Study of the choice of the weighting measure φ on the φ-wabl/ldev/rdev median

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Abstract

When summarizing the location of a random fuzzy number, some more robust approaches than the well-known Aumann-type mean have been proposed in the literature. Among them, the φ-wabl/ldev/rdev median extends the concept of median from the real-valued case. The characterization for fuzzy numbers and the distance the φ-wabl/ldev/redev median is based on will be recalled. This distance involves a weighting measure to distinguish the relevance of the different α-levels (φ). Since the φ-wabl/ldev/rdev median depends on such a weighting measure, a sensitivity analysis on the real influence of this choice on the estimate will be developed.

Keywords: Fuzzy data, φ-wabl/ldev/rdev median of a random fuzzy number, weighting measure, sensitivity analysis

1. Introduction

In the literature, several extensions of the median notion to random fuzzy numbers (or RFN, the also called 'fuzzy random variables' in Puri and Ralescu's sense, see [9]) can be found. In detail, [13] and [14] use L^1 metrics that make use of representations of fuzzy numbers for which necessary and sufficient conditions to characterize them are known, so the median can be defined as the fuzzy number minimizing the mean distance to all the values the RFN takes. A third alternative, the one considered in this paper, was introduced in [16] as a generalization of the Hausdorff-type median for random intervals (see [12]) following the same scheme.

In Section 2, the φ-wabl/ldev/rdev characterization for fuzzy numbers and the L^1-type distance between fuzzy numbers based on it will be recalled, including a study of the role the involved weighting measure in the metric plays. The φ-wabl/ldev/rdev median for an RFN will be explained in Section 3 and Section 4 will consist of the sensitivity analysis of the influence of the weighting measure φ on the estimate of the considered median. Finally, some conclusions and open problems will be commented in Section 5.

2. The φ-wabl/ldev/rdev characterization for RFNs and the associated L^1 metric

Let $\mathcal{F}_c(\mathbb{R})$ denote the class of fuzzy numbers with bounded support. The φ-wabl/ldev/rdev representation introduced in [15] takes into account an indicator of the 'center' (wabl$^\varphi$) along with two indicators of the 'shape' quantifying the deviation with respect to the center (ldev/rdev). The weighted averaging based on levels, denoted by wabl$^\varphi$, was first extended by [6] (see also [7]) from the concept introduced by [18]. For any $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$, it is defined as the real number in the interior set $\mathrm{int}(\widetilde{U}_0)$ such that

$$\mathrm{wabl}^\varphi(\widetilde{U}) = \int_{[0,1]} \mathrm{mid}\,\widetilde{U}_\alpha \, d\varphi(\alpha),$$

where φ is a weighting measure on the measurable space $([0,1], \mathcal{B}_{[0,1]})$ that can be formalized by means of an absolutely continuous probability measure with positive mass function on $(0,1)$ and mid represents the mid-point or centre

$$\mathrm{mid}\,\widetilde{U}_\alpha = \frac{\inf \widetilde{U}_\alpha + \sup \widetilde{U}_\alpha}{2}.$$

The wabl$^\varphi$ coincides with the well-known generalized Steiner point (or centroid) of a fuzzy number (see, for instance, [2, 3, 5]) by extending level-wise the Steiner points for convex sets (see [11]).

It should be pointed out that no stochastic meaning is actually associated with φ, but it allows us to weight the 'degrees of compatibility' given by the α-levels.

The other two components of the φ-wabl/ldev/rdev representation are level-wise indicators of the shape of a fuzzy number with respect to the considered center:

$$\mathrm{ldev}^\varphi_{\widetilde{U}}(\alpha) = \mathrm{wabl}^\varphi(\widetilde{U}) - \inf \widetilde{U}_\alpha, \text{ for all } \alpha \in [0,1],$$
$$\mathrm{rdev}^\varphi_{\widetilde{U}}(\alpha) = \sup \widetilde{U}_\alpha - \mathrm{wabl}^\varphi(\widetilde{U}), \text{ for all } \alpha \in [0,1].$$

Definition 2.1 [15] *Let φ be an absolutely continuous probability measure associated with the measurable space $([0,1], \mathcal{B}_{[0,1]})$ and having positive mass function on $(0,1)$. The **φ-wabl/ldev/rdev representation** of the fuzzy number $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$ is the vector-valued function $\boldsymbol{\upsilon}^\varphi_{\widetilde{U}} = (\upsilon^w_{\widetilde{U}}, \upsilon^l_{\widetilde{U}}, \upsilon^r_{\widetilde{U}}) : [0,1] \to \mathbb{R}^3$ such that $\upsilon^w_{\widetilde{U}}$ is constantly equal to* $\mathrm{wabl}^\varphi(\widetilde{U})$, *$\upsilon^l_{\widetilde{U}}(\alpha) = \mathrm{ldev}^\varphi_{\widetilde{U}}(\alpha)$ and $\upsilon^r_{\widetilde{U}}(\alpha) = \mathrm{rdev}^\varphi_{\widetilde{U}}(\alpha)$.*

The advantage of this representation with respect to the mid/spr one, in terms of the mid-point or centre and the spread or radius of all the α-levels of the fuzzy number (see e.g. [17]), is that necessary and sufficient conditions characterizing fuzzy numbers can be stated, what allows us to guarantee that the solution found for the minimization problem defining the corresponding median is indeed a fuzzy number.

