Construction of Typical Hesitant Triangular Norms regarding Xu-Xia-partial Order

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Abstract

Fuzzy Sets theory has been applied in many fields to handle uncertainty, which can be reflected on the membership degree of the objects belonging to a set. Moreover, Hesitant Fuzzy Sets are useful whenever there is indecision among several possible values for the preferences over objects in the process of decision-making. In this sense, the aim of this work is to consider the notion of aggregation functions regarding Xu-Xia-partial order for typical hesitant fuzzy elements and to study Typical Hesitant Triangular Norms and some of the properties usually demanded to such operator.

Keywords: Typical Hesitant Fuzzy Sets, t-norms, Xu-Xia-partial order

1. Introduction

Since the pioneering paper of Zadeh on fuzzy sets theory [27], many extensions have been proposed to deal with the problem of determining an exact membership degree, such as Type-2 Fuzzy Sets (T2FS) [28, 29], Set-Valued Fuzzy Sets (SVFSs) [13], Atanassov's Intuitionistic Fuzzy Sets (IFS) [1], Fuzzy Multisets (MSs) [26], or Hesitant Fuzzy Sets (HFSs) [19, 20]. In [9], it was observed that in fact the concept of HFSs are the same as SVFSs, as defined by Grattan-Guinness.

Anyways, in 2010 Vicenç Torra in [19, 20] proposed a new extension for fuzzy sets, namely hesitant fuzzy sets (HFSs), where the membership degree is given as a subset of the unit interval. For instance, if we consider that a group of indistinguishable experts must provide a membership degree for an element of the universe and there is not a consensus, it seems more appropriate to consider the set of possible values that takes into account the opinion of all experts. Many researches in decision making have used HFS theory (see for example [12, 22, 23, 24]). In particular, several weighted average and ordered weighted average (OWA)-like operators have been proposed to be used in decision making (as we can see in [6, 23, 30]).

Moreover, systems of inference modeled by Fuzzy Logic deal with information that can be formally discussed and compared in terms of the partial ordered sets defined consistently within the framework of lattice theory. In [6], hesitant aggregation functions (HAFs) were presented in accordance with the definition of aggregation functions valued in a complete bounded lattice, i.e. based on an order for the typical hesitant fuzzy values (THFV). They proposed to use that order on the THFV to define typical hesitant fuzzy negations (THFNs), and to study a way for obtaining THFNs from fuzzy negations, providing a characterization of strong and strict THFN, as well as some other results for THFN. However, that order seemed not completely appropriate for this, because the natural hesitant fuzzy negation $\mathcal{N}(X) = \{1 - x : x \in X\}$ does not satisfy the property of antitonicity [5]. This problem was overcome in [6] by weakening the antitonicity property, which resulted in a hard condition (see [6]) despite the good results. In [18], it was proposed to consider Xu-Xia-partial order introduced in [25] for the antitonicity property. So, our goal here is to introduce the construction of Hesitant Triangular Norms taking into account Xu-Xia-partial order and to generalize some of the properties usually demanded to such operator.

The outline of this paper is organized as follows. In Section 2 we present some preliminary definitions. Then, in Sections 3 we introduce relevant concepts related to Typical Hesitant Fuzzy Sets, where Xu-Xia-partial order is formally defined. Section 4 is devoted to present the results obtained. And, finally, we have some conclusions and references.

2. Preliminaries

In this section, basic concepts of aggregation functions on the unit interval $[0,1]$ are reviewed, and also some properties and examples are recalled, including the important class of triangular norms and fuzzy negation.

Definition 1. *Let $m \in \mathbb{N}$ such that $m \geq 2$. A function $A : [0,1]^m \to [0,1]$ is a m-ary aggregation operator,*

1. *If $x_i \leq y_i$ for each $i = 1, \ldots, m$, then $A(x_1, \ldots, x_m) \leq A(y_1, \ldots, y_m)$, for each $x_1, \ldots, x_m, y_1, \ldots, y_m \in [0,1]$;*
2. *$A(0, \ldots, 0) = 0$;*
3. *$A(1, \ldots, 1) = 1$.*

Definition 2. *A function $T : [0,1]^2 \to [0,1]$ is a t-norm if, for each $x, y, z \in [0,1]$, it satisfies (i)-(iv):*

(i.) *It is commutative:* $T(x,y) = T(y,x)$;
(ii.) *It is associative:* $T(x,T(y,z)) = T(T(x,y),z)$;
(iii.) *It is increasing in the first component (monotone): if* $x \leq y$ *then* $T(x,z) \leq T(y,z)$; *and*
(iv.) 1 *is the neutral element:* $T(x,1) = x$.

