On distances derived from symmetric difference functions

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Abstract

Once introduced a definition of symmetric difference function on the unit real interval [0,1], we consider a method to construct such functions based on a triplet formed by a t-norm, a t-conorm and a strong negation. Our main goal is to characterize those triplets that define symmetric difference functions which are distances.

Keywords: symmetric difference function, t-norm, t-conorm, strong negation, distance.

1. Introduction

Motivated by generalizations of the classical symmetric difference of sets, Alsina introduced in [3] (see also [5]) the idea of constructing distances from a t-norm T and its dual $T^* : T^*(a,b) = 1 - T(1-a, 1-b)$. Thus given a t-norm T, Alsina defines $d_T(a,b) = T^*(a,b) - T(a,b)$ if $a \neq b, d_T(a,a) = 0$, and proves that "if a t-norm T is a copula then d_T is a distance". There are examples of continuous non-strict Archimedean t-norms that are not copulas and that generate distances (see [1]), proving that for continuous t-norms the reciprocal of the Alsina's result is not true. In [1] a characterization of those t-norms having zero region $\{(a,b); a+b \leq 1\}$ that induce distances is given, however the complete characterization of those t-norms that induce distances is still an open problem. The problem of generating distances from a more general pair (S,T) of a t-conorm and a t-norm is also studied in [1].

In the same way that the linguistic "or" has the functional model given by t-conorms, a functional model for the linguistic "either or" by means of symmetric difference functions can be considered (see [4]). By generalizing the classical expression of set theory "either A or B" $= (A \cap B^c) \cup (B \cap A^c)$, we can consider a class of symmetric difference functions of the form $\triangle(a,b) = S(T(a, N(b)), T(b, N(a))$ where T, S, N are a t-norm, a t-conorm and a strong negation respectively. Our main concern in this paper is to give a characterization of those triplets (T,S,N) such that the symmetric difference functions $\triangle$ associated to them are distances.

In Section 2 basic definitions, examples and results are presented. Section 3 contains all the main results of the contribution.

2. Preliminaries

We begin with the definitions of t-norm, t-conorm and copula, and some properties and basic examples (see [5] and [7]).

Definition 1 *Let us consider functions* $T, S\colon [0,1]^2 \to [0,1]$. *We say that* T *is a t-norm if it is increasing in each variable, commutative, associative and has neutral element 1. We say that* S *is a t-conorm if it is increasing in each variable, commutative, associative and has neutral element 0.*

Definition 2 *A function* N *from* $[0,1]$ *onto itself is a strong negation if it is decreasing and involutive* ($N^2 = id$).

Given a strong negation N, the N-dual t-conorm of a t-norm T is $T^*(a,b) = N(T(N(a), N(b)))$. Given a t-norm T, a t-conorm S, and a strong negation N, we say that (T,S,N) is a De Morgan triplet if T and S are N-dual.

Example 1 *Basic t-norms are the minimum* $M(a,b) = \min(a,b)$, *the product* $\Pi(a,b) = ab$, *the Łukasiewicz t-norm* $W(a,b) = \max(a+b-1, 0)$ *and the drastic t-norm*

$$Z(a,b) = \begin{cases} a & \text{if } b = 1, \\ b & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Their dual t-conorms (with respect to the classical strong negation $N(a) = 1 - a$*) are, respectively, the maximum* $M^*(a,b) = \max(a,b)$, *the probabilistic sum* $\Pi^*(a,b) = a + b - ab$, *the Łukasiewicz t-conorm or bounded sum* $W^*(a,b) = \min(a+b, 1)$ *and the drastic t-conorm*

$$Z^*(a,b) = \begin{cases} a & \text{if } b = 0, \\ b & \text{if } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Note that, for any t-norm T and t-conorm S, $Z \leqslant T \leqslant M \leqslant M^* \leqslant S \leqslant Z^*$.

Proposition 1 *A continuous t-norm T is Archimedean ($T(a,a) < a$ for all a in $(0,1)$) if and only if it has an additive generator, that is, a strictly decreasing and continuous function $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$ such that*

$$T(a,b) = f^{(-1)}(f(a) + f(b)),$$

where $f^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of f, defined by

$$f^{(-1)}(a) = \begin{cases} f^{-1}(a) & \text{if } a \leqslant f(0), \\ 0 & \text{otherwise.} \end{cases}$$

An additive generator is defined up to a positive multiplicative constant. On the other hand, if f is an additive generator of a continuous Archimedean t-norm T, then T is strict (strictly increasing on $[0,1)^2$) if, and only if, $f(0) = \infty$.