Proposition 2.1 [15] *Given a fuzzy number $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$ there exist a value $m \in \mathbb{R}$ and two functions $l^* : [0,1] \to \mathbb{R}$ and $r^* : [0,1] \to \mathbb{R}$ satisfying that*

i) l^* and r^* are
- *left-continuous functions at any $\alpha \in (0,1]$,*
- *right-continuous at 0,*
- *and non-increasing on $[0,1]$,*

with

ii) $-l^*(1) \leq r^*(1)$,

and such that for all $\alpha \in [0,1]$,

$$\widetilde{U}_\alpha = [m - l^*(\alpha), m + r^*(\alpha)] .$$

Conversely, let $m \in \mathbb{R}$ and let $l^* : [0,1] \to \mathbb{R}$ and $r^* : [0,1] \to \mathbb{R}$ be functions satisfying Conditions i) and ii). Then there exists a unique $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$ such that for all $\alpha \in [0,1]$

$$\widetilde{U}_\alpha = [m - l^*(\alpha), m + r^*(\alpha)] .$$

Furthermore, if there is an absolutely continuous probability measure φ on $([0,1], \mathcal{B}_{[0,1]})$ with positive mass function on $(0,1)$ and such that

iii) $\displaystyle\int_{[0,1]} l^*(\alpha)\, d\varphi(\alpha) = \int_{[0,1]} r^*(\alpha)\, d\varphi(\alpha),$

*then, (m, l^*, r^*) is the φ-*wabl/ldev/rdev *representation of $\widetilde{U}$.*

Noticing that the φ-wabl/ldev/rdev representation coincides with the mid/spr one for symmetric fuzzy number-valued data, irrespective of φ, it is obvious that it is an extension of the mid/spr representation for interval-valued data. Therefore, this representation is suitable for extending the Hausdorff-type median (see [12]) to the fuzzy-valued case. In order to recall how this generalization has been done in the literature, the L^1 distance based on the φ-wabl/ldev/rdev representation and used in such concept will be specified now.

Definition 2.2 [16] *Given an absolutely continuous probability measure φ on the measurable space $([0,1], \mathcal{B}_{[0,1]})$ with positive mass function on $(0,1)$ and a parameter $\theta \in (0,1]$, the wabl/ldev/rdev-based L^1 metric is the mapping $\mathscr{D}_\theta^\varphi : \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R}) \to [0, +\infty)$ such that for $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$:*

$$\mathscr{D}_\theta^\varphi(\widetilde{U}, \widetilde{V}) = |\mathrm{wabl}^\varphi(\widetilde{U}) - \mathrm{wabl}^\varphi(\widetilde{V})|$$
$$+\frac{\theta}{2} \int_{[0,1]} |\mathrm{ldev}_{\widetilde{U}}^\varphi(\alpha) - \mathrm{ldev}_{\widetilde{V}}^\varphi(\alpha)|\, d\varphi(\alpha)$$
$$+\frac{\theta}{2} \int_{[0,1]} |\mathrm{rdev}_{\widetilde{U}}^\varphi(\alpha) - \mathrm{rdev}_{\widetilde{V}}^\varphi(\alpha)|\, d\varphi(\alpha).$$

Remark 2.1 *When we are working on $\mathcal{K}_c(\mathbb{R})$ and θ is equal to 1, the wabl/ldev/rdev-based L^1 metric coincides with the well-known Hausdorff metric:*
$d_H(K, K') = |\mathrm{mid}\, K - \mathrm{mid}\, K'| + |\mathrm{spr}\, K - \mathrm{spr}\, K'|.$

Another way to express the metric $\mathscr{D}_\theta^\varphi$ is the following:

$$\mathscr{D}_\theta^\varphi(\widetilde{U}, \widetilde{V}) = \int_{[0,1]} |\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha) - \boldsymbol{v}_{\widetilde{V}}^\varphi(\alpha)|_\theta^1\, d\varphi(\alpha),$$

where $|\cdot|_\theta^1$ denotes the L^1 norm in $\mathbb{R}^3$ given for $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ by

$$|\mathbf{x} - \mathbf{y}|_\theta^1 = |x_1 - y_1| + \frac{\theta}{2} \cdot |x_2 - y_2| + \frac{\theta}{2} \cdot |x_3 - y_3|.$$

The fact that the mapping $\mathscr{D}_\theta^\varphi$ is indeed a distance between fuzzy numbers is now stated and proved.

