Regular fuzzy equivalences and regular fuzzy quasi-orders

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Abstract

The notion of social roles is a centerpiece of most sociological theoretical considerations. Regular equivalences arise as a result of an attempt to capture the sociological notion of a relational or structural role. Regular equivalences were introduced by White and Reitz in [28] as the least restrictive among the most commonly used definitions of the equivalence in social network analysis. In recent years, with development of fuzzy social network theory, regular fuzzy equivalences are widely explored. In this paper we consider a generalization of this notion to a bipartite case. We define a pair of regular fuzzy equivalences on a two-mode social network and we provide an algorithm for computing the greatest pair of regular fuzzy equivalences.

Keywords: Two-mode fuzzy network, regular fuzzy equivalence, regular fuzzy quasi-order, fuzzy relation equation, residual of fuzzy relations, social network analysis.

1. Introduction

A very important task of the social network theory is to find such similarities between entities which indicate that some of them occupy the same position or have the same role in a network. Lorrain and White [28], Breiger et al. [9] and Burt [10] were the first to formalize these similarities. They did so by introducing the concept of the structural equivalence. Two entities are considered to be structurally equivalent if they have identical links to the rest of the network. Structural equivalences were extensively studied in [1, 5, 3, 18, 19, 20, 21]. Afterwards, White and Reitz [33] generalized the concept of a structural equivalence, by introducing the notion of a regular equivalence. The main difference between regular and structural equivalences is that the entities are considered to be regularly equivalent if they are equally related to equivalent others [6, 22], not equally related to the rest of the network. Regular equivalences have been studied in numerous of papers (cf. [24, 25]).

Regular equivalences arise as a result of an attempt to capture the sociological notion of a relational or structural role. Regular equivalence analysis identifies social roles by identifying regularities in the patterns of network ties. Regular equivalence classes are composed of entities who have similar relations to members of other regular equivalence classes. The concept does not refer to ties to specific other entities, in other words, entities are regularly equivalent if they have similar ties to any members of other classes.

In recent years, fuzzy social network theory have also received considerable attention. The fuzzy social network theory arises as an attempt to overcome vagueness which is present in social networks, and as the means to represent both the qualitative relationship and the degrees of interaction between entities. Fan, Liau and Lin [23] generalized the notion of regular equivalence to fuzzy social networks based on two alternative definitions of a regular equivalence. The first generalization, that they called the regular similarity, specifies the degree of similarity between entities in the social network, whereas the second generalization, called the generalized regular equivalence, determines the crisp partition of the entities in a fuzzy social network. The first concept has been discussed from another aspect in [11, 24, 25], where the name the regular fuzzy equivalence has been introduced.

In the present paper we generalize the notion of a regular equivalence to the bipartite case in the fuzzy framework. We introduce the notions of a pair of regular fuzzy equivalences and a pair of regular fuzzy-quasi orders. We consider a two-mode fuzzy network – an ordered triple $\mathcal{A} = (A, B, R)$, where A and B are non-empty sets and R is a fuzzy relation between A and B, and define a pair of regular fuzzy equivalences on $\mathcal{A}$ as a pair (E, F) of fuzzy equivalences on A and B, respectively, satisfying $E \circ R = R \circ F$. Similar fuzzy relation equations and inequalities have been recently extensively studied by Ćirić, Ignjatović and others in [13, 14, 15, 16, 17, 24, 25, 26, 27], where algorithms for computing their greatest solutions have been provided. Using the general ideas presented in these studies and of the well known Paige-Tarjan partition refinement algorithm [29], here we develop efficient procedures for computing the greatest pairs of regular fuzzy equivalences and regular fuzzy-quasi orders on two-mode fuzzy networks.

The paper is organized as follows. In Section 2 we give some basic properties of fuzzy relations, fuzzy equivalences and fuzzy-quasi orders. In particular, we define the right and left residuals of fuzzy relations and fuzzy sets. In Section 3 we define the notion of a regular fuzzy equivalence on a two-mode fuzzy network, and we provide an algorithm for computing the greatest pair of regular fuzzy equivalences on the given two-mode fuzzy network. Finally, in Section 4, we define the notion of a regular fuzzy-quasi order on a two-mode fuzzy network, and we provide a procedure for computing the greatest pair of regular fuzzy-quasi orders on the given two-mode fuzzy network.

