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Information retrieval from interval-valued fuzzy automata through K_α operators

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Abstract

Here we study the notions of lattice-valued finite state machine and lattice-valued fuzzy transformation semigroup when the lattice consists of all the closed subintervals contained in $[0, 1]$.

So, we can apply techniques of interval-valued fuzzy sets to fuzzy automata. In particular, we prove that Atanassov's K_α operators allow us to retrieve the information given by the transition functions of the interval-valued automata. Moreover, we analyze the functorial nature of these operators and characterize those K_α operators that provide a fuzzy transformation semigroup from an interval-valued one.

Keywords: Interval-valued fuzzy sets, finite state machines, Atanassov's K_α operators, transformation semigroups.

1. Introduction

Fuzzy finite state machines (ffsm) and fuzzy transformation semigroups (fts) have been widely studied in the literature [10]. Unlike the crisp case, the basic idea in their formulation is that they can switch from one state to another one to a certain truth degree between 0 and 1.

So, a ffsm is defined as a triple (Q, X, μ) where Q and X are finite sets and μ is a map from $Q \times X \times Q$ to $[0, 1]$. In a similar way, a fts is defined as a triple (Q, U, δ) where Q is a set, U is a finite semigroup and $\delta : Q \times U \times Q \to [0, 1]$ is a map satisfying

$$\delta(q, uv, p) = \bigvee\{\delta(q, u, r) \wedge \delta(r, v, p) \mid r \in Q\},$$

for any p, $q \in Q$ and u, $v \in U$.

Notice that neither initial nor final states are considered in these kinds of fuzzy automata.

Furthermore, the relationships between these automata have been deeply analyzed by different authors [10], [7], [8]. Other ones pave the way to consider any complete lattice as the truth structure of an automata (see [11], [9]). In [4] the truth structure of the automata previously described (ffsms and ftss) is extended to any complete lattice endowed with a t-norm and a t-conorm satisfying both a finiteness and a distributive properties. As a particular case the automata whose truth structure is the lattice of all the closed intervals contained in L are analyzed.

Many recent studies have shown that the use of intervals allows us to model some uncertainty situations occurring in practice [1], although the information retrieval from an interval lattice-valued automaton is not always an easy task. However, in the case that the truth structure consists of real intervals, it is possible to use some kind of operators to get a single value from each interval truth value given by the transition function of the machine.

Moreover, most of the practical situations which intervals are suitable in, use real number intervals. So, this paper focuses on fuzzy finite state machines and fuzzy transformation semigroups whose truth structure is the lattice consisting of all the closed intervals contained in $[0, 1]$. The novelty of this case is that the information provided by the transition functions of these automata can be retrieved by using a kind of Hurwicz aggregation functions [6], the so called Atanassov's K_α operators. By means of them, interval-valued fuzzy functions can be transformed into the corresponding fuzzy functions. A recent study of these operators can be found in [2] in the case of dimension two and in [3] in higher dimensions than two.

We study the behaviour of these operators acting on the transition functions of interval-valued automata. In particular, we characterize those K_α operators which allow us to obtain a fuzzy finite state machine or a fuzzy transformation semigroup starting from an interval-valued automaton.

Conversely, we characterize the functions between fuzzy sets that provide an interval-valued fuzzy transformation semigroup from two ordinary fuzzy finite state machines in the natural way.

2. Interval-valued fuzzy finite state machines and interval-valued fuzzy transformation semigroups

Throughout this paper, $\mathbf{L} = (L, \leq_L, \wedge, \vee)$ will be the lattice of all the closed subintervals contained in the real interval $[0, 1]$. Any closed interval contained in $[0, 1]$ is denoted $\mathbf{a} = [a_0, a_1]$ and the partial order $\leq_L$ in $\mathbf{L}$ is defined by

$$\mathbf{a} \leq_L \mathbf{b} \iff a_0 \leq b_0 \text{ and } a_1 \leq b_1.$$

In order to get distributivity of the t-norm with respect to the t-conorm , the t-conorm considered will be that given by the join, denoted by $\vee$ and the t-norm considered will be that given by the meet and denoted by $\wedge$.

We collect here the relevant material from [4] for the case $\mathbf{L} = (L, \leq_L, \wedge, \vee)$ in order to make our exposition self-contained.