Proposition 2.2 *Let φ be an arbitrarily fixed absolutely continuous probability measure on $([0,1], \mathcal{B}_{[0,1]})$ with positive mass function on $(0,1)$, and $\theta \in (0,1]$ be a weighting parameter. Then,*

i) $\mathscr{D}_\theta^\varphi$ is an L^1 metric on $\mathcal{F}_c(\mathbb{R})$, both translational and rotational invariant.

ii) For a fixed φ, the function $\boldsymbol{v}^\varphi : \mathcal{F}_c(\mathbb{R}) \to \mathbb{H}_1^\star = \{L^1$-type 3-dimensional vector-valued functions defined on $[0,1]\}$ satisfies that
- *$\boldsymbol{v}^\varphi$ is an isometry from $(\mathcal{F}_c(\mathbb{R}), \mathscr{D}_\theta^\varphi)$ into $\mathbb{H}_1^\star$,*
- *$\boldsymbol{v}^\varphi(\widetilde{U} + \widetilde{V}) = \boldsymbol{v}^\varphi(\widetilde{U}) + \boldsymbol{v}^\varphi(\widetilde{V})$ for all $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R})$,*
- *$\boldsymbol{v}^\varphi(\gamma \cdot \widetilde{U}) = \gamma \cdot \boldsymbol{v}^\varphi(\widetilde{U})$ for all $\widetilde{U} \in \mathcal{F}_c(\mathbb{R})$ and $\gamma > 0$;*

Thus, the $\boldsymbol{v}^\varphi$ function preserves the semilinearity of $\mathcal{F}_c(\mathbb{R})$ and relates the fuzzy arithmetic to the functional arithmetic, what implies that $\mathcal{F}_c(\mathbb{R})$ can be isometrically embedded into a convex cone of the Banach space $\left(\mathbb{H}_1^\star, \|\cdot\|_\theta^{\varphi\star}\right)$ with $\|f - g\|_\theta^{\varphi\star} = \int_{[0,1]} |f(\alpha) - g(\alpha)|_\theta^1\, d\varphi(\alpha)$.

Proof i) Indeed, $\mathscr{D}_\theta^\varphi$ satisfies

- the nonnegativity (or separation axiom); it is trivial that $\mathscr{D}_\theta^\varphi(\widetilde{U}, \widetilde{V}) \geq 0$ whatever the fuzzy numbers $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c(\mathbb{R}^p)$ may be;
- the identity of indiscernibles (or coincidence axiom); it is obvious that if $\widetilde{U} = \widetilde{V}$, then $\mathscr{D}_\theta^\varphi(\widetilde{U}, \widetilde{V}) = 0$ because wabl^φ, ldev and rdev characterize the fuzzy number. On the other hand, a necessary condition for having $\mathscr{D}_\theta^\varphi(\widetilde{U}, \widetilde{V}) = 0$ is that $\boldsymbol{\rho}_{\mathbf{1}}^\varphi(\widetilde{U}, \widetilde{V}) = 0$, where $\boldsymbol{\rho}_{\mathbf{1}}^\varphi$ denotes the following extension of the 1-norm distance defined by [1]:

$$\boldsymbol{\rho}_{\mathbf{1}}^\varphi(\widetilde{U}, \widetilde{V}) = \frac{1}{2} \int_{[0,1]} |\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha|\, d\varphi(\alpha)$$
$$+\frac{1}{2} \int_{[0,1]} |\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha|\, d\varphi(\alpha).$$

Therefore, a necessary condition for $\mathscr{D}_\theta^\varphi(\widetilde{U},\widetilde{V})$ to vanish is that $\boldsymbol{\rho}_{\mathbf{1}}^\varphi(\widetilde{U},\widetilde{V})$ also vanishes whence, because of $\boldsymbol{\rho}_{\mathbf{1}}^\varphi$ being a metric, $\widetilde{U}=\widetilde{V}$;

- the symmetry; it is trivial that we have $\mathscr{D}_\theta^\varphi(\widetilde{U},\widetilde{V}) = \mathscr{D}_\theta^\varphi(\widetilde{V},\widetilde{U})$ whatever the fuzzy numbers $\widetilde{U},\widetilde{V}\in\mathcal{F}_c(\mathbb{R}^p)$ may be;
- the subadditivity (or triangle inequality); if $\widetilde{U},\widetilde{V},\widetilde{W}\in\mathcal{F}_c(\mathbb{R}^p)$, then since $|\cdot|$ is a norm,

$$\mathscr{D}_\theta^\varphi(\widetilde{U},\widetilde{V}) \le |\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{W})|$$

$$+|\mathrm{wabl}^\varphi(\widetilde{W})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\frac{\theta}{2}\int_{[0,1]}|\mathrm{ldev}_{\widetilde{U}}^\varphi(\alpha)-\mathrm{ldev}_{\widetilde{W}}^\varphi(\alpha)|\,d\varphi(\alpha)$$

$$+\frac{\theta}{2}\int_{[0,1]}|\mathrm{ldev}_{\widetilde{W}}^\varphi(\alpha)-\mathrm{ldev}_{\widetilde{V}}^\varphi(\alpha)|)\,d\varphi(\alpha)$$