2. Preliminaries

We will use complete residuated lattices as the structures of membership (truth) values.

A *residuated lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,

(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,

(L3) $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \;\Leftrightarrow\; x \leqslant y \rightarrow z. \tag{1}$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathcal{L}$ is called a *complete residuated lattice*.

On the complete residuated lattice operation biimplication is defined as follows:

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

In the further text $\mathcal{L}$ will be a complete residuated lattice. A *fuzzy subset* of a set A *over* $\mathcal{L}$, or simply a *fuzzy subset* of A, is any function from A into L. The *equality* of f and g is defined as the usual equality of functions, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. Endowed with this partial order the set $\mathcal{F}(A)$ of all fuzzy subsets of A forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i\in I} f_i$ and the join (union) $\bigvee_{i\in I} f_i$ of an arbitrary family $\{f_i\}_{i\in I}$ of fuzzy subsets of A are functions from A into L defined by

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x),$$
$$\left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x),$$

and the *product* $f \otimes g$ is a fuzzy subset defined by $(f \otimes g)(x) = f(x) \otimes g(x)$, for every $x \in A$.

Let A and B be non-empty sets. A *fuzzy relation between sets* A and B is any function from $A \times B$ into L, and the equality, inclusion (ordering), joins and meets of fuzzy relations are defined as for fuzzy sets. In particular, a *fuzzy relation on a set* A is any function from $A \times A$ into L. Specially, universal fuzzy relation on A, denoted by U_A is defined in the following way

$$U_A(a, b) = 1, \quad \text{for any } a, b \in A.$$

The set of all fuzzy relations from A to B will be denoted by $\mathcal{R}(A, B)$, and the set of all fuzzy relations on a set A will be denoted by $\mathcal{R}(A)$. The *converse* (in some sources called *inverse* or *transpose*) of a fuzzy relation $R \in \mathcal{R}(A, B)$ is a fuzzy relation $R^{-1} \in \mathcal{R}(B, A)$ defined by $R^{-1}(b, a) = R(a, b)$, for all $a \in A$ and $b \in B$.

For fuzzy relation $R \in \mathcal{R}(\mathcal{A}, \mathcal{B})$ and $a \in A$ we define the fuzzy sets $R_a \in \mathcal{F}(B)$ and $R^a \in \mathcal{F}(B)$ by:

$$R_a(b) = R(a, b), \text{ for any } b \in B,$$
$$R^a(b) = R(b, a), \text{ for any } b \in B.$$

For non-empty sets A, B, C and fuzzy relations $\varphi \in \mathcal{R}(A, B)$ and $\psi \in \mathcal{R}(B, C)$, their *composition* $\varphi \circ \psi$ is a fuzzy relation from $\mathcal{R}(A, C)$ defined by

$$(\varphi \circ \psi)(a, c) = \bigvee_{b\in B} \varphi(a, b) \otimes \psi(b, c),$$

for all $a \in A$ and $c \in C$. Next, if $f \in \mathcal{F}(A)$, $\varphi \in \mathcal{R}(A, B)$ and $g \in \mathcal{F}(B)$, the compositions $f \circ \varphi$ and $\varphi \circ g$ are fuzzy subsets of B and A, respectively, which are defined by

$$(f \circ \varphi)(b) = \bigvee_{a\in A} f(a) \otimes \varphi(a, b),$$
$$(\varphi \circ g)(a) = \bigvee_{b\in B} \varphi(a, b) \otimes g(b),$$

for every $a \in A$ and $b \in B$.