Definition 1. An *interval-valued fuzzy finite state machine* (ivffsm) is a triple $M = (Q, X, \mu)$, where Q and X are nonempty finite sets, called the set of states and the set of input symbols respectively, and $\mu : Q \times X \times Q \to L$ is the membership function of an interval-valued fuzzy set.

Definition 2. Let $M = (Q, X, \mu)$ and $N = (P, Y, \nu)$ be ivffsms and let $f : Q \to P$ and $h : X \to Y$ be a pair of maps. Then $(f,\, h) : M \to N$ is an *ivffsm homomorphism* if

$$\mu(q, x, p) \leq_L \nu(f(q), h(x), f(p))$$

for any $q,\, p \in Q$ and $x \in X$.

An *ivffsm isomorphism* is an ivffsm homomorphism (f, h) such that both f and h are bijective.

The ivffsm homomorphism (f, h) is said to be *strong* if, for any $q,\, p \in Q$ and $x \in X$,

$$\nu(f(q), h(x), f(p)) = \bigvee\{\mu(q, x, r) \mid f(r) = f(p)\}.$$

Notice that if $(f,\, h) : M \to N$ is a strong ivffsm homomorphism and f is one-to-one, then

$$\mu(q, x, p) = \nu(f(q), h(x), f(p))$$

for any $q,\, p \in Q$ and $x \in X$.

Recall that any collection of *objects*, $\{A, B, C, \ldots\}$, together with a collection of morphisms, $\{\mathcal{C}(A, B),\, \mathcal{C}(B, C), \ldots\}$ satisfying both

1. For any object A, the identity map, id_A, belongs to $\mathcal{C}(A, A)$.
2. If $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, then the composition $g \circ f \in \mathcal{C}(A, C)$,

is a *category* (see [5]).

Theorem 3 ([4] Theor. 3.5)**.** *The set of all the interval-valued fuzzy finite state machines together with the ivffsm homomorphisms constitutes a category.*

Definition 4. An *interval-valued fuzzy transformation semigroup* (*ivfts*) is a triple $G = (Q, U, \delta)$, where Q is a nonempty finite set, U is a finite semigroup and $\delta : Q \times U \times Q \to L$ is a map satisfying the following conditions:
(TS1) If U is a semigroup with an identity element e then

$$\delta(q, e, p) = \begin{cases} 1_L & \text{if } p = q \\ 0_L & \text{if } p \neq q \end{cases} \quad (p,\, q \in Q).$$

(TS2) $\delta(q, uv, p) = \bigvee\{\delta(q, u, r) \wedge \delta(r, v, p) \mid r \in Q\}$ for any $p,\, q \in Q$ and $u,\, v \in U$.

An ivfts (Q, U, δ) is called *faithful* if, for any $u,\, v \in U$,

$$\delta(q, u, p) = \delta(q, v, p)$$

for any $q,\, p \in Q \Longrightarrow u = v$.

Remark 5 ([4] Prop. 4.6)**.** Let (Q, U, δ) be an ivfts. The equivalence relation $\sim$ defined in U by means of

$$u \sim v \Longleftrightarrow \delta(q, u, p) = \delta(q, v, p)$$

for every $q,\, p \in Q$ is a congruence and hence the triple $(Q, U/_{\sim}, \overline{\delta})$ is a faithful ivfts, where the map

$$\overline{\delta} : Q \times (U/_{\sim}) \times Q \to L$$

is defined by $\overline{\delta}(q, \bar{u}, p) = \delta(q, u, p)$ for any $p,\, q \in Q$, $u \in U$.

Definition 6. Let $G = (Q, U, \delta)$ and $H = (P, V, \varphi)$ be ivftss. An *ivfts homomorphism* from G to H is a pair (f, ψ) such that

1. $f : Q \to P$ is a map.
2. $\psi : U \to V$ is a semigroup homomorphism.
3. If both U and V are semigroups with identity elements, $e_1 \in U$ and $e_2 \in V$, then $\psi(e_1) = e_2$.
4. $\delta(q, u, p) \leq_L \varphi(f(q), \psi(u), f(p))$ for any $q,\, p \in Q$ and $u \in U$.

An ivfts homomorphism $(f, \psi) : (Q, U, \delta) \to (P, V, \varphi)$ is called an *ivfts isomorphism* if both f and ψ are bijective.