$$+\frac{\theta}{2}\int_{[0,1]}|\mathrm{rdev}_{\widetilde{U}}^\varphi(\alpha)-\mathrm{rdev}_{\widetilde{W}}^\varphi(\alpha)|\,d\varphi(\alpha)$$

$$+\frac{\theta}{2}\int_{[0,1]}|\mathrm{rdev}_{\widetilde{W}}^\varphi(\alpha)-\mathrm{rdev}_{\widetilde{V}}^\varphi(\alpha)|\,d\varphi(\alpha),$$

i.e., we have that

$$\mathscr{D}_\theta^\varphi(\widetilde{U},\widetilde{V}) \le \mathscr{D}_\theta^\varphi(\widetilde{U},\widetilde{W})+\mathscr{D}_\theta^\varphi(\widetilde{W},\widetilde{V}).$$

ii) Following the ideas in [10] on $\mathcal{K}_c(\mathbb{R})$ and [8] on $\mathcal{F}_c(\mathbb{R})$, the support functions of elements in these spaces allow us to embed isometrically each of these spaces into a convex cone of a Banach space of functions. □

The metric space $(\mathcal{F}_c(\mathbb{R}),\mathscr{D}_\theta^\varphi)$ is separable and it can be proved thanks to the topologically equivalence (in fact, strongly equivalence) of $\mathscr{D}_\theta^\varphi$ and $\boldsymbol{\rho}_{\mathbf{1}}^\varphi$.

Proposition 2.3 *Let $\theta\in(0,1]$ be a weight parameter and let φ be an arbitrarily fixed absolutely continuous probability measure on $([0,1],\mathcal{B}_{[0,1]})$ with positive mass function in $(0,1)$. The metric $\mathscr{D}_\theta^\varphi$ is uniformly equivalent to the metric $\boldsymbol{\rho}_{\mathbf{1}}^\varphi$ on $\mathcal{F}_c(\mathbb{R})$. More precisely,*

$$\theta\cdot\boldsymbol{\rho}_{\mathbf{1}}^\varphi(\widetilde{U},\widetilde{V}) \le \mathscr{D}_\theta^\varphi(\widetilde{U},\widetilde{V}) \le (2+3\,\theta)\cdot\boldsymbol{\rho}_{\mathbf{1}}^\varphi(\widetilde{U},\widetilde{V})$$

for all $\widetilde{U},\widetilde{V}\in\mathcal{F}_c(\mathbb{R})$.

Proof Indeed, because of the properties for the absolute value we can conclude for each α that

$$|\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha)-\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha)|_\theta^1 = |\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\frac{\theta}{2}\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\inf\widetilde{U}_\alpha-\mathrm{wabl}^\varphi(\widetilde{V})+\inf\widetilde{V}_\alpha|$$

$$+\frac{\theta}{2}\cdot|\sup\widetilde{U}_\alpha-\mathrm{wabl}^\varphi(\widetilde{U})-\sup\widetilde{V}_\alpha+\mathrm{wabl}^\varphi(\widetilde{V})|.$$

Therefore, on one hand

$$|\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha)-\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha)|_\theta^1 \ge |\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\frac{\theta}{2}\cdot|\inf\widetilde{U}_\alpha-\inf\widetilde{V}_\alpha|-\frac{\theta}{2}\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\frac{\theta}{2}\cdot|\sup\widetilde{U}_\alpha-\sup\widetilde{V}_\alpha|-\frac{\theta}{2}\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$=(1-\theta)\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\theta\cdot\left[\frac{1}{2}\cdot|\inf\widetilde{U}_\alpha-\inf\widetilde{V}_\alpha|+\frac{1}{2}\cdot|\sup\widetilde{U}_\alpha-\sup\widetilde{V}_\alpha|\right]$$

$$\ge\theta\cdot\left[\frac{1}{2}\cdot|\inf\widetilde{U}_\alpha-\inf\widetilde{V}_\alpha|+\frac{1}{2}\cdot|\sup\widetilde{U}_\alpha-\sup\widetilde{V}_\alpha|\right],$$

whence one derives the first inequality.

On the other hand, one can conclude for each α that

$$|\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha)-\boldsymbol{v}_{\widetilde{U}}^\varphi(\alpha)|_\theta^1 \le |\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\frac{\theta}{2}\cdot|\inf\widetilde{U}_\alpha-\inf\widetilde{V}_\alpha|+\frac{\theta}{2}\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\frac{\theta}{2}\cdot|\sup\widetilde{U}_\alpha-\sup\widetilde{V}_\alpha|+\frac{\theta}{2}\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$=(1+\theta)\cdot|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$+\theta\cdot\left[\frac{1}{2}\cdot|\inf\widetilde{U}_\alpha-\inf\widetilde{V}_\alpha|+\frac{1}{2}\cdot|\sup\widetilde{U}_\alpha-\sup\widetilde{V}_\alpha|\right],$$

