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A new class of fuzzy poverty measures

J.L. García-Lapresta[1] C. Lasso de la Vega[2] R.A. Marques Pereira[3] A. Urrutia[4]

[1]PRESAD Research Group, IMUVA, Dep. de Economía Aplicada, Universidad de Valladolid, Spain
[2]BRiDGE Research Group, Dep. de Economía Aplicada IV, Universidad del País Vasco UPV/EHU, Spain
[3]Dipartimento di Economia e Management, Università degli Studi di Trento, Italy
[4]BRiDGE Research Group, Dep. de Economía Aplicada IV, Universidad del País Vasco UPV/EHU, Spain

Abstract

In this paper, we introduce and analyze a class of fuzzy poverty measures based on exponential means. Since poverty is a vague notion, individuals should not be classified in poor or non-poor. In our proposal, we have associated a degree of poverty to each income through a fuzzy membership function. We have extended normalized gaps from the classical approach, where poverty is a dichotomous notion, to the fuzzy setting. The proposed family of fuzzy poverty measures decomposes into the three I's indicators: the incidence of poverty is captured through the headcount ratio, while intensity and inequality of poverty are measured by the core and the remainder, respectively, of a parameterized exponential mean over the normalized gaps of (somewhat) poor individuals. Taking into account the features of the dual decomposition of exponential means, we provide some properties of the proposed fuzzy poverty measures.

Keywords: poverty measures; aggregation functions; dual decomposition; exponential means.

1. Introduction

In this paper, we extend the results included in García-Lapresta *et al.* [15] to a fuzzy framework. While in the classical setting there is a drastic transition between poverty and non-poverty (individuals are poor or non-poor), in the present approach that transition is gradual. In this way, fuzziness in poverty is conducted by means of a poverty membership function that associates a degree of poverty between 0 and 1 to each income.

Some surveys of the literature on inequality, poverty, and welfare can be found in Chakravarty [7, 8], Zheng [31, 32] and Lambert [22]. Different analyses on the fuzzy approach to the poverty measurement can be found in Lemmi and Betti [23].

Following Cerioli and Zani [6] and Chakravarty [9], two poverty lines should be fixed[1]. Thus, incomes can be sorted into three types: extreme poverty, moderate poverty and non-poverty, whenever incomes are below the smaller poverty line, between both poverty lines or above the highest poverty line, respectively. The membership function of incomes to poverty assigns 1 to extreme poverty, 0 to non-poverty, and intermediate values to moderate poverty, through a continuous and decreasing function. Other contributions of poverty measurement within the fuzzy approach can be found in Shorrocks and Subramanian [29], Chiappero-Martinetti [11], Qizilbash [26], Subramanian [30], Belhadj [1, 2], Belhadj and Limam [3] and Zheng [33], among others.

The rest of the paper is organized as follows. Section 2 is devoted to introduce the main ingredients of poverty identification and the corresponding gaps, both in the classical and the fuzzy approaches. Section 3 contains notions and properties of aggregation functions and the dual decomposition, with special attention to the class of exponential means. In Section 4 we introduce and analyze a family of fuzzy poverty measures. Finally, Section 5 includes some comments about further research.

2. Measuring poverty

We consider a population consisting of n individuals, with $n \geq 2$. An *income distribution* is represented by a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0, \infty)^n$, where x_i is the income of individual i.

According to the Nobel Laureate Amartya K. Sen [28], every poverty measure must include a method to identify the poor and an aggregation procedure for merging the deprivation degrees of poor individuals. In the classical approach poverty is a dichotomous concept and individuals are poor or non-poor. However, the fuzzy approach recognizes poverty as a vague predicate[2]. Thus, being poor is a matter of degree.

2.1. The classical approach

In the classical approach, the identification step requires the specification of a poverty line that splits society into two groups: the poor and the non-poor people.

This step requires the specification of a *poverty line* $z \in (0, \infty)$, which represents the necessary income to maintain a minimum level of living. Given

[1]Cheli and Lemmi [10] criticized the use of these two thresholds.