The t-norm Π is strict with additive generator $f(a) = -\log a$, and the t-norm W is non-strict with additive generator $f(a) = 1 - a$.

If T is a non-strict continuous Archimedean t-norm with additive generator f, then $N(a) = f^{-1}(f(0) - f(a))$ is a strong negation that we call associated to T. Note that $T(a,b) = 0$ if, and only if, $b \leq N(a)$.

We recall here also the definition of distance.

Definition 3 *A function $d\colon X \times X \to [0,\infty)$ is a* distance *on the set X if the following properties are satisfied, for all $a,b,c \in X$:*

1) $d(a,b) = 0$ if and only if $a = b$,
2) $d(a,b) = d(b,a)$,
3) $d(a,b) \leqslant d(a,c) + d(c,b)$.

3. Symmetric difference functions and distances

Definition 4 *A function $\triangle : [0,1]\times[0,1] \longrightarrow [0,1]$ is a* symmetric difference function *(SDF) if it satisfies for any $a,b \in [0,1]$:*

$\triangle$1) *$\triangle(a,b) = \triangle(b,a)$,*
$\triangle$2) *$\triangle(a,a) = 0$, $\triangle(a,0) = a$, $N(a) = \triangle(a,1)$ is a strong negation.*

Definition 5 *Given T,S,N a t-norm, a t-conorm, and a strong negation (not necessarily a De Morgan triplet), we define the function:*

$$\triangle(a,b) = S(T(a,N(b)),T(N(a),b)) \qquad (1)$$

Next result was mentioned without proof in [2]. For the sake of completeness, we have included it in this paper.

Proposition 2 *$\triangle$ is a SDF if, and only if, $T(a,N(a)) = 0$ $\forall a \in [0,1]$.*

In this case we say that $\triangle$ is the SDF associated to the triplet (T,S,N).

Proof If $\triangle$ is a SDF, then $0 = \triangle(a,a) = S(T(a,N(a)),T(N(a),a))$ and thus $T(a,N(a)) = 0$ $\forall a \in [0,1]$.
Let us suppose now that $T(a,N(a)) = 0, \forall a \in [0,1]$. Then

$$\triangle(a,a) = S(T(a,N(a)),T(N(a),a)) = S(0,0) = 0$$

On the other hand,

$$\triangle(a,0) = S(T(a,N(0)),T(N(a),0)) = S(a,0) = a$$

and

$$\begin{aligned} \triangle(a,1) &= S(T(a,N(1)),T(N(a),1)) \\ &= S(0,N(a)) = N(a) \end{aligned}$$

Finally,

$$\begin{aligned} \triangle(a,b) &= S(T(a,N(b)),T(N(a),b)) \\ &= S(T(N(a),b),T(a,N(b))) \\ &= S(T(b,N(a)),T(N(b),a)) = \triangle(b,a) \end{aligned}$$

■

Example 2 *The SDF associated to the triplet $(W,W^*,1-id)$ is the usual distance on $[0,1]$: $\triangle(a,b) = |a-b|$.*

We are interested in those triplets (T,S,N) such that $\triangle$ defined in (1) is a distance.

Proposition 3 *Given a triplet (T,S,N), the function $\triangle$ defined in (1) is a distance if, and only if, the following conditions hold:*

i) $T(a,b) = 0$ if, and only if, $b \leq N(a)$.
ii) For all $x \in [0,1]$ and any $\epsilon,\delta \in \mathbb{R}$ such that $0 \leq \epsilon \leq 1-x, 0 \leq \delta \leq 1-N(x)$, the following inequality holds (see Figure 1)

$$T(x+\epsilon,N(x)+\delta) \leq T(x,N(x)+\delta)+T(x+\epsilon,N(x)) \qquad (2)$$

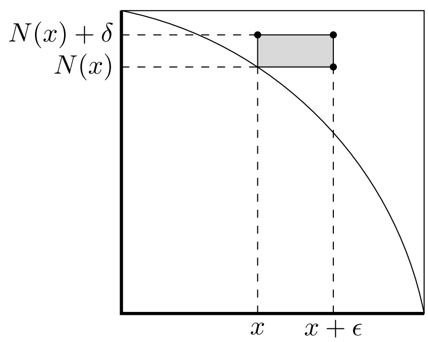

Figure 1: The points involved in the condition (2).