whence

$$|\mathrm{wabl}^\varphi(\widetilde{U})-\mathrm{wabl}^\varphi(\widetilde{V})|$$

$$=\left|\int_{[0,1]}\left[\mathrm{mid}\,\widetilde{U}_\alpha-\mathrm{mid}\,\widetilde{V}_\alpha\right]\,d\varphi(\alpha)\right|$$

$$\le\int_{[0,1]}|\mathrm{mid}\,\widetilde{U}_\alpha-\mathrm{mid}\,\widetilde{V}_\alpha|\,d\varphi(\alpha)$$

$$\le\int_{[0,1]}d_H(\widetilde{U}_\alpha,\widetilde{U}_\alpha)\,d\varphi(\alpha)=\boldsymbol{d}_{\mathbf{1}}^\varphi(\widetilde{U},\widetilde{V}),$$

and due to the fact that $\boldsymbol{d}_{\mathbf{1}}^\varphi(\widetilde{U},\widetilde{V})\le 2\cdot\boldsymbol{\rho}_{\mathbf{1}}^\varphi(\widetilde{U},\widetilde{V})$, one can easily derive the second inequality. □

3. The φ-wabl/ldev/rdev median for an RFN

The population and sample φ-wabl/ldev/rdev medians for random fuzzy numbers are defined as follows:

Definition 3.1 [16] *Given a probability space $(\Omega,\mathcal{A},P)$, an absolutely continuous probability measure φ on the measurable space $([0,1],\mathcal{B}_{[0,1]})$ with positive mass function on $(0,1)$, $\theta>0$ and an associated RFN $\mathcal{X}:\Omega\to\mathcal{F}_c(\mathbb{R})$, the* ***population*** *φ-wabl/ldev/rdev median(s) of $\mathcal{X}$ is (are) the fuzzy number(s)*

$$\widetilde{\mathrm{M}}^\varphi(\mathcal{X})=\arg\min_{\widetilde{U}\in\mathcal{F}_c(\mathbb{R})}E\left(\mathscr{D}_\theta^\varphi(\mathcal{X},\widetilde{U})\right),$$

whenever these expectations exist.

Definition 3.2 [16] *Given a probability space* $(\Omega, \mathcal{A}, P)$*, an absolutely continuous probability measure* φ *on the measurable space* $([0,1], \mathcal{B}_{[0,1]})$ *with positive mass function on* $(0,1)$*,* $\theta > 0$*, an associated RFN* $\mathcal{X}$ *and a simple random sample* $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ *obtained from* $\mathcal{X}$*, the* φ*-wabl/ldev/rdev median(s) of* $\mathcal{X}$ *is(are) the fuzzy number-valued statistic(s)*

$$\widehat{\widetilde{\mathrm{M}}^{\varphi}(\mathcal{X})}_n = \arg \min_{\widetilde{U} \in \mathcal{F}_c(\mathbb{R})} \frac{1}{n} \sum_{i=1}^{n} \left(\mathscr{D}_{\theta}^{\varphi}(\mathcal{X}_i, \widetilde{U}) \right).$$

Indeed, the following result guarantees the existence of at least one such median and simplifies its computation a lot.

Theorem 3.1 [16] *Given a probability space* $(\Omega, \mathcal{A}, P)$*, an absolutely continuous probability measure* φ *on the measurable space* $([0,1], \mathcal{B}_{[0,1]})$ *with positive mass function on* $(0,1)$ *and an associated RFN* $\mathcal{X}$*, for any* $\alpha \in [0,1]$*, the fuzzy number* $\widetilde{\mathrm{M}}^{\varphi}(\mathcal{X}) \in \mathcal{F}_c(\mathbb{R})$ *such that*

$$\left(\widetilde{\mathrm{M}}^{\varphi}(\mathcal{X})\right)_{\alpha} = \left[\mathrm{Me}\big(\mathrm{wabl}^{\varphi}(\mathcal{X})\big) - \mathrm{Me}\big(\mathrm{ldev}_{\mathcal{X}}^{\varphi}(\alpha)\big)\right., \quad \left.\mathrm{Me}\big(\mathrm{wabl}^{\varphi}(\mathcal{X})\big) + \mathrm{Me}\big(\mathrm{rdev}_{\mathcal{X}}^{\varphi}(\alpha)\big)\right]$$

(where in case $\mathrm{Me}\big(\mathrm{wabl}^{\varphi}(\mathcal{X})\big)$*,* $\mathrm{Me}\big(\mathrm{ldev}_{\mathcal{X}}^{\varphi}(\alpha)\big)$ *or* $\mathrm{Me}\big(\mathrm{rdev}_{\mathcal{X}}^{\varphi}(\alpha)\big)$ *are non-unique, the most usual convention for real-valued medians of choosing the mid-point of the interval of medians is considered) is a population* φ*-wabl/ldev/rdev median of* $\mathcal{X}$*.*

As it can be noticed, neither the population φ-wabl/ldev/rdev median nor its sample version depend on the parameter θ.