[2]Qizilbash [27] justifies that using fuzzy set theory to study poverty is motivated by the vagueness of that predicate.

an income distribution $\boldsymbol{x}$, person i is considered as poor if $x_i < z$. Otherwise, this person is non-poor.

The set of poor people is denoted by

$$Q(\boldsymbol{x}, z) = \{i \in \{1, \ldots, n\} \mid x_i < z\},$$

and $q(\boldsymbol{x}, z)$ denotes the number of the poor, i.e., $q(\boldsymbol{x}, z) = \#Q(\boldsymbol{x}, z)$.

Once the poor people have been identified, the second stage to determine the extent of the poverty is the aggregation step.

The first poverty measure introduced in the literature is the *headcount ratio*,

$$H : [0, \infty)^n \times (0, \infty) \longrightarrow [0, 1],$$

defined as

$$H(\boldsymbol{x}, z) = \frac{q(\boldsymbol{x}, z)}{n},$$

that measures the percentage of poor people in the society.

For every $z \in (0, \infty)$, the *normalized gap* of a poor person is the fractional shortfall of his income from the poverty line. It is given by the function $g_z : [0, \infty) \longrightarrow [0, 1]$, defined as

$$g_z(x) = \begin{cases} \dfrac{z - x}{z}, & \text{if } x < z, \\ 0, & \text{if } x \geq z. \end{cases}$$

Notice that $g_z(x) = 0 \Leftrightarrow x \geq z$, and $g_z(x) = 1 \Leftrightarrow x = 0$.

Every poverty measure should be expressed as a function of three poverty indicators: incidence, intensity and inequality of the poverty, the so called *three I's of poverty* (see Sen [28] and Jenkins and Lambert [20]).

2.2. A fuzzy approach

Cerioli and Zani [6] propose the cardinality of the fuzzy set of poor individuals as poverty measure, that is a natural extension of the headcount ratio and the poverty measures introduced by Foster *et al.* [13] and Hagenaars [19].

Let $\boldsymbol{z} = (z_1, z_2) \in (0, \infty)^2$ be a pair of poverty lines such that $z_1 \leq z_2$ and a continuous and decreasing function $h : [z_1, z_2] \longrightarrow [0, 1]$ such that $h(z_1) = 1$ and $h(z_2) = 0$. The membership function of incomes to poverty with respect to $\boldsymbol{z} = (z_1, z_2)$ and h, $\mu_{\boldsymbol{z}} : [0, \infty) \longrightarrow [0, 1]$, is defined as (see also Fig. 1)

$$\mu_{\boldsymbol{z}}(x) = \begin{cases} 1, & \text{if } 0 \leq x < z_1, \\ h(x), & \text{if } z_1 \leq x < z_2, \\ 0, & \text{if } x \geq z_2. \end{cases}$$

Incomes in $[0, z_1)$, $[z_1, z_2)$ and $[z_2, \infty)$ correspond to extreme poverty, moderate poverty and no poverty, respectively.

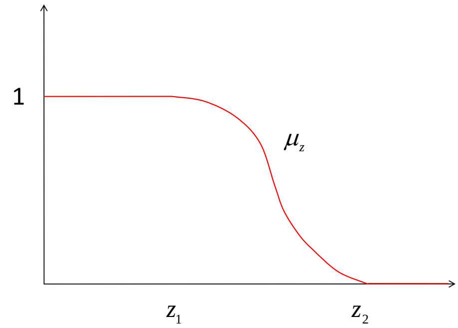

Figure 1: Membership function.