The ivfts homomorphism (f, ψ) is said to be *strong* if, in addition, for any $q,\, p \in Q$ and $u \in U$,

$$\varphi(f(q), \psi(u), f(p)) = \bigvee\{\delta(q, u, r) \mid f(r) = f(p)\}.$$

Notice that if $(f, \psi) : (Q, U, \delta) \to (P, V, \varphi)$ is a strong ivfts homomorphism and f is one to one, then for any $q,\, p \in Q$ and $u \in U$,

$$\delta(q, u, p) = \varphi(f(q), \psi(u), f(p)).$$

Theorem 7 ([4] Theor. 4.3)**.** *The set of all the interval-valued fuzzy transformation semigroups together with the ivfts homomorphisms constitutes a category.*

Let $M = (Q, X, \mu)$ be an ivffsm. From now on we will denote by

1. X^* the set of all the words of elements of X of finite length, including the empty word λ. The binary operation given by the concatenation makes X^* a non-finite semigroup with λ as its identity element.
2. $|u|$ the length of any word $u \in X^*$.
3. $\mu^* : Q \times X^* \times Q \to L$ the map defined for any $q,\, p \in Q$, as follows
   (a)
$$\mu^*(q, \lambda, p) = \begin{cases} 1_L & \text{if } p = q \\ 0_L & \text{if } p \neq q \end{cases}$$

(b) $\mu^*(q, ux, p) = \bigvee\{\mu^*(q, u, r) \wedge \mu(r, x, p) \mid r \in Q\}$ for every $u \in X^*$ and $x \in X$.

Notice that $\mu^*(q, x, p) = \mu(q, x, p)$ for any $x \in X$ and any $q, p \in Q$.

4. $X^*/_\sim$ and $\overline{\mu^*} : Q \times (X^*/_\sim) \times Q \to L$ are defined like in Remark 5.

Theorem 8 ([4] Theorem 5.1). *Let $M = (Q, X, \mu)$ be any ivffsm. The triple $(Q, X^*/_\sim, \overline{\mu^*})$ is a faithful ivfts, denoted by $\mathbf{G}(M)$.*

Remark 9. Recall that a fuzzy finite state machine (ffsm) is a triple (Q, X, μ) with $\mu : Q \times X \times Q \to [0, 1]$.

If we consider the fuzzy finite state machines (ffsms) with the ffsm homomorphisms, it is clear that they constitute a category.

Similarly, a fuzzy transformation semigroup is a triple (Q, U, δ) where the map $\delta : Q \times U \times Q \to [0, 1]$ satisfies (TS1) and (TS2) with respect to the t-norm $\wedge$ and the t-conorm $\vee$ in $[0, 1]$.

It is easy to see that the set of fuzzy transformation semigroups (ftss) together with the fts homomorphisms constitutes another category.

Moreover, in the same way as in the case of intervals (see [10]), we can define a map G mapping each ffsm $M = (Q, X, \mu)$ to the faithful fts $G(M) = (Q, X^*/_\approx, \overline{\mu^*})$.

For simplicity of notation, we use the same symbol $*$ for both the interval-valued and the fuzzy set cases.

Notation: For any set Z, any map $\mu : Z \to L$ and any $z \in Z$, we will write $\mu(z) = [\mu_0(z), \mu_1(z)]$. So we denote $\mu = [\mu_0, \mu_1]$, where $\mu_i : Z \to [0, 1]$ for $i \in \{0, 1\}$.

Lemma 10 ([4] Lemma 6.12). *Let Q be a finite set and U a finite semigroup. Consider a map $\delta : Q \times U \times Q \to L$. Then, (Q, U, δ) is an interval-valued fuzzy transformation semigroup if and only if both triples, (Q, U, δ_0) and (Q, U, δ_1), are fuzzy transformation semigroups. Moreover, if either (Q, U, δ_0) or (Q, U, δ_1) is faithful, then (Q, U, δ) is faithful as an ivfts.*

Proposition 11 ([4] Prop. 6.14). *Let (Q, U, δ) and (P, V, φ) be ivftss. Let $f : Q \to P$ be a map and $\psi : U \to V$ a semigroup homomorphism (verifying that $\psi(e_1) = e_2$ whenever U and V have identity elements, e_1 and e_2, respectively). Then:*