The φ-wabl/ldev/rdev median of a random fuzzy number fulfills most of the basic properties of the median of a random variable, like the equivariance under 'linear' transformations, the symmetry about a real number when the random fuzzy number is symmetric and the strong consistency of the sample version under some sufficient conditions (for more details, see [16]). With respect to the robustness of this proposal, the finite sample breakdown point of the sample φ-wabl/ldev/rdev median was calculated. Taking into account that this measure represents the minimum proportion of observations we have to contaminate in the sample in order to make the estimate increase arbitrarily and that its value for this median has been proved to be $\frac{1}{n} \cdot \lfloor \frac{n+1}{2} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function, the new approach has been shown to be more robust than the Aumann-type mean (whose finite sample breakdown point is $1/n$) for samples of size at least 2.

4. The influence of φ on the φ-wabl/ldev/rdev median

The simulations to study the influence of φ on the φ-wabl/ldev/rdev median have been based on the scheme in [14] (Case 1), only considering trapezoidal fuzzy numbers:

Step 1. A sample of $n = 100$ trapezoidal fuzzy numbers has been simulated in such a way that

- to generate the trapezoidal fuzzy data, we have considered four real-valued random variables as follows: $X_1 = \mathrm{mid}\mathcal{X}_1$, $X_2 = \mathrm{spr}\mathcal{X}_1$, $X_3 = \inf \mathcal{X}_1 - \inf \mathcal{X}_0$, $X_4 = \sup \mathcal{X}_0 - \sup \mathcal{X}_1$. Therefore, we are dealing with the trapezoidal fuzzy numbers $\mathrm{Tra}(X_1 - X_2 - X_3,$ $X_1 - X_2, X_1 + X_2, X_1 + X_2 + X_4)$.;
- the case in which random variables X_i are independent has been considered. More specifically, we assume that $X_1 \sim \mathcal{N}(0,1)$ and $X_2, X_3, X_4 \sim \chi_1^2$.

Step 2. We have considered as weighting measure the Beta distribution $\varphi = B(p,q)$, where both parameters p and q range in $\{.5, 1, 2, 3, 5\}$. For each considered weighting measure, the estimates of the sample φ-wabl/ldev/rdev median have been computed.

In Figure 1, it can be seen how the estimates of the sample φ-wabl/ldev/rdev for each fixed value of p vary with respect to the value of q. The main conclusion looking at these plots is that neither the location nor the shape of the sample φ-wabl/ldev/rdev are very influenced by the chosen Beta distribution, but they are scarcely affected.

Notice that, for any fixed p, increasing the value of q is equivalent to assigning more importance, when computing the wabl^{φ}, to the smaller α-levels. Since only trapezoidal fuzzy numbers are involved, we could have a look at the corresponding formulas. After simplification:

$$\mathrm{wabl}^{\varphi}(\mathcal{X}) = X_1 + \frac{X_4 - X_3}{2} \cdot \frac{q}{p+q} = \mathrm{mid}\, \mathcal{X}_{\frac{p}{p+q}}.$$

That is to say, when computing the $\mathrm{wabl}^{B(p,q)}$ of one of these observations for a fixed p, the higher the q, the smaller the $\mathrm{wabl}^{B(p,q)}$ (recall that $\mathrm{mid}\, \mathcal{X}_{\alpha}$ of a trapezoidal fuzzy number is a linear function of α). With respect to the deviations in shape:

$$\mathrm{ldev}_{\alpha}^{\varphi} = X_2 + \left(1 - \alpha - \frac{q}{2(p+q)}\right) X_3 + \frac{q}{2(p+q)} X_4,$$

$$\mathrm{rdev}_{\alpha}^{\varphi} = X_2 + \frac{q}{2(p+q)} X_3 + \left(1 - \alpha - \frac{q}{2(p+q)}\right) X_4,$$

i.e., the higher the q, the higher the influence X_4 and X_3 have on $\mathrm{ldev}_{\mathcal{X}}^{\varphi}(\alpha)$ and $\mathrm{rdev}_{\mathcal{X}}^{\varphi}(\alpha)$, respectively (and therefore, the smaller the influence X_3 and X_4 have on $\mathrm{ldev}_{\mathcal{X}}^{\varphi}(\alpha)$ and $\mathrm{rdev}_{\mathcal{X}}^{\varphi}(\alpha)$).

However, when computing the medians of the real-valued random variables wabl^{φ}, $\mathrm{ldev}_{\mathcal{X}}^{\varphi}(\alpha)$ and $\mathrm{rdev}_{\mathcal{X}}^{\varphi}(\alpha)$, any possible change could happen, like it is checked in Figure 1.

We will now consider as a real-life example the following.