In particular, for fuzzy subsets f and g of A we write

$$f \circ g = \bigvee_{a\in A} f(a) \otimes g(a).$$

A fuzzy relation R on A is said to be:

(R) *reflexive* (or *fuzzy reflexive*) if $R(a, a) = 1$, for every $a \in A$;

(S) *symmetric* (or *fuzzy symmetric*) if $R(a, b) = R(b, a)$, for all $a, b \in A$;

(T) *transitive* (or *fuzzy transitive*) if we have that $R(a, b) \otimes R(b, c) \leqslant R(a, c)$, for all $a, b, c \in A$.

It can easily be shown, that $R \circ R = R$ holds for any reflexive and transitive fuzzy relation R on A.

A reflexive and transitive fuzzy relation on A is called a *fuzzy quasi-order*. A reflexive, symmetric and transitive fuzzy relation on A is called a *fuzzy equivalence*. With the respect to the inclusion of fuzzy relations, the set $\mathcal{E}(A)$ of all fuzzy equivalences on A is a complete lattice.

Let A and B be non-empty sets and $\eta \in \mathcal{F}(A)$ and $\xi \in \mathcal{F}(B)$ be fuzzy subsets of A and B, respectively. The *right residual* of ξ by η is a fuzzy relation $\eta\backslash\xi \in \mathcal{R}(A, B)$ defined by

$$(\eta\backslash\xi)(a, b) = \eta(a) \rightarrow \xi(b), \tag{2}$$

for all $a \in A$ and $b \in B$, and the *left residual* of ξ by η is a fuzzy relation $\xi/\eta \in \mathcal{R}(B, A)$ defined by

$$(\xi/\eta)(b, a) = \eta(a) \rightarrow \xi(b)\,, \tag{3}$$

for all $a \in A$ and $b \in B$. Clearly, $\eta \backslash \xi = (\xi/\eta)^{-1}$.

The right and left residual are very important in determining the solution to certain fuzzy relational inequalities:

Lemma 2.1 *Let A and B be non-empty sets and let $\eta \in \mathcal{F}(A)$ and $\xi \in \mathcal{F}(B)$.*

(a) *The set of all solutions to the inequality*

$$\eta \circ \chi \leqslant \xi,$$

where χ denotes an unknown taking values in $\mathcal{R}(A, B)$, is the principal ideal of the lattice $\mathcal{R}(A, B)$ generated by the fuzzy relation $\eta \backslash \xi$.

(b) *The set of all solutions to the inequality*

$$\chi \circ \xi \leqslant \eta,$$

where χ denotes an unknown taking values in $\mathcal{R}(A, B)$, is the principal ideal of the lattice $\mathcal{R}(A, B)$ generated by the fuzzy relation η/ξ.

Note that $(\eta\backslash\xi) \wedge (\eta/\xi) = \eta \,|\, \xi$, where $\eta \,|\, \xi$ is a fuzzy relation between A and B, called the right-left (or left-right) residual of η by ξ, defined by

$$(\eta \,|\, \xi)(a, b) = \eta(a) \leftrightarrow \xi(b), \tag{4}$$

for arbitrary $a \in A$ and $b \in B$. It is interesting to consider the right-left residual in case $A = B$ and $\eta = \xi$. Then, we have the following result from [2]:

Lemma 2.2 *Let f be a fuzzy subset of a set A. Then the fuzzy relation $f \,|\, f$ is a fuzzy equivalence.*

In the sequel we recall some well-known results [2] concerning fuzzy equivalence relations, which will be needed in the further work:

Lemma 2.3 *Let E and F be fuzzy equivalences on a set A. Then the fuzzy relation $E \wedge F$ is also a fuzzy equivalence on A.*

Lemma 2.4 *Let E and F be fuzzy equivalences on a set A such that $E \leqslant F$. Then $E \circ F = F \circ E = F$.*

As a consequence of Lemma 2.4 and the adjunction property, we have the following result:

Lemma 2.5 *Let E and F be fuzzy equivalences on a set A such that $E \leqslant F$. Then*

$$E \leqslant \bigwedge_{a \in A} (F^a \,|\, F^a).$$

3. Regular fuzzy equivalences

Let $\mathcal{A} = (A, B, R)$ be a two-mode fuzzy network, and let E and F be fuzzy equivalences on A and B, respectively. The pair (E, F) is called a *pair of regular fuzzy equivalences* on the network $\mathcal{A}$ if

$$E \circ R = R \circ F. \tag{5}$$

We have the following:

Theorem 3.1 *Let $\mathcal{A} = (A, B, R)$ be a two-mode fuzzy network, and let E and F be fuzzy equivalences on A and B, respectively. Then the pair (E, F) is a pair of regular fuzzy equivalences if and only if the following is true:*

$$E \circ R \circ F = E \circ R \wedge R \circ F. \tag{6}$$

Proof. Let (E, F) be a pair of regular fuzzy equivalences. Then (E, F) satisfies (5), so

$$E \circ R \circ F = R \circ F \circ F = R \circ F,$$

and similarly $E \circ R \circ F = E \circ R$. Thus, (6) holds.

On the other hand, let (6) holds. Hence,

$$E \circ R \circ F \leqslant R \circ F \text{ and } E \circ R \circ F \leqslant E \circ R$$

holds. Directly from the fact E and F are reflexive we obtain $R \circ F \leqslant E \circ R \circ F$ and $E \circ R \leqslant E \circ R \circ F$. So

$$R \circ F = E \circ R \circ F \text{ and } E \circ R = E \circ R \circ F,$$

which means that (5) holds. □

Theorem 3.2 *Let $\mathcal{A} = (A, B, R)$ be a two-mode fuzzy network, and let E and F be fuzzy equivalences on A and B, respectively. Then the pair (E, F) is a pair of regular fuzzy equivalences if and only if the following is true:*

$$E \leqslant \bigwedge_{b \in B} (R \circ F^b \,|\, R \circ F^b),$$
$$F \leqslant \bigwedge_{a \in A} (E^a \circ R \,|\, E^a \circ R). \tag{7}$$

Proof. Let (E, F) be a pair of regular fuzzy equivalences. Then according to Theorem 3.1 we have that the pair (E, F) satisfies (6). Therefore,

$$E \circ R \circ F \leqslant R \circ F \text{ and } E \circ R \circ F \leqslant E \circ R.$$

Therefore, we have:

$$\bigvee_{c \in A} E(a, c) \otimes (R \circ F)(c, b) \leqslant (R \circ F)(a, b),$$
$$\bigvee_{c \in B} (E \circ R)(a, c) \otimes F(c, b) \leqslant (E \circ R)(a, b),$$

for all $a \in A$ and $b \in B$. As a consequence of the adjunction property we have:

$$E(a, c) \leqslant (R \circ F)(c, b) \to (R \circ F)(a, b),$$

for all $a, c \in A$, $b \in B$, and

$$F(c, b) \leqslant (E \circ R)(a, c) \to (E \circ R)(a, b),$$

for all $a \in A$, $b, c \in B$, that is

$$E(a, c) \leqslant ((R \circ F^b)/(R \circ F^b))(a, c),$$

for all $a, c \in A$, $b \in B$, and

$$F(c, b) \leqslant ((E^a \circ R)\backslash(E^a \circ R))(c, b)$$

for all $a \in A$, $b, c \in B$. Using these formulas and the fact that E and F are symmetric, it follows that

$$E(a,c) \leqslant ((R \circ F^b) \backslash (R \circ F^b))(a,c),$$

for all $a, c \in A$, $b \in B$, and

$$F(c,b) \leqslant ((E^a \circ R)/(E^a \circ R))(c,b),$$

for all $a \in A$, $b, c \in B$. Hence, (7) holds.

On the other hand, let (7) holds. Using a similar procedure to the one described above we obtain

$$E \circ R \circ F \leqslant R \circ F \text{ and } E \circ R \circ F \leqslant E \circ R.$$

According to the fact that E and F are reflexive we obtain

$$R \circ F \leqslant E \circ R \circ F \text{ and } E \circ R \leqslant E \circ R \circ F.$$

Therefore, we have that

$$R \circ F = E \circ R \circ F \text{ and } E \circ R = E \circ R \circ F,$$

which means that (5) holds, that is, (E, F) is a pair of regular fuzzy equivalences. □

Let (A, B, R) be a two-mode fuzzy network and let E and F be fuzzy equivalences on A and B, respectively. With $\mathcal{L}(E, F, R)$ we will denote the subalgebra of $\mathcal{L}$ generated by all membership values taken by E, F and R.