Proof We know that $\triangle$ is symmetric. Let us suppose now that the conditions i) and ii) hold and let us prove that $\triangle$ is a distance. First of all, we have from condition i) that $\triangle(a,a) = S(T(a,N(a)),T(N(a),a)) = S(0,0) = 0$. On the other hand, if $\triangle(a,b) = 0$, then $T(a,N(b)) = T(N(a),b) = 0$ and from condition i), $a \leq b$ and $b \leq a$, thus $a = b$. Now we have to prove the triangular inequality, that is, $\triangle(a,b) \leq \triangle(a,c) + \triangle(c,b)$. From i) we can write

$$\triangle(a,b) = \begin{cases} T(a,N(b)) & if\ b \leq a \\ T(N(a),b) & if\ a \leq b \end{cases}$$

By symmetry, we can suppose that $a < b$; thus we have to consider three cases: $a < b < c, c < a < b$ and $a < c < b$. The triangular inequality for the first two cases follows immediately from the increasingness of T. Let us consider now the case $a < c < b$. We have to prove that $T(N(a),b) \leq T(N(a),c) + T(N(c),b)$. This inequality follows from (2) just by taking $x = N(c), \epsilon = N(a) - N(c)$ and $\delta = b - c$.

Conversely, let us suppose now that $\triangle$ is a distance. Since $\triangle(a,a) = 0$, we have that $T(a,N(a)) = 0$ for all a, and the monotonicity of T gives that $T(a,b) = 0$ if $b \leq N(a)$. Now, if $T(a,b) = 0$ for $b > N(a)$, let $c = N(b)$. Thus $\triangle(b,c) = S(T(a,N(c)),T(N(a),c)) = S(0,0) = 0$, which is impossible since $a > c$. Now we have to prove the condition ii). Let us consider $x \in [0,1], 0 < \epsilon \leq 1 - x, 0 < \delta \leq 1 - N(x)$ and $a = N(x+\epsilon), b = N(x) + \delta$ and $c = N(x)$. Thus we have that $a < c < b$ and the triangular inequality gives

$$\begin{aligned} T(x+\epsilon, N(x)+\delta) &= T(N(a),b) \\ &\leq T(N(a),c) + T(N(c),b) \\ &= T(x+\epsilon, N(x)) + T(x, N(x)+\delta) \end{aligned}$$

which is (2). ■

Remark 1

- *i) If we take* $\epsilon = 1 - a, \delta = 1 - N(a)$ *in (3), we have* $1 \leq a + N(a)$*, that is,* $N \geq 1 - id$*.*
- *ii) If we take* $\delta = 1 - N(a)$*, we have* $a + \epsilon \leq a + T(a+\epsilon, N(a))$*, that is,* $T(a+\epsilon, N(a)) \geq \epsilon$*. Analogously,* $T(a, N(a)+\delta) \geq \delta$*. Thus, if we take* $N = 1 - id$*, we have* $T \geq W$*.*
- *iii) If* $\triangle$ *is a distance, then*

$$\triangle(a,b) = \begin{cases} 0 & if\ a = b \\ T(N(a),b) & if\ a < b \\ T(a,N(b)) & if\ a > b \end{cases}$$

  Observe that the values of $\triangle$ *do not depend on the t-conorm* S*.*
- *iv) Condition (2) can be expressed as a condition of "restricted subadditivity":*

$$T(u \oplus v) \leq T(u) + T(v) \qquad (3)$$

  for any $u = w + \overrightarrow{\epsilon}, v = w + \overrightarrow{\delta}$*, where* $w = (a, N(a))$ *is a vector "on the negation N",* $\overrightarrow{\epsilon} = (\epsilon, 0)$*,* $\overrightarrow{\delta} = (0, \delta)$*,* $0 \leq \epsilon \leq 1 - a, 0 \leq \delta \leq 1 - N(a)$*, and* $u \oplus v = u + v - w$*.*

Proposition 4 *Let* (T, N) *satisfying the conditions in Proposition 3. If* T *is continuous on the graph of* N $(\{(x, N(x)); x \in [0,1]\})$*, then it is continuous on all its domain.*