1. *$(f, \psi) : (Q, U, \delta) \to (P, V, \varphi)$ is an ivfts homomorphism if and only if $(f, \psi) : (Q, U, \delta_0) \to (P, V, \varphi_0)$ and $(f, \psi) : (Q, U, \delta_1) \to (P, V, \varphi_1)$ are homomorphisms between the fuzzy transformation semigroups considered.*
2. *$(f, \psi) : (Q, U, \delta) \to (P, V, \varphi)$ is a strong homomorphism between ivftss if and only if the maps $(f, \psi) : (Q, U, \delta_0) \to (P, V, \varphi_0)$ and $(f, \psi) : (Q, U, \delta_1) \to (P, V, \varphi_1)$ are strong homomorphisms between the fuzzy transformation semigroups considered.*

If $M = (Q, X, \mu)$ is an ivffsm, we can consider the ffsms $M_0 = (Q, X, \mu_0)$ and $M_1 = (Q, X, \mu_1)$. Hence $G(M_0) = (Q, X^*/_{\approx_0}, \overline{\mu_0^*})$ is a faithful fuzzy transformation semigroup with $u \approx_0 v \iff$

$$\mu_0^*(q, u, p) = \mu_0^*(q, v, p) \text{ for every } q,\, p \in Q.$$

Analogously $G(M_1) = (Q, X^*/_{\approx_1}, \overline{\mu_1^*})$ is a faithful fuzzy transformation semigroup with $u \approx_1 v \iff$

$$\mu_1^*(q, u, p) = \mu_1^*(q, v, p) \text{ for every } q,\, p \in Q.$$

The following result shows the relationship between the pair of ftss $G(M_0)$, $G(M_1)$ and the ivfts $\mathbf{G}(M)$ obtained from the same ivffsm M.

Theorem 12 ([4] Theor. 6.20 and 6.21). *Let $M = (Q, X, \mu)$ be an ivffsm and $\mathbf{G}(M) = (Q, X^*/_\sim, \overline{\mu^*})$ as defined in Theorem 8. Consider $M_0 = (Q, X, \mu_0)$ and $M_1 = (Q, X, \mu_1)$, ffsms and $G(M_0) = (Q, X^*/_{\approx_0}, \overline{\mu_0^*})$ and $G(M_1) = (Q, X^*/_{\approx_1}, \overline{\mu_1^*})$, the faithful ivftss defined in Remark 9. Then $\mu^* = [\mu_0^*, \mu_1^*]$ and the quotient $X^*/_\sim = X^*/_{(\approx_0 \wedge \approx_1)}$.*

3. Generation of ivffsms and ivfts from fuzzy finite state machines and fuzzy transformation semigroups

Throughout this section, for each set Z, we denote by $\mathcal{FS}(Z)$ the set of all the fuzzy sets defined on Z.

We consider two fuzzy finite state machines, $M = (Q, X, \mu)$ and $N = (Q, X, \nu)$, and two maps, $f,\, g : \mathcal{FS}(Z) \times \mathcal{FS}(Z) \to \mathcal{FS}(Z)$, satisfying that $f(\mu, \nu) \le g(\mu, \nu)$.

We provide a way of building an ivffsm, $(Q, X, [f(\mu, \nu),\, g(\mu, \nu)])$ and then a faithful ivfts from it, $(Q, X^*/_\sim, \overline{[f(\mu, \nu),\, g(\mu, \nu)]^*})$. Moreover we characterize the cases in which, starting from the maps μ^* and ν^*, the same ivfts is obtained.

Theorem 13. *Let $M = (Q, X, \mu)$ and $N = (Q, X, \nu)$ be fuzzy finite state machines and let $f,\, g : \mathcal{FS}(Z) \times \mathcal{FS}(Z) \to \mathcal{FS}(Z)$ be maps satisfying that $f(\mu, \nu) \le g(\mu, \nu)$. Then*

1. *The triple $(Q, X, [f, g]_{(\mu,\nu)})$ is an interval-valued fuzzy finite state machine, where $[f, g]_{(\mu,\nu)}(q, x, p) = [f(\mu, \nu)(q, x, p),\, g(\mu, \nu)(q, x, p)]$.*
2. *The map $[f, g]^*_{(\mu,\nu)} : Q \times X^* \times Q \to L$ is equal to $[f^*_{(\mu,\nu)},\, g^*_{(\mu,\nu)}]$, where*
$$f^*_{(\mu,\nu)} = (f(\mu, \nu))^*\,,\; g^*_{(\mu,\nu)} = (g(\mu, \nu))^*$$
and $[f, g]^*_{(\mu,\nu)} = [f(\mu, \nu), g(\mu, \nu)]^*$.
3. *If $f(\mu^*, \nu^*) \le g(\mu^*, \nu^*)$, the map $[f,\, g]_{(\mu^*,\nu^*)} : Q \times X^* \times Q \to L$ satisfies (TS1) and (TS2) if and only if*
$$[f,\, g]_{(\mu^*,\nu^*)} = [f^*_{(\mu,\nu)},\, g^*_{(\mu,\nu)}].$$
In that case, the faithful ivfts $(Q, X^*/_\sim, \overline{[f^*_{(\mu,\nu)}, g^*_{(\mu,\nu)}]})$ defined as in Remark 5 is the faithful quotient of the ivfts