Recall the notion of descending chain condition, which will be needed in further work.

A partially ordered set P is said to satisfy the *descending chain condition* (briefly *DCC*) if every descending sequence of elements of P eventually terminates, i.e., if for every descending sequence $\{a_k\}_{k\in\mathbb{N}}$ of elements of P there is $k \in \mathbb{N}$ such that $a_k = a_{k+l}$, for all $l \in \mathbb{N}$. In other words, P satisfies DCC if there is no infinite descending chain in P.

The next theorem provides a method for computing the pair of greatest regular fuzzy equivalences on a given two-mode fuzzy network.

Theorem 3.3 *Let (A, B, R) be a two-mode fuzzy network and let E and F be fuzzy equivalences on A and B, respectively.*

Let us define the sequences $\{(E_k, F_k)\}_{k\in\mathbb{N}}$ and $\{(X_k, Y_k)\}_{k\in\mathbb{N}}$ as follows: Initially for $k = 1$

$$(X_1, Y_1) = (U_A, U_B), \tag{8}$$
$$(E_1, F_1) = (E, F) \wedge \big((R \circ U_B^b) \,|\, (R \circ U_B^b), (U_A^a \circ R) \,|\, (U_A^a \circ R)\big),$$

where $a \in A$ and $b \in B$ are arbitrary elements.

Further, for each $k \in \mathbb{N}$ repeat the following: Find $a \in A$ and $b \in B$ such that $(X_k^a, Y_k^b) \neq (E_k^a, F_k^b)$ and set

$$(X_{k+1}, Y_{k+1}) = (X_k, Y_k) \wedge (E_k^a \,|\, E_k^a, F_k^b \,|\, F_k^b), \tag{9}$$

$$(E_{k+1}, F_{k+1}) = (E_k, F_k) \wedge \tag{10}$$
$$\Big(\bigwedge_{b'\in B} (R \circ Y_{k+1}^{b'} | R \circ Y_{k+1}^{b'}), \bigwedge_{a'\in A} (X_{k+1}^{a'} \circ R \,|\, X_{k+1}^{a'} \circ R)\Big),$$

until $(X_k, Y_k) = (E_k, F_k)$. Then:

(a) *Sequences $\{(E_k, F_k)\}_{k\in\mathbb{N}}$ and $\{(X_k, Y_k)\}_{k\in\mathbb{N}}$ are descending;*
(b) *For every $k \in \mathbb{N}$, $E_k \leqslant X_k$ and $F_k \leqslant Y_k$;*
(c) *For every $k \in \mathbb{N}$ the following holds:*

$$E_k \leqslant \bigwedge_{b'\in B} (R \circ Y_k^{b'}) \,|\, (R \circ Y_k^{b'}),$$
$$F_k \leqslant \bigwedge_{a'\in A} (X_k^{a'} \circ R) \,|\, (X_k^{a'} \circ R);$$

(d) *If there exists $n \in \mathbb{N}$ such that $(X_n, Y_n) = (E_n, F_n)$, then (E_n, F_n) is the greatest pair of regular fuzzy equivalences contained in (E, F);*
(e) *If $\mathcal{L}(E, F, R)$ satisfies DCC, then there exists $n \in \mathbb{N}$ such that $(X_n, Y_n) = (E_n, F_n)$.*

Proof. (a) This follows directly from the definition of these sequences;

(b) We will prove only that $E_k \leqslant X_k$ holds for every $k \in \mathbb{N}$, since the opposite inequality can be proved analogously.

We prove this inequality by induction on $k \in \mathbb{N}$.

For $k = 1$, it is clear that $E_1 \leqslant X_1$.

Suppose that $E_m \leqslant X_m$, for some $m \in \mathbb{N}$. We have to prove that $E_{m+1} \leqslant X_{m+1}$.