Observe that each t-norm is an aggregation operator.

For an arbitrary t-norm T, we can consider the following properties:

- $[SM]$ T is said to be strictly monotonic if $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$.
- $[CL]$ T satisfies the cancellation law if $T(x,y) = T(x,z)$ implies that $x = 0$ or $y = z$.
- $[ZD]$ An element $a \in]0,1[$ is said to be a zero divisor of T if there is some $b \in]0,1[$ such that $T(a,b) = 0$.
- $[TI]$ An element $a \in [0,1]$ is said to be an idempotent element of T if $T(a,a) = a$. The numbers 0 and 1 (which are idempotent elements for each t-norm T) are said to be trivial idempotent elements of T.

Definition 3. *A function* $N : [0,1] \to [0,1]$ *is a fuzzy negation if* $N(0) = 1, N(1) = 0$ *and it is decreasing, i.e. for each* $x, y \in [0,1]$ *if* $x \leq y$ *then* $N(y) \leq N(x)$. *A fuzzy negation* N *is strict if it is continuous and* $N(x) < N(y)$ *when* $y < x$. *A fuzzy negation* N *is strong if it is involutive, i.e.* $N(N(x)) = x$ *for each* $x \in [0,1]$.

The most common strong fuzzy negation is $N_S(x) = 1 - x$ known as the standard or Zadeh negation. Each strong fuzzy negation is strict but the converse does not hold. For example, the negation $N(x) = 1 - \sqrt{x}$ is strict but it is not strong.

Let $n \geq 1$, so we consider the n-dimensional upper simplex:

$$L_n([0,1]) = \{(x_1, \ldots, x_n) \in [0,1]^n : x_1 \leq \cdots \leq x_n\}. \tag{1}$$

Note that $L_1([0,1]) = [0,1]$ and $L_2([0,1])$ reduces to the usual lattice $L([0,1])$ of all the closed subintervals of $[0,1]$. Elements of $L_n([0,1])$ are called n-dimensional intervals [3].

For each $i = 1, \ldots, n$, the i-th projection of $L_n([0,1])$ is the function $\pi_i : L_n([0,1]) \to [0,1]$ defined by $\pi_i(x_1, \ldots, x_n) = x_i$.

The product order on $L_n([0,1])$ is defined as follows:

$$\mathbf{x} \leq_{L_n} \mathbf{y} \text{ iff } \pi_i(\mathbf{x}) \leq \pi_i(\mathbf{y}) \text{ for each } i = 1, \ldots, n.$$

Observe that $\langle L_n([0,1]), \leq \rangle$ is a continuous lattice, hence it is a distributive complete lattice ([14]). The meet and join operations on $L_n([0,1])$ are defined by:

$$\mathbf{x} \vee \mathbf{y} = (\max(x_1,y_1), \ldots, \max(x_n,y_n)) \text{ and}$$
$$\mathbf{x} \wedge \mathbf{y} = (\min(x_1,y_1), \ldots, \min(x_n,y_n))$$

3. Typical Hesitant Fuzzy Sets

3.1. A brief historical overview

Fuzzy sets theory has been applied in many fields to handle uncertainty, which can be reflected on the membership degree of the objects belonging to a set. However, when different sources of fuzziness appear simultaneously it is hard to manipulate such uncertainty. In this sense, different extensions of fuzzy sets have been presented in the literature in order to overcome that difficulty.

In order to understand the history behind Hesitant Fuzzy Sets, we will briefly discuss here some of the ideas given by Bustince et al. in [9] which analyzed the relationships among different types of fuzzy sets (FSs). Besides, they also observed that in fact the concept of HFSs are the same as SVFSs, as defined by Grattan-Guinness in 1976 [13].

In this sense, SVFSs were defined as FSs where the membership degrees were understood as subsets of $[0,1]$, formally:

Definition 4. *A SVFS* A *on* X *is a mapping* $A : X \to \wp([0,1]) \setminus \{\emptyset\}$.

And here $\wp([0,1])$ is the set of all subsets of $[0,1]$, i.e., the power set of $[0,1]$.

From the notions given by Zadeh in [28, 29], a type-2 fuzzy set (T2FS) can be defined as follows:

Definition 5. *A T2FS* A *on* X *is a mapping* $A : X \to FS([0,1])$, *where* $FS([0,1])$ *is the class of fuzzy sets over the referential* $[0,1]$.

A SVFS A can also be seen as a T2FS B_A by the following equation (defined in [9]), for each $x \in X$, the FS $B_A(x) : [0,1] \to [0,1]$:

$$B_A(x)(u) = \begin{cases} 1, & \text{if } u \in A(x); \\ 0, & \text{otherwise.} \end{cases}$$

In such case the union and intersection of FSs (as proposed by Zadeh) can not be recovered and it still is an interesting open problem to find a lattice structure on the class of all SVFSs on X in a way that such case holds when restricted to FSs.

