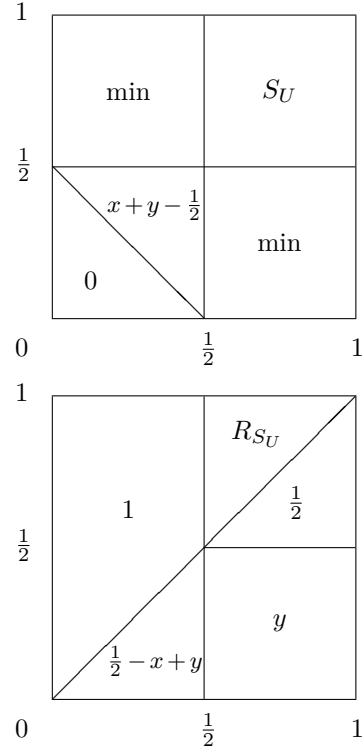


Figure 4: Structure of U (top) and I_U (bottom) when $U \in \mathcal{U}_{\min}$, $T_U = T_{\mathbf{L}}$ and $e = \frac{1}{2}$.

Proposition 10 Let U be a uninorm with neutral element $e \in]0, 1[$ and underlying t -norm given by the ordinal sum $T_U = (\langle \frac{a_i}{e}, \frac{b_i}{e}, T_i \rangle)_{i \in I}$ with $0 \leq a_i < b_i \leq e$ and T_i Archimedean for all $i \in I$. Let I_U be the RU -implication derived from U and T a continuous t -norm. Then I_U is a T -conditional if and only if the following items hold:

- i) The set of idempotent elements of T , that are less than or equal to e , are contained in $[0, e] \setminus \{\cup_{i \in I} (a_i, b_i)\}$.
- ii) The t -norm T is an ordinal sum of the form $T = (\langle c_j, d_j, T'_j \rangle)_{j \in J}$ in such a way that for all $i \in I$ there exists a $j \in J$ such that
 - $(a_i, b_i) \subseteq (c_j, d_j)$, and if T_i is nilpotent with associated negation N_i for some $i \in I$ and $a_i = c_j$ then T'_j must be also nilpotent with associated negation N_j such that N_j covers N_i .
 - If φ_i and φ_j are the respective additive generators of T_i and T'_j , then the function g_{ij} that is defined from $[0, \varphi_i(0)]$ to $[\varphi_j(\frac{b_i - c_j}{d_j - c_j}), \varphi_j(\frac{a_i - c_j}{d_j - c_j})]$ given by

$$g_{ij}(u) = \varphi_j\left(\frac{a_i + (b_i - a_i)\varphi_i^{-1}(u) - c_j}{d_j - c_j}\right)$$

is sub-additive.

From the previous result we can easily derive the following result.

Proposition 11 Let U be a uninorm with neutral element $e \in]0, 1[$ and underlying t-norm given by the ordinal sum $T_U = (\langle \frac{a_i}{e}, \frac{b_i}{e}, T_i \rangle)_{i \in I}$ with $0 \leq a_i < b_i \leq e$ and T_i Archimedean for all $i \in I$. Let T be the ordinal sum $T = (\langle a_i, b_i, T'_i \rangle)_{i \in I}$ with T'_j Archimedean for all $i \in I$ and suppose that $I_{T'_i}$ is T_i -conditional for all $i \in I$. Then I_U is a T -conditional.

Example 4 Let U be a uninorm in \mathcal{U}_{\min} with neutral element $e = \frac{1}{2}$ and underlying t-norm T_U given by the ordinal sum $T_U = (\langle 0, \frac{1}{2}, T_P \rangle, \langle \frac{1}{2}, 1, T_L \rangle)$ that is, U is given by the expression

$$U(x, y) = \begin{cases} 4xy & \text{if } (x, y) \in [0, \frac{1}{4}]^2, \\ \max(\frac{1}{4}, x + y - \frac{1}{2}) & \text{if } (x, y) \in [\frac{1}{4}, \frac{1}{2}]^2, \\ \frac{1 + S_U(2x - 1, 2y - 1)}{2} & \text{if } (x, y) \in [\frac{1}{2}, 1]^2, \\ \min(x, y) & \text{otherwise,} \end{cases}$$

where S_U can be any t-conorm.

Let us consider the t-norm T given by the expression

$$T(x, y) = \begin{cases} T_P & \text{if } (x, y) \in [0, \frac{1}{2}]^2, \\ T_L & \text{if } (x, y) \in [\frac{1}{2}, 1]^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Using Proposition 11, I_U is a T -conditional because, I_{T_P} is a T_P -conditional and I_{T_L} is a T_L -conditional.

In Figure 5 we can see the structure of this general uninorm in \mathcal{U}_{\min} as in this example and the corresponding RU -implication.

4. Conclusions and future work

Forward inference schemes in approximate reasoning are based on the *Modus Ponens* property, also called T -conditionality. Thus, fuzzy implication functions used in the inference process of any fuzzy rule based system are required to satisfy this property, which becomes essential in approximate reasoning and fuzzy control. Fixed a continuous t-norm T modelling the conjunction, we studied in this paper which fuzzy implication functions satisfy T -conditionality among a special kind of implications derived from uninorms: RU -implications. In this case we have characterized all the solutions of the Modus Ponens property with respect to a continuous t-norm T and from these characterizations we obtain a lot of new fuzzy implication functions satisfying the T -conditionality.

Moreover, we want to extend this study to the cases of other classes of fuzzy implication functions derived from uninorms, like (U, N) -implications and QL and D -implications. It is worth to point out that we have already started with the case of (U, N) -implications obtaining again a lot of new solutions. Moreover, contrarily to what happens with RU -implications, T -conditionality for (U, N) -implications only depends on the underlying t-conorm S_U and only in some cases, depending on

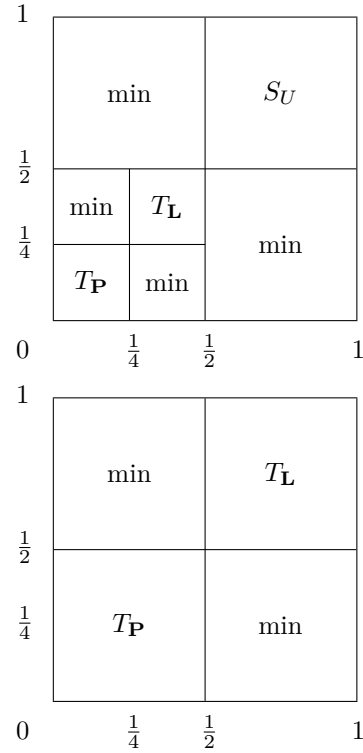


Figure 5: Structure of U (top) and T (bottom) such that $U \in \mathcal{U}_{\min}$, and I_U is a T -conditional.

the value

$$\alpha_N = \inf\{z \in [0, 1] \mid N(z) = e\},$$

where N is the negation used to derive the corresponding (U, N) -implication, and e is the neutral element of the uninorm U . Another possible extension that deserves to be investigated is related to the Modus Ponens with respect to a conjunctive uninorm U instead of a continuous t-norm T .

Of course that, as future work, it should be also included the study of the Modus Tollens for all these classes of implications.

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