By Lemma 2.5 it follows $E_m \leqslant (E_m^a \,|\, E_m^a)$, and by the induction hypothesis we have $E_m \leqslant X_m$. Thus, $E_m \leqslant X_m \wedge (E_m^a \,|\, E_m^a) = X_{m+1}$, and since $\{E_k\}_{k\in\mathbb{N}}$ is descending, we have that $E_{m+1} \leqslant X_{m+1}$, which was to be proved.

(c) We will prove only the first inequality, the second one can be proved in an analogous way. We will consider only the case $k = 1$. For $k > 1$ it is evident from the definition of (E_{k+1}, F_{k+1}). Let us first note that all Y_1^c of Y_1 are equal to each other, that is, for any $c \in B$, Y_1^c is defined by $Y_1^c(b) = 1$, for every $b \in A$. According to this fact and the definition of E_1 we have :

$$E_1 \leqslant (R \circ Y_1^c \,|\, R \circ Y_1^c),$$

for every $c \in A$, and hence, (c) holds for $k = 1$.

(d) If $(E_k, F_k) = (X_k, Y_k)$, for some $k \in \mathbb{N}$, then according to (c) the following holds:

$$(E_k, F_k) \leqslant$$
$$\Big(\bigwedge_{b'\in B} (R \circ F_k^{b'}) \,|\, (R \circ F_k^{b'}), \bigwedge_{a'\in A} (E_k^{a'} \circ R) \,|\, (E_k^{a'} \circ R)\Big).$$

which means that (E_k, F_k) is a pair of regular fuzzy equivalences. In order to show that (E_k, F_k) is the greatest pair, consider an arbitrary pair (E', F') of regular fuzzy equivalences on $\mathcal{A}$ contained in (E, F).

By induction on n we will prove that $(E', F') \leqslant (E_n, F_n)$, for every $n \in \mathbb{N}$,

For $n = 1$ we have that (E', F') and (U_A, U_b) are fuzzy equivalences such that $(E', F') \leqslant (U_A, U_B) = (X_1, Y_1)$. According to Theorem 3.1, the fuzzy equivalence E' satisfies $E' \circ R \circ F' \leqslant R \circ F'$. By $F' \leqslant Y_1$, multiplying this inequality with Y_1 on the right, we obtain:

$$E' \circ R \circ F' \circ Y_1 \leqslant R \circ F' \circ Y_1.$$

Next, according to Lemma 2.4, we have

$$E' \circ R \circ Y_1 \leqslant R \circ Y_1,$$

and by the proof of Theorem 3.2 we have that

$$E' \leqslant (R \circ Y_1^{b'}) \,|\, (R \circ Y_1^{b'}),$$

for all $b' \in B$, and since $E' \leqslant E$ we conclude that $E' \leqslant E_1$. In a similar way we show that $F' \leqslant F_1$.

Suppose that $(E', F') \leqslant (E_m, F_m)$ holds for some $m \in \mathbb{N}$, and prove that $(E', F') \leqslant (E_{m+1}, F_{m+1})$.

Since $E' \leqslant E_m$, according to (b) we obtain that $E' \leqslant X_m$, and by Lemma 2.5 it follows $E' \leqslant X_{m+1}$, and similarly, $F' \leqslant Y_{m+1}$. Now, if we again use the inequality $E' \circ R \circ F' \leqslant R \circ F'$ and the fact that $F' \leqslant Y_{m+1}$, we obtain $E' \leqslant X_{m+1}$, and similarly, $F' \leqslant F_{m+1}$, which was to be proved.

(e) Let $\mathcal{L}(E, F, R)$ satisfy DCC. Then fuzzy relations from the sequence $\{X_k\}_{k\in\mathbb{N}}$ can be considered as fuzzy matrices with entries in $\mathcal{L}(E, F, R)$, and for any pair $(a, b) \in A \times A$, the (a, b)-entries of these matrices form a decreasing sequence $\{X_k(a, b)\}_{k\in\mathbb{N}}$ of elements of $\mathcal{L}(E, F, R)$. By the hypothesis, all these sequences stabilize, and since there is a finite number of such sequences, there exists $s \in \mathbb{N}$ such that after s steps all these sequences stabilize. This means that the sequence $\{X_k\}_{k\in\mathbb{N}}$ of fuzzy equivalences also stabilizes after s steps, i.e., $X_s = X_{s+1}$.