And as Grattan-Guinness did not foresee that, Torra in [20] provided a possible solution for this problem, giving birth to Hesitant Fuzzy Sets.

3.2. Hesitant Fuzzy Sets and Typical Hesitant Fuzzy Sets

As mentioned before Hesitant Fuzzy Sets (HFSs), presented in [19, 20], represent the membership degree of an element to a set by means of a subset of $[0,1]$. HFSs are useful to handle situations where there is indecision among several possible values for the preferences over objects in the process of decision-making.

Let $\wp([0,1])$ be the power set of $[0,1]$. A HFS A defined over U, where U is a non-empty set, is given by:

$$A = \{(x, \mu_A(x)) : x \in U\} \tag{2}$$

where $\mu_A : U \to \wp([0,1])$.

We have a particular case if $\mu_A(x)$ is finite and non-empty for each $x \in U$, which leads us to the following definition of Typical Hesitant Fuzzy Set (THFS).

Definition 6. *[6] Let $\mathbb{H} = \{X \subseteq [0,1] : X$ is finite and $X \neq \emptyset\}$. A THFS A defined over U is given by Eq. (2), where $\mu_A : U \to \mathbb{H}$.*

Each $X \in \mathbb{H}$ is named Typical Hesitant Fuzzy Element (THFE) of $\mathbb{H}$ and the cardinality of X (number of its elements) is referred to as $\#X$.

Some examples of THFSs, where $X, Y \in \mathbb{H}$, could be: $X = \{0.1, 0.2, 0.9\}$ and $Y = \{0.6, 0.5\}$ where the cardinality of $\#X = 3$ and $\#Y = 2$.

The set of all unitary subsets on $\wp([0,1])$ called **diagonal** elements of $\mathbb{H}$, is denoted by $\mathcal{D}_{\mathbb{H}}$, i.e. $\mathcal{D}_{\mathbb{H}} = \{X \in \mathbb{H} : \#X = 1\}$. Moreover, interval-valued and Atanassov intuitionistic fuzzy values[1] and also $\mathcal{D}_{\mathbb{H}}$ can be seen as THFEs from the mathematical point of view.

3.3. Partial orders on $\mathbb{H}$

First of all, we must consider the necessity of a partial order on the THFS. The reasons why the usual inclusion order $\subseteq$, restricted to the set $\mathbb{H} \subseteq \wp([0,1])$, is not suitable for comparing THFSs were discussed in [6]. It was shown that $\subseteq$ does not match the usual order on $[0,1]$ when we have to deal with degenerate elements of $\mathbb{H}$, for instance the complete bounded lattice $\langle \mathcal{D}_{\mathbb{H}}, \subseteq \rangle \not\cong \langle [0,1], \leq \rangle$. Besides, they considered two ways of the normalization process on subsets of $\mathbb{H}$, α-normalization and β-normalization – consisting of removing elements of the set having more elements and adding elements to the set with a lower cardinality, respectively.

We suggest using Xu-Xia-partial order [25], where for the purpose of comparing two HFEs of different cardinalities, they assume the pessimistic scenario where the decision makers' expect unfavorable outcomes. So, they repeat the shortest element of the HFE with lower cardinality until both HFEs have the same cardinality[2] (a kind of β-normalization according to [6]) and then, they order these elements and compare the greatest elements of both HFEs and so on, which means comparing the elements until reaching the lowest of both HFEs. Formally, this will be defined as follows and to do so, we denote $\mathbb{N}_k = \{1, \ldots, k\}$.

Definition 7. *Given $X \in \mathbb{H}$, let $\sigma_X : \mathbb{N}_{\#X} \to X$ be a mapping such that for any $i \in \mathbb{N}_{\#X-1}$,*

$$\sigma_X(i) < \sigma_X(i+1).$$

This is equivalent to consider an increasing re-ordering (via permutation) of the elements of X.

Definition 8. *For any $n \in \mathbb{N}^+ = \mathbb{N} - \{0\}$, let $\beta_n : \mathbb{H} \to L_n$ be the function defined by*

$$\beta_n(X) = \begin{cases} [\underbrace{\sigma_X(1), \ldots, \sigma_X(1)}_{n\text{-}m+1 \text{ times}}, \sigma_X(2), \ldots, \sigma_X(m)], & \text{if } m \leq n; \\ [\sigma_X(m-n+1), \ldots, \sigma_X(m)], & \text{otherwise.} \end{cases} \quad (3)$$

where $m = \#X$.