$(Q, X^*/_{\approx}, \overline{[f, g]}_{(\mu^*, \nu^*)})$, *where the equivalence relation* $\approx$ *is defined in* X^* *by*

$$u \approx v \Longleftrightarrow \left\{ \begin{array}{l} \mu^*(q,u,p) = \mu^*(q,v,p) \text{ and} \\ \nu^*(q,u,p) = \nu^*(q,v,p) \end{array} \right\}$$

for every $q, p \in Q$.

Sketch of proof. The first item is immediate. For the second one, it is enough to prove that

$$[f,\, g]^*_{(\mu,\nu)}(q,u,p) = [f^*_{(\mu,\nu)}(q,u,p),\; g^*_{(\mu,\nu)}(q,u,p)],$$

which can be made by using induction on the length of the word u.

For the left-to-right proof of the third item, Lemma 10 assures that both $f_{(\mu^*,\,\nu^*)}$ and $g_{(\mu^*,\,\nu^*)}$ satisfy (TS1) and (TS2). Then, by using induction on the length of u, it is easy to prove that $f_{(\mu^*,\,\nu^*)}(q,u,p) = f^*_{(\mu,\nu)}(q,u,p)$ for every $q,\, p \in Q$ and $u \in X^*$, and the analogous equality for the map g.

The right-to-left proof is based on that $[f,\, g]^*_{(\mu,\nu)}$ satisfies (TS1) and (TS2) and, consequently, $[f,\, g]_{(\mu^*,\,\nu^*)}$ also does.

In this case, an equivalence relation can be well-defined in $X^*/_{\approx}$ by $[u]_{\approx} \equiv [v]_{\approx} \Leftrightarrow$

$$\overline{[f,g]}_{(\mu^*,\nu^*)}(q,[u]_{\approx},p) = \overline{[f,g]}_{(\mu^*,\nu^*)}(q,[v]_{\approx},p)$$
for every $q,p \in Q$
$$\Leftrightarrow [f,g]_{(\mu^*,\nu^*)}(q,u,p) = [f,g]_{(\mu^*,\nu^*)}(q,v,p)$$
for every $q,p \in Q$.

It can be carefully checked that the faithful quotient given by this equivalence relation, $(Q, [X^*/_{\approx}]_{\equiv}, \overline{[\overline{[f,g]}_{(\mu^*,\nu^*)}]_{\equiv}})$, agrees with $(Q, X^*/_{\sim}, \overline{[f^*_{(\mu,\nu)},\, g^*_{(\mu,\nu)}]})$ defined in the statement above.

□

Remark 14. Let $M = (Q, X, \mu)$ and $N = (Q, X, \nu)$ be fuzzy finite state machines.

1. Assume that $\mu \leq \nu$. We consider the operators f and g given respectively by $f(\mu,\nu) = \mu$ and $g(\mu,\nu) = \nu$ for any $\mu,\, \nu \in \mathcal{FS}(Q \times X \times Q)$. Then $(Q, X, [\mu, \nu])$ is an ivffsm and $(Q, X^*/_{\sim}, \overline{[\mu,\nu]^*})$ is a faithful ivfts such that
$$(Q, X^*/_{\sim}, \overline{[\mu,\nu]^*}) = (Q, X^*/_{\sim}, \overline{[\mu^*,\nu^*]}).$$
2. The triple $(Q, X, [\mu, \mu\vee\nu])$ is an ivffsm and $(Q, X^*/_{\sim}, \overline{[\mu,\mu\vee\nu]^*})$ is a faithful ivfts equal to $(Q, X^*/_{\sim}, \overline{[\mu^*,(\mu\vee\nu)^*]})$. However, in general
$$[\mu^*,\, (\mu\vee\nu)^*] \neq [\mu^*,\, \mu^*\vee\nu^*],$$
which does not always satisfy (TS2).
3. The triple $(Q, X, [\mu\wedge\nu,\, \mu])$ is an ivffsm and $(Q, X^*/_{\sim}, \overline{[\mu\wedge\nu,\, \mu]^*})$ is a faithful ivfts equal to $(Q, X^*/_{\sim}, \overline{[(\mu\wedge\nu)^*,\, \mu^*]})$. However, in general
$$[(\mu\wedge\nu)^*,\, \mu^*] \neq [\mu^*\wedge\nu^*,\, \nu^*],$$
which does not always satisfy (TS2).