Proof Let us suppose that T is discontinuous at (a,b) with $b > N(a)$. Thus either $T(\cdot, b)$ is discontinuous at a or $T(a, \cdot)$ is discontinuous at b. Let us suppose first that $T(\cdot, b)$ is right-discontinuous at a. Then there exists $\lambda > 0$ such that

$$T(a+\epsilon, b) - T(a,b) \geq \lambda\ \forall \epsilon > 0$$

From condition (2) we have $T(a+\epsilon) \leq T(a,b) + T(a+\epsilon, N(a))$, that is

$$\begin{aligned} \lambda &\leq T(a+\epsilon, b) - T(a,b) \\ &\leq T(a+\epsilon, N(a)) \\ &= T(a+\epsilon, N(a)) - T(a, N(a)) \end{aligned}$$

for all $\epsilon > 0$, and thus T is not right-continuous at $(a, N(a))$.

Let us suppose now that $T(\cdot, b)$ is left-discontinuous at a. Then there exists $\lambda > 0$ such that

$$\forall \epsilon > 0, T(a,b) - T(a-\epsilon, b) \geq \lambda$$

From condition (2) we have $T(a,b) \leq T(a-\epsilon, b) + T(a, N(a-\epsilon))$, that is

$$\begin{aligned} \lambda &\leq T(a,b) - T(a-\epsilon, b) \\ &\leq T(a, N(a-\epsilon)) \\ &= T(a, N(a-\epsilon)) - T(a, N(a)) \end{aligned}$$

for all $\epsilon > 0$. Then, from the continuity of N, we have

$$\lambda \leq T(a, N(a)+\epsilon)) - T(a, N(a))$$

for all $\epsilon > 0$, and thus T is not right-continuous at $(a, N(a))$.

The proof for the case of $T(a, \cdot)$ is completely analogous and it has been omitted. ■

Proposition 5 *Given a triplet* (T, S, N) *with* T *a continuous t-norm, the function* $\triangle$ *defined in (1) is a distance if, and only if, the following conditions hold:*

- *i)* T *is a non-strict archimedean t-norm with associated negation* N*.*
- *ii) The function* $f^{-1}(1 - id)$ *is subadditive, where* f *is the normalized additive generator of* T $(f(0) = 1)$*.*

Proof If $\triangle$ is a distance, then the condition i) of Proposition 3 proves that T is a non-strict archimedean t-norm with associated negation N. Let now f be the additive generator of T with $f(0) = 1$. Thus $N(a) = f^{-1}(1 - f(a))$ and the expression of $\triangle$ becomes

$$\triangle(a,b) = \begin{cases} 0 & if\ a = b \\ f^{-1}(1 - f(a) + f(b)) & if\ a < b \\ f^{-1}(1 + f(a) - f(b)) & if\ a > b \end{cases}$$

that is, $\triangle(a,b) = f^{-1}(1 - |f(a) - f(b)|)$. Now the triangular inequality for the case $a < c < b$ becomes $f^{-1}(1 - (f(a) - f(b))) \leq f^{-1}(1 - (f(a) - f(c))) + f^{-1}(1 - (f(c) - f(b)))$. If we take $u = f(a) - f(b)$ and $v = f(c) - f(b)$, we obtain

$$f^{-1}(1 - (u + v)) \leq f^{-1}(1 - u) + f^{-1}(1 - v)$$

for all $u, v \geq 0$ such that $u+v \leq 1$. Thus $f^{-1}(1-id)$ is subadditive.

Conversely, let us suppose now that the conditions i) and ii) hold. From previous results, we only have to prove the triangular inequality of $\triangle$. But this result comes immediately from the above reasoning. ∎

Remark 2

i) Note that (T, S, N) does not need to be a De Morgan triplet.
ii) The function $f^{-1}(1 - id)$ is subadditive if, and only if, $1 - f$ is superadditive.
iii) If the t-norm T is a copula and $\triangle$ is a distance, then f is convex and thus $f^{-1}(1 - id)$ is superadditive. Then $f^{-1}(1 - id)$ is additive, thus $f^{-1}(1 - id) = id$ and therefore $f = 1 - id$, that is $T = W$.
iv) Let us observe that the if $f^{-1}(1 - id)$ is subadditive then $N \geq 1 - id$.