In [18], Xu-Xia-partial order denoted by $\leq_{xx}$ was defined as follows:

Definition 9. *Given $X, Y \in \mathbb{H}$, we say $X \leq_{xx} Y$ if and only if $\beta_n(X) \leq_{L_n} \beta_n(Y)$, where $n = \max(\#X, \#Y)$.*

As an example of how that works, let $X = \{0.1, 0.2, 0.9\}$ and $Y = \{0.6, 0.5\}$. By reordering Y according to Def. 7, we get $Y = \{0.5, 0.6\}$. And by applying Eq. 3 on Y, we obtain $Y = \{0.5, 0.5, 0.6\}$. Observe that X and Y are not comparable.

Proposition 1. *$\langle \mathbb{H}, \leq_{xx} \rangle$ is a complete lattice with the bottom and top elements given by $\mathbf{0}_{\mathbb{H}} = \{0\}$ and $\mathbf{1}_{\mathbb{H}} = \{1\}$, respectively.*

Proof. Clearly, $\mathbf{0}_{\mathbb{H}} \leq_{xx} X \leq_{xx} \mathbf{1}_{\mathbb{H}}$, for each $X \in \mathbb{H}$ and $\leq_{xx}$ is a partial order. Consider $X, Y \in \mathbb{H}$ and $n = \max(\#X, \#Y)$. Equations (4) and (5) determine the join and meet operations of X and Y.

$$X \vee Y = \{\max(\pi_i(\beta_n(X)), \pi_i(\beta_n(Y))) : i = 1, \ldots, n\} \quad (4)$$

$$X \wedge Y = \{\min(\pi_i(\beta_n(X)), \pi_i(\beta_n(Y))) : i = 1, \ldots, n\} \quad (5)$$

Observe that in these equations it is not suitable to allow repetitions of the projections π_i.

So, $\langle \mathbb{H}, \leq_{xx} \rangle$ is a bounded lattice. Let $\{X_i\}_{i \in I}$ be a family of THFEs indexed by a set of index I and $n = \max\{\#X_i : i \in I\}$. Then $\bigvee_{i \in I} X_i = \{\max_{i \in I} \pi_j(\beta_n(X_i)) : j = 1, \ldots, n\}$ is the join operation in the lattice $\langle \mathbb{H}, \leq_{xx} \rangle$ of the family $\{X_i\}_{i \in I}$. Therefore, $\langle \mathbb{H}, \leq_{xx} \rangle$ is also a complete lattice. □

Proposition 2. *Let $X, Y \in \mathbb{H}$. If $X \wedge Y \in \mathcal{D}_{\mathbb{H}}$ ($X \vee Y \in \mathcal{D}_{\mathbb{H}}$) then $X \wedge Y = X$ or $X \wedge Y = Y$ ($X \vee Y = X$ or $X \vee Y = Y$).*

Proof. Straightforward from Eqs. (4) and (5). □

Note that the converse does not hold. Just take, for instance, $X = \{0.3, 0.4\}$ and $Y = \{0.5, 0.6\}$. Then $X \wedge Y = X$, but clearly $X, Y \notin \mathcal{D}_{\mathbb{H}}$.

[1] I) Intervals may be recovered since it is enough to give the lower bound and the upper bound. In this case, there is no problem with degenerate intervals since they are identified, correctly, to points. II). A IFSs may be recovered taking into account the mathematical equivalence between them and IVFSs.[9]

[2] Notice that the result of this process is not a THFE, but a finite and non-empty multiset over $[0,1]$.

3.4. $\mathbb{H}$-automorphisms

Automorphisms play an important role in fuzzy logic and its extensions as they permit a simple characterization of some classes of operators (see for example [2, 7, 21]).

Definition 10. *A mapping $\phi : [0,1] \to [0,1]$, is an automorphism if it is bijective and monotonic, i.e. $x \leq y \Rightarrow \phi(x) \leq \phi(y)$ [16, 15]. An equivalent definition was given in [10], where automorphisms are continuous and strictly increasing functions $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$.*

We consider a similar notion of automorphism presented in [6] for typical hesitant elements and other results that can also be found in that paper. However, in here we use Xu-Xia-partial order.