4. Generating ffsms from ivffsms and ftss from ivftss.

We study here the way of generating ffsms and fts starting from an ivffsm and an ivts respectively.

Throughout this section we consider the family of Attanasov's K_α operators, which is the family of maps
$$\{K_\alpha : L \to [0,1] \mid 0 \leq \alpha \leq 1\},$$
where each K_α is defined by $K_\alpha([a_0,a_1]) = a_0 + \alpha(a_1 - a_0)$.

It is easy to check that this family satisfies the following properties:

1. If $a_0 = a_1$, then $K_\alpha([a_0,a_1]) = a_0$ for any $\alpha \in [0,1]$.
2. $K_0([a_0,a_1]) = a_0$ and $K_1([a_0,a_1]) = a_1$ for any $[a_0,a_1] \in L$.
3. Whenever $[a_0,a_1] \leq_L [b_0,b_1]$, then $K_\alpha([a_0,a_1]) \leq K_\alpha([b_0,b_1])$ for any $\alpha \in [0,1]$.
4. If $0 \leq \alpha \leq \beta \leq 1$, then $K_\alpha([a_0,a_1]) \leq K_\beta([a_0,a_1])$ for any $[a_0,a_1] \in L$.

Notation: Let Z be any set and $\mu : Z \to L$ any map. For any $\alpha \in [0,1]$, denote by $K_\alpha(\mu)$ the map from Z to $[0,1]$ given by
$$(K_\alpha(\mu))\,(z) = K_\alpha\,(\mu(z))\,.$$

Lemma 15. *Let* $\alpha \in [0,1]$. *The following assertions are equivalent:*

1. $K_\alpha\,(\mathbf{a}\vee\mathbf{b}) = K_\alpha\,(\mathbf{a}) \vee K_\alpha\,(\mathbf{b})$ *for any* $\mathbf{a},\, \mathbf{b} \in L$.
2. $K_\alpha = K_0$ *or* $K_\alpha = K_1$.
3. $K_\alpha\,(\mathbf{a}\wedge\mathbf{b}) = K_\alpha\,(\mathbf{a}) \wedge K_\alpha\,(\mathbf{b})$ *for any* $\mathbf{a},\, \mathbf{b} \in L$.

Using the previous Lemma, we can obtain the next result

Theorem 16. *Let* $\alpha \in [0,1]$. *The following statements are equivalent:*

1. $(Q, U, K_\alpha(\delta))$ *is a fuzzy transformation semigroup for every ivfts* $G = (Q, U, \delta)$.
2. $K_\alpha = K_0$ *or* $K_\alpha = K_1$.

The next results analyze the functorial behaviour of operators K_α. Recall [5] that a *functor* F from the category $\mathcal{C}$ to the category $\mathcal{D}$ is an operator satisfying:

1. F assigns an object $F(A)$ in $\mathcal{D}$ to each object A in $\mathcal{C}$,
2. F assigns a morphism $F(f) \in \mathcal{D}\,(F(A), F(B))$ to each morphism $f \in \mathcal{C}(A,B)$,
3. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for any object A in $\mathcal{C}$.
4. $F(g \circ f) = F(g) \circ F(f)$ to each $f \in \mathcal{C}(A,B)$, $g \in \mathcal{C}(B,C)$.

If F and G are functors from the category $\mathcal{C}$ to the category $\mathcal{D}$, a *natural transformation* from F to G is a family of morphisms in $\mathcal{D}$,
$$\{\delta_A : F(A) \to G(A) \mid A \text{ is an object of } \mathcal{C}\},$$

such that, for any $f \in \mathcal{C}(A, B)$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \delta_A} & & \downarrow{\scriptstyle \delta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes.