The *normalized gap* of income x is now defined as

$$g_{\boldsymbol{z}}(x) = \begin{cases} \dfrac{\int_x^{z_2} \mu_{\boldsymbol{z}}(y)\, dy}{\int_0^{z_2} \mu_{\boldsymbol{z}}(y)\, dy}, & \text{if } x < z_2, \\ 0, & \text{if } x \geq z_2. \end{cases}$$

If $z_1 = z_2 = z$, then we obtain the classical case:

$$\mu_z(x) = \begin{cases} 1, & \text{if } x < z, \\ 0, & \text{if } x \geq z \end{cases}$$

and

$$g_z(x) = \begin{cases} \dfrac{z - x}{z}, & \text{if } x < z, \\ 0, & \text{if } x \geq z. \end{cases}$$

Assumption: From now on suppose h is linear. Therefore, the membership function $\mu_{\boldsymbol{z}}$ has the following expression (see also Fig. 2)

$$\mu_{\boldsymbol{z}}(x) = \begin{cases} 1, & \text{if } 0 \leq x < z_1, \\ \dfrac{z_2 - x}{z_2 - z_1}, & \text{if } z_1 \leq x < z_2, \\ 0, & \text{if } x \geq z_2. \end{cases}$$

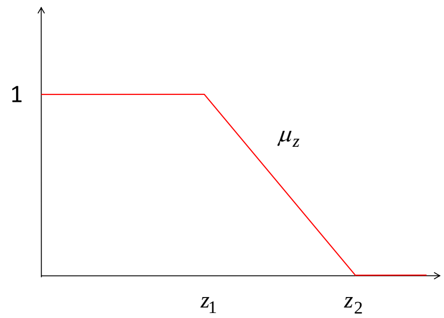

Figure 2: Linear membership function.

Then, the normalized gap of income x is expressed as (see also Fig. 3)

$$g_{\boldsymbol{z}}(x) = \begin{cases} \dfrac{z_1 + z_2 - 2x}{z_1 + z_2}, & \text{if } 0 \le x < z_1, \\[2ex] \dfrac{z_2 - x}{z_1 + z_2}, & \text{if } z_1 \le x < z_2, \\[2ex] 0, & \text{if } x \ge z_2. \end{cases}$$

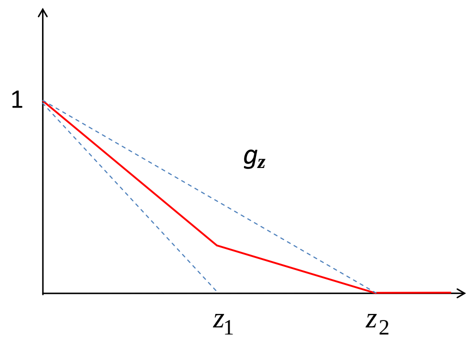

Figure 3: Normalized gap in the linear case.

Notice that the slope of $g_{\boldsymbol{z}}$ in $[0, z_1)$ is between the slopes of the dashes lines, and its absolute value is double with respect to the slope of $g_{\boldsymbol{z}}$ in $[z_1, z_2)$.

3. Aggregation functions

We begin by defining standard properties of real functions on $\mathbb{R}^n$.

Vectors in $[0,\infty)^n$ are denoted as $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{0} = (0, \ldots, 0)$, $\boldsymbol{1} = (1, \ldots, 1)$. Accordingly, for every $x \in [0,\infty)$, we have $x \cdot \boldsymbol{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$, by $\boldsymbol{x} \ge \boldsymbol{y}$ we mean $x_i \ge y_i$ for every $i \in \{1, \ldots, n\}$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \ge \boldsymbol{y}$ and $\boldsymbol{x} \ne \boldsymbol{y}$. Given $\boldsymbol{x} \in [0,\infty)^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \le \cdots \le x_{(n)}$ and $x_{[1]} \ge \cdots \ge x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. Clearly, $x_{[k]} = x_{(n+1-k)}$ for every $k \in \{1, \ldots, n\}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The arithmetic mean of $\boldsymbol{x}$ is denoted by $\mu(\boldsymbol{x})$.

We begin by defining standard properties of real functions on $\mathbb{R}^n$. For further details the interested reader is referred to Fodor and Roubens [12], Calvo *et al.* [5], Beliakov *et al.* [4], García-Lapresta and Marques Pereira [17] and Grabisch *et al.* [18].