Definition 11. *A function $\Phi : \mathbb{H} \to \mathbb{H}$ is a $\mathbb{H}$-automorphism if it is bijective and, for each $X, Y \in \mathbb{H}$, $X \leq_{xx} Y$ if and only if $\Phi(X) \leq_{xx} \Phi(Y)$.*

Example 1. *Consider the mapping $\Phi : \mathbb{H} \to \mathbb{H}$, for every $X = \{x_1, \ldots, x_n\} \in \mathbb{H}$, defined by: $\Phi(X) = \{x_1^2, \ldots, x_n^2\}$. Then Φ is a $\mathbb{H}$-automorphism.*

Proposition 3. *Let $\Phi : \mathbb{H} \to \mathbb{H}$ be a $\mathbb{H}$-automorphism. Then:*

1. $\Phi(X) = \mathbf{1}_{\mathbb{H}}$ *if and only if* $X = \mathbf{1}_{\mathbb{H}}$.
2. $\Phi(X) = \mathbf{0}_{\mathbb{H}}$ *if and only if* $X = \mathbf{0}_{\mathbb{H}}$.
3. $\Phi(X \vee Y) = \Phi(X) \vee \Phi(Y)$, *for each* $X, Y \in \mathbb{H}$.
4. $\Phi(X \wedge Y) = \Phi(X) \wedge \Phi(Y)$, *for each* $X, Y \in \mathbb{H}$.

Proof. It follows from the fact that Φ is an automorphism on the lattice $\langle \mathbb{H}, \leq_{xx} \rangle$ and lattice automorphism on bounded lattices are isomorphisms (see [4, 8, 17]). ☐

Proposition 4. *[6] Let $Aut(\mathbb{H})$ be the set of all $\mathbb{H}$-automorphisms. Then $(Aut(\mathbb{H}), \circ)$ is a group.*

Proof. By Definition 11, $\mathbb{H}$-automorphisms are closed under composition and the inverse of a $\mathbb{H}$-automorphism is also a $\mathbb{H}$-automorphism. So, once the composition of a function is associative and the identity function trivially is a $\mathbb{H}$-automorphism which is neutral with respect to the composition, then $(Aut(\mathbb{H}), \circ)$ is a group. ☐

Theorem 1. *[18] $\Phi : \mathbb{H} \to \mathbb{H}$ is a $\mathbb{H}$-automorphism if and only if there exists an automorphism $\phi : [0,1] \to [0,1]$ such that, for all $X \in \mathbb{H}$, it follows that*

$$\Phi(X) = \{\phi(x) : x \in X\}. \tag{6}$$

4. Typical Hesitant Triangular Norms regarding Xu-Xia-partial order

In [6] it was presented the extension of the notion of t-norms for typical hesitant elements and we recall this notion in this section. However, notice that on the next definition we use $\leq_{xx}$ in the third item. Our motivation comes from the fact that the partial order used in [6] does not guarantee the antitonicity property needed to define negations in the context of typical hesitant fuzzy elements. In this sense, Typical Hesitant Fuzzy Negations endowed with Xu-Xia-partial order was introduced in [18] and in here we intend to extend the notions of t-norms for typical hesitant elements.

Definition 12. *Let $\mathcal{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$. $\mathcal{T}$ is a typical hesitant triangular norm, $\mathbb{H}$-t-norm in short, if*

(i.) *It is commutative:* $\mathcal{T}(X, Y) = \mathcal{T}(Y, X)$;
(ii.) *It is associative:* $\mathcal{T}(X, \mathcal{T}(Y, Z)) = \mathcal{T}(\mathcal{T}(X, Y), Z)$;
(iii.) *It is monotone, i.e., if* $X \leq_{xx} Y$ *then* $\mathcal{T}(X, Z) \leq_{xx} \mathcal{T}(Y, Z)$; *and*
(iv.) $\mathbf{1}_{\mathbb{H}}$ *is the neutral element:* $\mathcal{T}(X, \mathbf{1}_{\mathbb{H}}) = X$.

Now, let $\mathcal{T}$ be a $\mathbb{H}$-t-norm in $\mathbb{H}$. It follows that:

- $[SM_{\mathbb{H}}]$ $\mathcal{T}$ is said to be strictly monotonic if $\mathcal{T}(X, Y) <_{xx} \mathcal{T}(X, Z)$ whenever $\mathbf{0}_{\mathbb{H}} <_{xx} X$ and $Y <_{xx} Z$.
- $[CL_{\mathbb{H}}]$ $\mathcal{T}$ satisfies the cancellation law if $\mathcal{T}(X, Y) = \mathcal{T}(X, Z)$ implies that $X = \mathbf{0}_{\mathbb{H}}$ or $Y = Z$.
- $[ZD_{\mathbb{H}}]$ $\mathcal{T}$ has a zero divisor if there are $X, Y \in \mathbb{H}$ such that $\mathcal{T}(X, Y) = \mathbf{0}_{\mathbb{H}}$, $X \neq \mathbf{0}_{\mathbb{H}}$ and $Y \neq \mathbf{0}_{\mathbb{H}}$.
- $[TI_{\mathbb{H}}]$ An element $A \in [0,1]$ is said to be an idempotent element of $\mathcal{T}$ if $\mathcal{T}(A, A) = A$. $\mathbf{0}_{\mathbb{H}}$ and $\mathbf{1}_{\mathbb{H}}$ (which are idempotent elements for each t-norm $\mathcal{T}$) are said to be trivial idempotent elements of $\mathcal{T}$.