A natural transformation is called a *natural equivalence* if δ_A is an isomorphism for any object A in $\mathcal{C}$.

Proposition 17. *Consider $\alpha \in \{0, 1\}$ and the map F_α which maps each ivfts $G = (Q, U, \delta)$ to $F_\alpha(G) = (Q, U, K_\alpha(\delta))$ and each ivfts homomorphism (f, h) to $F_\alpha(f, h) = (f, h)$. Then F_α is a functor from the category of the ivftss to the category of the ftss.*

Proposition 18. *For any $\alpha \in [0, 1]$, the map D_α which maps each ivffsm $M = (Q, X, \mu)$ to the ffsm $D_\alpha(M) = (Q, X, K_\alpha(\mu))$ and each ivffsm homomorphism (f, h) to the ffsm homomorphism $D_\alpha(f, h) = (f, h)$ is a functor from the category of the ivffsms to the category of the ffsms.*

Moreover, for any ivffsm $M = (Q, X, \mu)$, the pair (id_Q, id_X) is a ffsm homomorphism from $D_\alpha(M)$ to $D_\beta(M)$ if $0 \leq \alpha \leq \beta \leq 1$. In fact, there exists a natural equivalence from the functor D_α to the functor D_β.

The previous results allow us to build several faithful ftss starting from an ivffsm M by using K_α operators.

Theorem 19. *Let $M = (Q, X, \mu)$ be an ivffsm. For any $\alpha \in [0, 1]$, the triple $G(D_\alpha(M)) = (Q, X^*/_{\approx_\alpha}, \overline{(K_\alpha(\mu))^*})$ is a faithful fts.*

Proof. $D_\alpha(M) = (Q, X, K_\alpha(\mu))$ is a ffsm by Proposition 18. Then the triple $G(D_\alpha(M)) = (Q, X^*/_{\approx_\alpha}, \overline{(K_\alpha(\mu))^*})$ is a faithful fts by Remark 9. Notice that, for any $\alpha \in [0, 1]$, the equivalence relation $\approx_\alpha$ in X^* is given by

$$u \approx_\alpha v \Longleftrightarrow (K_\alpha(\mu))^* (q, u, p) = (K_\alpha(\mu))^* (q, v, p)$$

for every $q, p \in Q$. □

Another way of getting ftss from an ivffsm M is by means of the ivfts $\mathbf{G}(M)$.

Theorem 20. *Let $M = (Q, X, \mu)$ be an ivffsm. For $\alpha \in \{0, 1\}$, the triple $\overline{F}_\alpha(\mathbf{G}(M)) = (Q, [X^*/_\sim]_\alpha, [K_\alpha(\overline{\mu^*})]_\alpha)$ is a faithful fts.*

Proof. $\mathbf{G}(M) = (Q, X^*/_\sim, \overline{\mu^*})$ is a faithful ivfts by Theorem 8, where the equivalence relation $\sim$ is given in X^* by

$$u \sim v \Longleftrightarrow \mu^*(q, u, p) = \mu^*(q, v, p) \text{ for every } q,\, p \in Q.$$

Proposition 17 assures that the triple $F_\alpha(\mathbf{G}(M)) = (Q, X^*/_\sim, K_\alpha(\overline{\mu^*}))$ is a fts for $\alpha \in \{0, 1\}$, but it is not necessarily faithful. At last, the quotient $\overline{F}_\alpha(\mathbf{G}(M)) = (Q, [X^*/_\sim]_\alpha, [K_\alpha(\overline{\mu^*})]_\alpha)$ is a faithful fts, where $[\overline{u}]_\alpha = [\overline{v}]_\alpha \Longleftrightarrow K_\alpha(\overline{\mu^*})(q, \overline{u}, p) = K_\alpha(\overline{\mu^*})(q, \overline{v}, p)$ for every $q, p \in Q$ and $[K_\alpha(\overline{\mu^*})]_\alpha(q, [\overline{u}]_\alpha, p) = K_\alpha(\overline{\mu^*})(q, \overline{u}, p)$ for any $q, p \in Q$ and $u \in X^*$. □

In order to compare the ftss obtained in Theorems 19 and 20, we need the following lemma.