Example We have adapted several questions from the TIMSS-PIRLS survey (international assessments of mathematics and science and reading at the fourth and eighth grades, whose responses have to be chosen among those in a Likert scale with 4

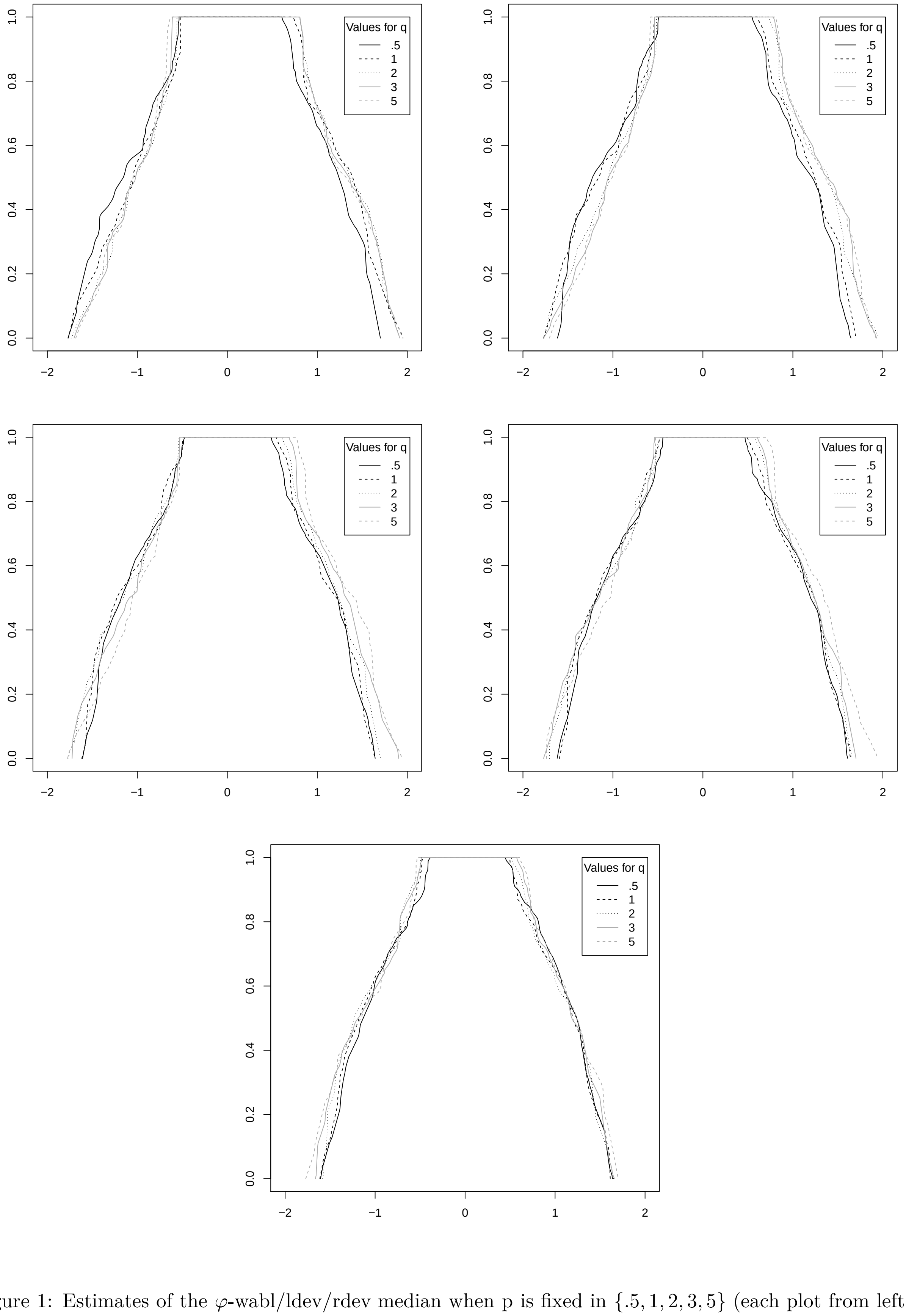

Figure 1: Estimates of the φ-wabl/ldev/rdev median when p is fixed in $\{.5, 1, 2, 3, 5\}$ (each plot from left to right and from top to bottom) and for different values of q

MS3			
inf $\widetilde{A}_0$	sup $\widetilde{A}_0$	sup $\widetilde{A}_1$	inf $\widetilde{A}_1$
5.2	6	7	7
0	0	0	2.95
6	6.425	7.35	7.875
0	0	0.775	1.225
0	0	2.15	2.15
10	10	10	10
7.9	7.95	8.7	8.7
9	9	9.4	10
9	10	10	10
4.975	4.975	5.325	5.4
7.6	7.6	8.35	8.65
2	2	4	4
0	0	0	0.45
9.975	10	10	10
0	0	1.575	1.575
2.225	2.225	3.125	3.125
1.9	1.95	3	3.15
4.875	5.05	5.45	5.625
6.15	6.15	6.75	6.75
3.45	3.45	4.425	4.425
2.5	3.2	3.3	4.45
3	3.6	4.2	5.05
3	3	3	3
0	0.6	1.25	1.65
0	0.5	0.5	1
2.5	3	3.6	3.6
3	3.2	3.6	4.2
10	10	10	10
10	10	10	10
6.975	6.975	7.925	7.925
0	0	2.575	2.575
6	6.45	7.4	8
2.35	2.8	3.25	3.5
3.15	3.4	3.6	4
0	0.625	2.725	2.75
0	0	1	2
4.925	5.025	5.95	6.3
10	10	10	10
0	0.825	2.425	2.425
0	0.325	1.475	1.475
5.15	5.35	6.15	6.15
8.55	8.85	9.625	10
0	0	0	0.725
0	0	0	0
10	10	10	10
7	7.4	8.2	8.4
4.05	4.05	4.7	4.775
10	10	10	10
0	0.85	1.5	1.825
1.6	1.825	2.425	3.075
3.125	3.275	3.7	4.05
9.9	9.9	10	10
0	0	1.125	1.125
6	6	7	8
1	1.8	2.35	3.1
0	0.075	1	1.35
0	0	0	0
0	0.4	0.95	1.75
10	10	10	10
8.8	8.8	9.5	9.575
4.6	6.15	6.15	6.85
3.6	3.925	4.575	4.575
3.875	3.875	5.6	5.6
0	0.25	1.025	1.025
10	10	10	10
0.3	0.45	1.15	1.5
5.5	6.1	6.9	7.4
6.325	6.925	7.175	7.65