Theorem 2. *Let $\mathcal{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a binary function and $\Phi : \mathbb{H} \to \mathbb{H}$ be a $\mathbb{H}$-automorphism defined by ϕ in Eq.(6). It follows that $\mathcal{T}^{\Phi}$ is a $\mathbb{H}$-t-norm if and only if $\mathcal{T}$ is a $\mathbb{H}$-t-norm.*

Proposition 5. *Let $\mathcal{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a $\mathbb{H}$-t-norm, then:*

(i.) If $\mathcal{T}$ satisfies the cancellation law, then $\mathcal{T}$ is strictly monotonic;
(ii.) If $\mathcal{T}$ is strictly monotonic then it has only trivial idempotent elements;
(iii.) If $\mathcal{T}$ is strictly monotonic, then it has no zero divisors.

Proof. Let $\mathcal{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a $\mathbb{H}$-t-norm and $X, Y, Z \in \mathbb{H}$.

(i.) If $X \neq \mathbf{0}_{\mathbb{H}}$ and $Y <_{xx} Z$ then by the monotonic property, $\mathcal{T}(X, Y) \leq_{xx} \mathcal{T}(X, Z)$. If $\mathcal{T}$ satisfies $[CL_{\mathbb{H}}]$ and $\mathcal{T}(X, Y) = \mathcal{T}(X, Z)$, then $X = \mathbf{0}_{\mathbb{H}}$ or $Y = Z$, which is a contradiction. So, $\mathcal{T}$ is strictly monotonic.

(ii.) The fact that $\mathbf{1}_{\mathbb{H}}$ is the neutral element and that $\mathcal{T}$ being strictly monotonic implies that $\mathcal{T}(X, X) <_{xx} \mathcal{T}(X, \mathbf{1}_{\mathbb{H}}) = X$ for all $X \in \mathbb{H} \setminus \{\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}\}$, it is enough to state that $\mathcal{T}$ has only trivial idempotent elements.

(iii.) If Y is a zero divisor, i.e. $\mathcal{T}(Y,X) = \mathbf{0}_{\mathbb{H}}$ for some $X \in \mathbb{H} \setminus \{\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}\}$, then $Z = \{\frac{x}{n} : x \in X\} \in \mathbb{H} \setminus \{\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}\}$ and $Z <_{xx} X$. So, because $\mathcal{T}$ is strictly monotonic, we have that $\mathcal{T}(Y,Z) <_{xx} \mathcal{T}(Y,X) = \mathbf{0}_{\mathbb{H}}$, which is an absurd. □

Corollary 1. *If $\mathcal{T}$ satisfies $[CL_{\mathbb{H}}]$, then:*

(vi.) $\mathcal{T}$ also satisfies $[TI_{\mathbb{H}}]$.
(v.) $\mathcal{T}$ does not satisfy $[ZD_{\mathbb{H}}]$.

Proof. Straightforward from Prop. 5 □

Bearing in mind the notion of t-representability given by Deschrijver et al. in [11], we can say that a $\mathbb{H}$-t-norm $\mathcal{T}$ is said to be t-representable if there are t-norms $T_1, \ldots, T_n$ such that:

$$\mathcal{T}(X,Y) = \bigcup_{i=1}^{n} \{T_i(x,y) : x \in X \text{ and } y \in Y\}.$$

The following examples show that there exist t-representable $\mathbb{H}$-t-norms and non-t-representable $\mathbb{H}$-t-norms.

Example 2. *T-representable $\mathbb{H}$-t-norm.*

$$\mathcal{T}(X,Y) = \{xy : x \in X \text{ and } y \in Y\}.$$

Example 3. *Non-t-representable $\mathbb{H}$-t-norm.*

$$\mathcal{T}(X,Y) = \{\min(\pi_i(\beta_n(X)), \pi_i(\beta_n(Y))) : i=1,\ldots,n\}$$

where $n = \max(\#X, \#Y)$. For example,
$\mathcal{T}(\{0.3, 0.5, 0.7, 0.8\}, \{0.6, 0.9\}) = \{0.3, 0.5, 0.6, 0.8\}$ *and*
$\mathcal{T}(\{0.3, 0.5, 0.7, 0.8\}, \{0.2, 0.6, 0.9\}) = \{0.2, 0.6, 0.8\}$.