Next, we will prove that if $X_s = X_{s+1}$, for some $s \in \mathbb{N}$, then $E_s = X_s$. If $X_s = X_{s+1}$ then

$$X_s = X_{s+1} = X_s \wedge (E_s^a \,|\, E_s^a),$$

and thus, $X_s \leqslant E_s^a \,|\, E_s^a$. Consequently, $X_s^a \leqslant E_s^a$, and since $E_s^a \leqslant X_s^a$ we obtain $X_s^a = E_s^a$. This means that there is no class X_s^a of X_s such that $X_s^a \neq E_s^a$, or equivalently $X_s = E_s$. □

Using Theorem 3.3 we construct an algorithm for computing the greatest pair of regular fuzzy equivalences on a fuzzy network (A, B, R) contained in a given pair (E, F) of fuzzy equivalences.

Algorithm 3.4 (*Construction of the greatest pair of regular fuzzy equivalences*) The input of this algorithm is a two-mode fuzzy network (A, B, R) and fuzzy equivalences E and F on A and B, respectively. The output of the algorithm is the greatest pair of regular fuzzy equivalences contained in (E, F).

The procedure is to construct the sequences of fuzzy relations $\{(X_k, Y_k)\}_{k\in\mathbb{N}}$ and $\{(E_k, F_k)\}_{k\in\mathbb{N}}$, in the following way:

(A1) In the first step we compute (X_1, Y_1) and (E_1, F_1) using formula (8).

(A2) After the kth step let the pairs (X_k, Y_k) and (E_k, F_k) have been constructed.

If $(X_k, Y_k) = (E_k, F_k)$, then the procedure terminates and (E_k, F_k) is the greatest pair of regular fuzzy equivalences contained in (E, F).

Otherwise, if $(X_k, Y_k) \neq (E_k, F_k)$, we construct fuzzy relations (X_{k+1}, Y_{k+1}) and (E_{k+1}, F_{k+1}) by means of formulas (9) and (10).

Note again that in the case when $\mathcal{L}(E, F, R)$ satisfies DCC, this algorithm terminates in a finite number of steps.

Example 3.5 *Let (A, B, R) be a two-mode Boolean network with $A = \{a_1, a_2, \ldots, a_8\}$, $B = \{b_1, b_2, \ldots, b_{12}\}$, and a relation R given by the following matrix:*

$$R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Initial matrix E and F are given by:

$$E_{start} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$F_{start} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Using the previous algorithm we obtain the greatest pair of regular equivalences (E', F')

$$E' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$F' = \begin{pmatrix} 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 1&1&1&1&1&1&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$$

4. Regular fuzzy quasi-orders

Note that if we consider inequality (5) and require that E and F are fuzzy quasi-orders, we obtain even greater solutions than in the case when solutions are required to be fuzzy equivalences.

The following theorem can be proved in a similar way as Theorem 3.3, so we will omit the proof.

Theorem 4.1 *Let $\mathcal{A} = (A, B, R)$ be a two-mode fuzzy network and let $P \in \mathcal{Q}(A)$ and $Q \in \mathcal{Q}(B)$ be fuzzy quasi-oreds.*

Let us define the sequences $\{(P_k, Q_k)\}_{k\in N}$ and $\{(X_k, Y_k)\}_{k\in N}$ as follows: Initially for $k = 1$

$$\begin{aligned} &(X_1, Y_1) = (U_A, U_B), \\ &(P_1, Q_1) = (P, Q)\wedge \\ &\quad \big((R \circ U_B^b) \,/\, (R \circ U_B^b), (U_A^a \circ R) \setminus (U_A^a \circ R)\big), \end{aligned} \tag{11}$$

where $a \in A$ and $b \in B$ are arbitrary elements.