Proposition 6 *If the additive generator f of T is concave, then the condition ii) of Proposition 5 holds. The converse is not true, in general.*

Proof If f is concave, then f^{-1} and $h = f^{-1}(1-id)$ are also concave. Since $h(0) = 0$, the function h is subadditive. To prove that the converse is not true, let us consider $f = g^{-1}$, where $g(a) = -a^3 + a^2 - a + 1$. The function f is not concave (since g is not concave), but $f^{-1}(1-id) = g(1-id)$ is subadditive. Thus the concavity of f is not a necessary condition for (2) to hold. ∎

Example 3 *The generator of the Yager t-norms, $f(a) = (1 - a)^\lambda$ where $0 \leq \lambda \leq 1$, is a concave function. The associated distance is*

$$\begin{aligned} \triangle(a,b) &= f^{-1}(1 - |f(a) - f(b)|) \\ &= 1 - (1 - |(1 - a)^\lambda - (1 - b)^\lambda|)^{1/\lambda} \end{aligned}$$

The generator of the Sugeno-Weber t-norms, $t_\lambda(a) = 1 - \frac{\ln(1+\lambda a)}{\ln(1+\lambda)}$, is a concave function for any $\lambda \in (-1, 0)$. The associated distance is

$$\begin{aligned} \triangle(a,b) &= f^{-1}(1 - |f(a) - f(b)|) \\ &= \frac{1+\lambda}{\lambda}\, exp\{\frac{1}{\ln(1+\lambda)} \cdot |\ln \frac{1+\lambda b}{1+\lambda a}|\} \end{aligned}$$

Proposition 7 *Let us consider a triplet (T, S, N) such that $T(a,b) = 0$ if, and only if, $b \leq N(a)$. If N is concave and T is concave in each variable on its positive region, then the condition (2) holds.*

Proof Let $a \in [0,1]$, and $\epsilon, \delta \in \mathbb{R}$ such that $0 \leq \epsilon \leq 1 - a, 0 \leq \delta \leq 1 - N(a)$. If we take $\alpha = \frac{N(a)-N(a+\epsilon)}{N(a)+\delta-N(a+\epsilon)}$, then we can write $(a+\epsilon, N(a)) = \alpha \cdot (a+\epsilon, N(a)+\delta) + (1-\alpha) \cdot (a+\epsilon, N(a+\epsilon))$.

Analogously, we have $(a, N(a) + \delta) = \alpha' \cdot (a+\epsilon, N(a)+\delta) + (1-\alpha') \cdot (N(N(a)+\delta), N(a)+\delta)$, where $\alpha' = \frac{a-N(N(a)+\delta)}{a+\epsilon-N(N(a)+\delta)}$.

Since T is concave and it equals 0 on the negation N, we have $T(a+\epsilon, N(a)) \geq \alpha \cdot T(a+\epsilon, N(a)+\delta)$ and $T(a, N(a)+\delta) \geq \alpha' \cdot T(a+\epsilon, N(a)+\delta)$. By adding this two inequalities, we obtain $T(a+\epsilon, N(a)) + T(a, N(a)+\delta) \geq (\alpha+\alpha') \cdot T(a+\epsilon, N(a)+\delta)$. Thus, if we prove that $\alpha + \alpha' \geq 1$, we will have the condition (2). Now a straightforward calculation proves that $\alpha + \alpha' \geq 1$ is equivalent to

$$(a - N(N(a) + \delta)) \cdot (N(a) - N(a+\epsilon)) \geq \epsilon \cdot \delta$$

and this inequality holds since N is concave (see Figure 2). ∎

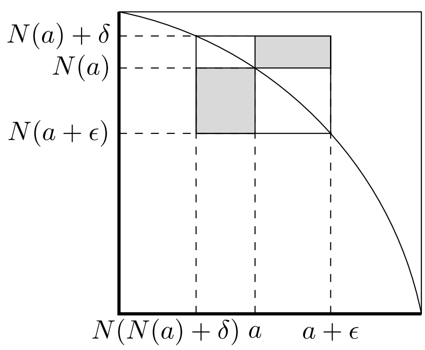

Figure 2: The points involved in the proof of Proposition 7.