Lemma 21. *Let $\alpha \in [0, 1]$. The following assertions are equivalent:*

1. *$K_\alpha(\mu^*) = (K_\alpha(\mu))^*$ for any ivffsm $M = (Q, X, \mu)$.*
2. *$K_\alpha(\mu^*)$ satisfies (TS1) and (TS2) in $[0, 1]$ for any ivffsm $M = (Q, X, \mu)$.*
3. *$K_\alpha = K_0$ or $K_\alpha = K_1$.*

Now we can prove

Theorem 22. *Let $M = (Q, X, \mu)$ be an ivffsm. Then for $\alpha \in \{0, 1\}$*

$$G(D_\alpha(M)) = \overline{F}_\alpha(\mathbf{G}(M)),$$

i.e, are equal as faithful ftss.

Proof. Consider $\alpha \in \{0, 1\}$. We have to show that the following diagram commutes:

$$\begin{array}{ccc} (Q, X, \mu) & \xrightarrow{D_\alpha} & (Q, X, K_\alpha(\mu)) \\ \downarrow{\scriptstyle \mathbf{G}} & & \downarrow{\scriptstyle G} \\ (Q, X^*/_\sim, \overline{\mu^*}) & \xrightarrow{\overline{F}_\alpha} & (Q, X^*/_{\approx_\alpha}, \overline{(K_\alpha(\mu))^*}) \end{array}$$

Firstly we obtain

$$G(D_\alpha(M)) = (Q, X^*/_{\approx_\alpha}, \overline{(K_\alpha(\mu))^*})$$

as in Theorem 19, where for $u, v \in X^*$, $u \approx_\alpha v$ if and only if $(K_\alpha(\mu))^* (q, u, p) = (K_\alpha(\mu))^* (q, v, p)$ for every $q, p \in Q$.

Now, if we consider $\overline{F}_\alpha(\mathbf{G}(M)) = (Q, [X^*/_\sim]_\alpha, [K_\alpha(\overline{\mu^*})]_\alpha)$ as in Theorem 20, then $u \sim v$

$$\Longleftrightarrow \mu^*(q, u, p) = \mu^*(q, v, p) \text{ for every } q, p \in Q$$
$$\Longleftrightarrow \text{for every } q, p \in Q$$
$$\begin{cases} K_0(\mu^*(q, u, p)) = K_0(\mu^*(q, v, p)) \text{ and} \\ K_1(\mu^*(q, u, p)) = K_1(\mu^*(q, v, p)) \end{cases}$$

Moreover, u is equivalent to v in $[X^*/_\sim]_\alpha$ if and only if $K_\alpha(\overline{\mu^*(q, \overline{u}, p)}) = K_\alpha(\overline{\mu^*(q, \overline{v}, p)})$ for every $q, p \in Q$ or equivalently

$$K_\alpha(\mu^*(q, u, p)) = K_\alpha(\mu^*(q, v, p)) \text{ for every } q, p \in Q.$$

Since $K_\alpha(\mu^*) = (K_\alpha(\mu))^*$ by using Lemma 21, we conclude that the quotients $X^*/_{\approx_\alpha}$ and $[X^*/_\sim]_\alpha$ agree and so do the triples

$$(Q, X^*/_{\approx_\alpha}, \overline{(K_\alpha(\mu))^*}) \text{ and } (Q, [X^*/_\sim]_\alpha, [K_\alpha(\overline{\mu^*})]_\alpha).$$

□

5. Conclusions and future research

In this paper the concepts of a lattice-valued fuzzy finite state machine and a lattice-valued fuzzy transformation semigroup are studied in the particular case that the lattice is the set of all the closed intervals contained in $[0, 1]$ with the usual order. The relationships obtained in [4] between these two concepts are particularized to the special case of these interval-valued fuzzy automata.

In addition, starting from an interval-valued fuzzy finite state machine, it is proven that Atanassov's K_α operators provide a fuzzy finite state machine and then a fuzzy transformation semigroup for each $\alpha \in [0, 1]$.

However, starting from an interval-valued transformation semigroup, the only Atanassov's K_α operators that provide a fuzzy transformation semigroup are K_0 and K_1.

Conversely, the paper provides two different ways of getting an interval-valued transformation semigroup starting from two suitable fuzzy finite state machines. It is proven that one of them is the faithful quotient of the other one.

Regarding future lines of research, we will generalize the use of K_α operators or generalized K_α operators to the study of fuzzy finite state machines whose transition functions take values in the set of the intervals contained in any lattice where the use of weighted means has sense.

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