4.1. Typical Hesitant Fuzzy Negations

In [18], it was presented the notion of Typical Hesitant Fuzzy Negations (THFNs) using Xu-Xia-partial order, $\leq_{xx}$, as in Definition 9. In this way, as we can see in the following definition, a THFN must fulfil [BC] and [D] properties.

Definition 13. *Let $\mathcal{N} : \mathbb{H} \to \mathbb{H}$ be a function. $\mathcal{N}$ is a THFN if*

[BC] $\mathcal{N}(\mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ and $\mathcal{N}(\mathbf{1}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$ (boundary condition).

[D] And if $X \leq_{xx} Y$ then $\mathcal{N}(Y) \leq_{xx} \mathcal{N}(X)$ (decreasing).

A THFN $\mathcal{N}$ is strong if it is involutive, i.e. if for each $X \in \mathbb{H}$ it satisfies [In]: $\mathcal{N}(\mathcal{N}(X)) = X$.

Proposition 6. *Let $\mathcal{N} : \mathbb{H} \to \mathbb{H}$ be a strong THFN. Then for each $X, Y \in \mathbb{H}$,*

1. *$\mathcal{N}(X \vee Y) = \mathcal{N}(X) \wedge \mathcal{N}(Y)$.*
2. *$\mathcal{N}(X \wedge Y) = \mathcal{N}(X) \vee \mathcal{N}(Y)$.*

Proof. Let $X, Y \in \mathbb{H}$. Since, $X \leq_{xx} X \vee Y$ and $Y \leq_{xx} X \vee Y$, then $\mathcal{N}(X \vee Y) \leq_{xx} \mathcal{N}(X)$ and $\mathcal{N}(X \vee Y) \leq_{xx} \mathcal{N}(Y)$. So, (*) $\mathcal{N}(X \vee Y) \leq_{xx} \mathcal{N}(X) \wedge \mathcal{N}(Y)$.

Analogously, it is possible to prove that (**) $\mathcal{N}(X) \vee \mathcal{N}(Y) \leq_{xx} \mathcal{N}(X \wedge Y)$.

Suppose that for some $X, Y \in \mathbb{H}$, $\mathcal{N}(X \vee Y) <_{xx} \mathcal{N}(X) \wedge \mathcal{N}(Y)$. So, as $\mathcal{N}(\mathcal{N}(X) \wedge \mathcal{N}(Y)) <_{xx} \mathcal{N}(\mathcal{N}(X \vee Y))$, then by (**) and [In], $X \vee Y = \mathcal{N}(\mathcal{N}(X)) \vee \mathcal{N}(\mathcal{N}(Y)) \leq_{xx} \mathcal{N}(\mathcal{N}(X) \wedge \mathcal{N}(Y)) <_{xx} X \vee Y$ which is a contradiction. Therefore, $\mathcal{N}(X \vee Y) = \mathcal{N}(X) \wedge \mathcal{N}(Y)$ for each $X, Y \in \mathbb{H}$. In an analogous way, it is possible to prove that $\mathcal{N}(X \wedge Y) = \mathcal{N}(X) \vee \mathcal{N}(Y)$.

□

4.2. Typical Hesitant Triangular Conorms regarding Xu-Xia-partial order

In the same way, the extension of the notion of t-conorms for typical hesitant elements can also be done. Thus,

Definition 14. *Let $\mathcal{S} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$. $\mathcal{S}$ is a typical hesitant triangular conorm, $\mathbb{H}$-t-conorm in short, if for each $X, Y, Z \in \mathbb{H}$ the following properties are satisfied:*

(i.) It is commutative: $\mathcal{S}(X,Y) = \mathcal{S}(Y,X)$;
(ii.) It is associative: $\mathcal{S}(X, \mathcal{S}(Y,Z)) = \mathcal{S}(\mathcal{S}(X,Y), Z)$;
(iii.) It is monotone: if $X \leq_{xx} Y$ then $\mathcal{S}(X,Z) \leq_{xx} \mathcal{S}(Y,Z)$; and
(iv.) $\mathbf{0}_{\mathbb{H}}$ is the neutral element: $\mathcal{S}(X, \mathbf{0}_{\mathbb{H}}) = X$.

It is also possible to obtain the dual $\mathbb{H}$-t-conorm ($\mathcal{S}_{\mathcal{T}}$) of the $\mathbb{H}$-t-norm $\mathcal{T}$ using the notion of negations for typical hesitant elements using Xu-Xia-partial order introduced in [18].