Next, for each $k \in \mathbb{N}$ do the following: Find $a \in A$ and $b \in B$ such that $(X_k^a, Y_k^b) \neq (P_k^a, Q_k^b)$ and set

$$\begin{aligned} (X_{k+1}, Y_{k+1}) = (X_k, Y_k) \wedge \\ (P_k^a \,/\, P_k^a, Q_k^b \setminus Q_k^b), \end{aligned} \tag{12}$$

$$\begin{aligned} (P_{k+1}, Q_{k+1}) = (P_k, Q_k) \wedge \\ \Big(\bigwedge_{b'\in B} (R \circ Y_{k+1}^{b'} / R \circ Y_{k+1}^{b'}) \\ \bigwedge_{a'\in A} (X_{k+1}^{a'} \circ R \setminus X_{k+1}^{a'} \circ R)\Big), \end{aligned} \tag{13}$$

until $(X_k, Y_k) = (P_k, Q_k)$. Then:

(a) *Sequences $\{(P_k, Q_k)\}_{k\in N}$ and $\{(X_k, Y_k)\}_{k\in N}$ are descending;*

(b) *For every $k \in \mathbb{N}$, $P_k \leqslant X_k$ and $Q_k \leqslant Y_k$;*

(c) *For every $k \in \mathbb{N}$, the following holds :*

$$P_k \leqslant \bigwedge_{b'\in B} (R \circ Y_k^{b'}) \,/\, (R \circ Y_k^{b'}),$$

$$Q_k \leqslant \bigwedge_{a'\in A} (X_k^{a'} \circ R) \setminus (X_k^{a'} \circ R);$$

(d) *If there is $n \in \mathbb{N}$ such that $(X_n, Y_n) = (P_n, Q_n)$ then (P_n, Q_n) is the greatest pair of regular fuzzy quasi-orders contained in (P, Q);*

(e) *If $\mathcal{L}(P, Q, R)$ satisfies DCC, then there exists $n \in \mathbb{N}$ such that $(X_n, Y_n) = (P_n, Q_n)$.*

Based on Theorem 4.1 we construct an algorithm which computes the greatest pair of regular fuzzy quasi-orders on a fuzzy network (A, B, R) contained in (P, Q), where (P, Q) is a given pair of fuzzy-quasi orders on A and B, respectively.

Algorithm 4.2 (*Construction of the greatest pair of regular fuzzy-quasi orders*) The input of the algorithm is a two-mode fuzzy network (A, B, R) and fuzzy-quasi orders $P \in \mathcal{Q}(A)$ and $Q \in \mathcal{Q}(B)$. The output of the algorithm is the greatest pair of regular fuzzy-quasi orders contained in (P, Q).

The procedure is to construct the sequences of fuzzy relations $\{(X_k, Y_k)\}_{k\in\mathbb{N}}$ and $\{(P_k, Q_k)\}_{k\in\mathbb{N}}$, in the following way:

(A1) In the first step we compute (X_1, Y_1) and (P_1, Q_1) using formula (11).

(A2) After the kth step let the pairs (X_k, Y_k) and (P_k, Q_k) have been constructed.

If $(X_k, Y_k) = (P_k, Q_k)$, then the procedure terminates and (P_k, Q_k) is the greatest pair of regular fuzzy-quasi orders contained in (P, Q).

Otherwise, if $(X_k, Y_k) \neq (P_k, Q_k)$, then we construct the pairs (X_{k+1}, Y_{k+1}) and (P_{k+1}, Q_{k+1}) by means of formulas (12) and (13).

In the case when $\mathcal{L}(P, Q, R)$ satisfies DCC, this algorithm terminates in a finite number of steps.

5. Conclusion

In this paper we developed an efficient algorithm for computing the greatest pair of regular equivalences on a two-mode fuzzy network. The method computes the exact pair of regular equivalence. However when dealing with social networks it is usually better to have a good approximation of the solution than the exact one, therefore in the further work we plan to develop this algorithm to provide some good approximations.

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