Remark 3 *Under the conditions above, condition (2) plus T concave in each variable in its positive region do not imply that N is concave. Moreover, condition (2) plus N concave do not imply that T is concave in each variable in its positive region. Let see two examples.*

Example 4 *Let N be a strong negation. Let us consider the (left-continuous but not continuous) t-norm M_N given by*

$$M_N(a,b) = \begin{cases} 0 & if\, b \le N(a) \\ \min(a,b) & if\, b > N(a) \end{cases} \quad (4)$$

It can be proved that for any t-conorm S, (M_N, S, N) defines a distance through (1) if, and only if, $N \ge 1 - id$. This distance is given by

$$\triangle(a,b) = \begin{cases} 0 & if\, a = b \\ \min(N(a), b) & if\, a < b \\ \min(a, N(b)) & if\, a > b \end{cases}$$

In the case when $N = 1 - id$ ($M_{1-id} = T^{nM}$, the nilpotent minimum), this distance becomes

$$\triangle(a,b) = \begin{cases} 0 & if\, a = b \\ b & if\, a < b, a+b \le 1 \\ 1-a & if\, a < b, a+b \ge 1 \\ a & if\, a > b, a+b \le 1 \\ 1-b & if\, a > b, a+b \ge 1 \end{cases}$$

(see Figure 3).

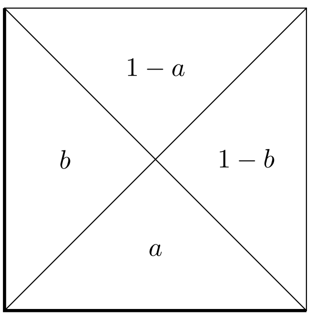

Figure 3: Structure of the distance $\triangle(a,b)$ in Example 4, for the case $N = 1 - id$.

The following result can be found in [6].

Proposition 8 *Given a strong negation N, the t-norms T such that*

1) *$T(a,b) = 0$ when $b \le N(a)$*
2) *T is positive and continuous in the region $\{(a,b) : b > N(a)\}$*

have the form

$$T(a,b) = \begin{cases} 0, \\ if\;\, b \le N(a) \\ \\ \alpha + (\beta-\alpha)\, T_1\left(\frac{a-\alpha}{\beta-\alpha}, \frac{b-\beta}{\beta-\alpha}\right), \\ if\;\, b > N(a), \max(a,b) < \beta\, (\alpha \ne \beta) \\ \\ \min(a,b), \\ if\;\, b > N(a), \max(a,b) \ge \beta \end{cases} \quad (5)$$

where $0 \le \alpha \le \beta \le 1, N(\alpha) = \beta$, and T_1 is a continuous and non-strict archimedean t-norm with zero region $\{(a,b) : b \le N_\alpha^\beta(a)\}$, where N_α^β is the strong negation defined by $N_\alpha^\beta(a) = \frac{N((\beta-\alpha)a+\alpha)-\alpha}{\beta-\alpha}$ (see Figure 4).

Remark 4

i) *If $\alpha = 0$ and $\beta = 1$, then $N_0^1 = N$, and T is a continuous and non-strict archimedean t-norm with zero region $\{(a,b) : b \le N(a)\}$.*
ii) *The case $\alpha = \beta$ (point of symmetry of the negation N) means that T has the form:*

$$T(a,b) = \begin{cases} 0 & if\, b \le N(a) \\ \min(a,b) & if\, b > N(a), \\ & \max(a,b) \ge \beta \end{cases}$$

that is, $T = M_N$.

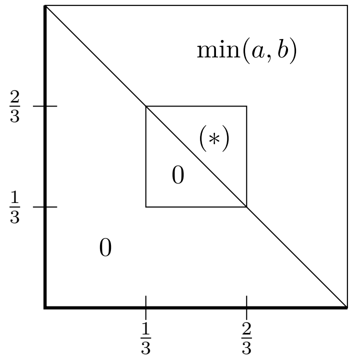

Figure 4: The structure of the t-norm in Proposition 8 for $\alpha = 1/3, \beta = 2/3$ and $N = 1 - id$, where $(*)$ stands for $\alpha + (\beta-\alpha)\, T_1\left(\frac{a-\alpha}{\beta-\alpha}, \frac{b-\beta}{\beta-\alpha}\right)$.

According to Proposition 3, Proposition 5, and Example 4, we have

Proposition 9 *For the t-norms T of the form given in (5), the function $\triangle$ defined in (1) is a distance if, and only if, the following conditions hold:*

i) *$N \ge 1 - id$.*
ii) *$1-f$ is superadditive, where f is the normalized additive generator of the t-norm T_1 (with $\alpha \ne \beta$).*

4. Conclusions

We present a full description of those triplets (T, S, N), T a t-norm, S a t-conorm and N a strong negation, such that the symmetric difference function $\triangle(a,b) = S(T(a,N(b)), T(b,N(a))$ is a distance.

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