Proposition 7. *Let $\mathcal{N}$ be a strong THFN and $\mathcal{T}$ be a $\mathbb{H}$-t-norm, then the function $\mathcal{S}_{\mathcal{T}} : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ defined by*

$$\mathcal{S}_{\mathcal{T}}(X,Y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(Y)))$$

is a $\mathbb{H}$-t-conorm called the dual $\mathbb{H}$-t-conorm of $\mathcal{T}$.

Proof. The four properties given in Def. 14 must be held for $\mathcal{S}_{\mathcal{T}}$. So we have that:

(i.) For each $X, Y \in \mathbb{H}$, $\mathcal{S}_{\mathcal{T}}$ is commutative:

$$\begin{aligned} \mathcal{S}_{\mathcal{T}}(X,Y) &= \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(Y))) \\ &= \mathcal{N}(\mathcal{T}(\mathcal{N}(Y), \mathcal{N}(X))) \\ &= \mathcal{S}_{\mathcal{T}}(Y,X). \end{aligned}$$

(ii.) For each $X, Y, Z \in \mathbb{H}$, $\mathcal{S}_{\mathcal{T}}$ is associative:

$\mathcal{S}_\mathcal{T}(X, \mathcal{S}_\mathcal{T}(Y, Z)) =$

$$
\begin{aligned}
&= \mathcal{S}_\mathcal{T}(X, (\mathcal{N}(\mathcal{T}(\mathcal{N}(Y), \mathcal{N}(Z))))) \\
&= \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(\mathcal{N}(\mathcal{T}(\mathcal{N}(Y), \mathcal{N}(Z)))))) \\
&= \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{T}(\mathcal{N}(Y), \mathcal{N}(Z)))) \\
&= \mathcal{N}(\mathcal{T}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(Y)), \mathcal{N}(Z))) \\
&= \mathcal{N}(\mathcal{T}(\mathcal{N}(\mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(Y)))), \mathcal{N}(Z))) \\
&= \mathcal{N}(\mathcal{T}(\mathcal{N}(\mathcal{S}_\mathcal{T}(X, Y)), \mathcal{N}(Z))) \\
&= \mathcal{S}_\mathcal{T}(\mathcal{S}_\mathcal{T}(X, Y), Z).
\end{aligned}
$$

(iii.) For each $X, Y, Z \in \mathbb{H}$, $\mathcal{S}_\mathcal{T}$ is monotone because, if $X \leq_{xx} Y$ from the monotonicity of $\mathcal{T}$ we have that: $\mathcal{T}(\mathcal{N}(Y), \mathcal{N}(Z)) \leq_{xx} \mathcal{T}(\mathcal{N}(X), \mathcal{N}(Z))$. Besides, from Def. 13 we know [D] holds for $\mathcal{N}$, so: $\mathcal{S}_\mathcal{T}(X, Z) = \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(Z)) \leq_{xx} \mathcal{N}(\mathcal{T}(\mathcal{N}(Y), \mathcal{N}(Z))) = \mathcal{S}_\mathcal{T}(Y, Z)$.

(iv.) For each $X \in \mathbb{H}$, $\mathbf{0}_\mathbb{H}$ is the neutral element as:

$$
\begin{aligned}
\mathcal{S}_\mathcal{T}(X, \mathbf{0}_\mathbb{H}) &= \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathcal{N}(\mathbf{0}_\mathbb{H}))) \\
&= \mathcal{N}(\mathcal{T}(\mathcal{N}(X), \mathbf{1}_\mathbb{H})) \\
&= \mathcal{N}(\mathcal{N}(X)) \\
&= X.
\end{aligned}
$$

□

5. Conclusions

As stated in [11], triangular norms and conorms are an important notion in the Fuzzy Set Theory since they can be used for many purposes, such as to generalize the union and intersection in fuzzy sets or to extend the composition of fuzzy relations. Besides, using t-norms and t-conorms in fuzzy logic controllers provides more flexibility as well as reliability once it is possible to derive from those operators several fuzzy implication functions (for instance, S-implications, R-implications, QL-implications, D-implications). The importance of that relies not only in the fact that they are used in rules (such as If-Then rules) in fuzzy systems, but also because they are used in performing inferences in approximate reasoning and fuzzy control. Thus, we proposed to consider the notion of aggregation functions regarding Xu-Xia-partial order for typical hesitant fuzzy elements and used such functions to present the Typical Hesitant Triangular Norms, THFNs and Typical Hesitant Triangular Conorms and some properties demanded for such operators.

As future works we are looking forward to continue this study by considering automorphisms and the notion of negation-preserving automorphism introduced by Navara in [15] and generalized by Bedregal in [2], in the context of typical hesitant fuzzy elements. Besides, we also intend to study equilibrium points for THFNs in the same sense.

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