Table 1: Fuzzy rating scale-based responses given by 4th grade students in Colegio San Ignacio (Oviedo, Spain)

points, namely, DISAGREE A LOT, DISAGREE A LITTLE, AGREE A LITTLE and AGREE A LOT. See see http://timss.bc.edu/timss2011/downloads/T11_StuQ_4.pdf) in order to get answers in the fuzzy rating scale introduced by [4], which allows to combine a free-response format with a fuzzy valuation.

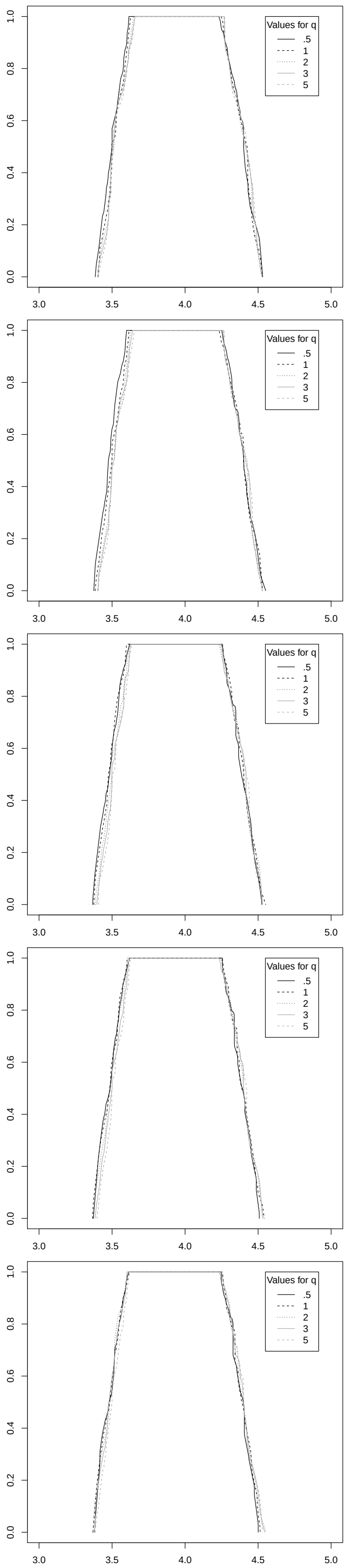

Figure 2: Estimates of the φ-wabl/ldev/rdev median when p is fixed in $\{.5, 1, 2, 3, 5\}$ (each plot from left to right and from top to bottom) and for different values of q in the *Example*

The questionnaire has been conducted on the fourth grade students of the Colegio San Ignacio in Oviedo-Asturias (Spain), being formulated with a double-type response (namely, Likert scale and fuzzy rating scale-based). Data from one these adapted questions will be considered to show how the φ-wabl/ldev/rdev median is not too influenced by the choice of the weighting measure $\varphi = B(p, q)$. The considered question from the survey has been *MS3. Mathematics is harder for me than any other subject* and the collected data is shown in Table 1.

The results are shown in Figure 2. Again, the use of different parameters when considering the Beta distribution as weighting measure φ scarcely affects the results, what makes this proposal, together with its high finite sample breakdown point, a robust alternative to the Aumann-type mean and scarcely dependent on the choice of φ.

5. Concluding remarks

The recently introduced φ-wabl/ldev/rdev median for random fuzzy numbers (see [16]) has been recalled, as well as the distance this notion is based on. Two properties of such metric have been proved in this manuscript. The aim of studying in a preliminary way the influence the choice of the weighting measure φ has on the computation of the φ-wabl/ldev/rdev median has been carried out by generating a random sample like in the literature (see [14] and by considering an applied example.

However, this study is clearly preliminary and more detailed simulations should be performed. Although trapezoidal fuzzy numbers are very common when dealing with real life examples because of being easier-to-handle, other shapes could be taken into account, as well as many others weighting measures that are left as open problems.

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