Definition 1 *Let $A : \mathbb{D}^n \longrightarrow \mathbb{R}$ be a function with $\mathbb{D} = [0,1]$ or $\mathbb{D} = [0,\infty)$.*

1. *A is* idempotent *if for every $x \in \mathbb{D}$ it holds $A(x \cdot \boldsymbol{1}) = x$.*
2. *A is* symmetric *if for every permutation σ on $\{1, \ldots, n\}$ and every $\boldsymbol{x} \in \mathbb{D}^n$ it holds $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$.*
3. *A is* monotonic *if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ it holds $\boldsymbol{x} \ge \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \ge A(\boldsymbol{y})$.*
4. *A is* strictly monotonic *if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ it holds $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$.*
5. *A is* compensative *(or* internal*) if for every $\boldsymbol{x} \in \mathbb{D}^n$ it holds $x_{(1)} \le A(\boldsymbol{x}) \le x_{(n)}$.*
6. *A is* self-dual *if $\mathbb{D} = [0,1]$ and for every $\boldsymbol{x} \in [0,1]^n$ it holds $A(\boldsymbol{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x})$.*
7. *A is* anti-self-dual *if $\mathbb{D} = [0,1]$ and for every $\boldsymbol{x} \in [0,1]^n$ it holds $A(\boldsymbol{1} - \boldsymbol{x}) = A(\boldsymbol{x})$.*
8. *A is* invariant for translations *if for every $\boldsymbol{x} \in \mathbb{D}^n$ it holds $A(\boldsymbol{x} + t \cdot \boldsymbol{1}) = A(\boldsymbol{x})$ for all $t \in \mathbb{R}$ such that $\boldsymbol{x} + t \cdot \boldsymbol{1} \in \mathbb{D}^n$.*
9. *A is* stable for translations *(or* shift-invariant*) if for every $\boldsymbol{x} \in \mathbb{D}^n$ it holds $A(\boldsymbol{x} + t \cdot \boldsymbol{1}) = A(\boldsymbol{x}) + t$ for all $t \in \mathbb{R}$ such that $\boldsymbol{x} + t \cdot \boldsymbol{1} \in \mathbb{D}^n$.*

Definition 2 *Let $\succcurlyeq$ be the binary relation on $[0,\infty)^n$ defined as*

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Leftrightarrow \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \text{ and } \sum_{i=1}^{k} x_{(i)} \le \sum_{i=1}^{k} y_{(i)}$$

for every $k \in \{1, \ldots, n-1\}$.

1. *A is* Schur-convex *if for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$:*
$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \ge A(\boldsymbol{y}).$$
2. *A is* Schur-concave *if for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$:*
$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \le A(\boldsymbol{y}).$$

Definition 3 *Given $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$, we say that $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a* progressive transfer *if there exist two individuals $i, j \in \{1, \ldots, n\}$ and $h > 0$ such that $x_i < x_j$, $y_i = x_i + h \le x_j - h = y_j$ and $y_k = x_k$ for every $k \in \{1, \ldots, n\} \setminus \{i, j\}$.*

A classical result (see Marshall and Olkin [24, Ch. 4, Prop. A.1]) establishes that $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ if and only if $\boldsymbol{y}$ can be derived from $\boldsymbol{x}$ by means of a finite sequence of permutations and/or progressive transfers.

Definition 4 *Let $\{A^{(k)}\}_{k \in \mathbb{N}}$ be a sequence of functions, with $A^{(k)} : \mathbb{D}^k \longrightarrow \mathbb{R}$ and $A^{(1)}(x) = x$ for every $x \in \mathbb{D}$, where $\mathbb{D} = [0,1]$ or $\mathbb{D} = [0,\infty)$. $\{A^{(k)}\}_{k \in \mathbb{N}}$ is* invariant for replications *(or* strongly idempotent*) if for all $\boldsymbol{x} \in \mathbb{D}^n$ and any number of replications $m \in \mathbb{N}$ of $\boldsymbol{x}$:*

$$A^{(mn)}(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m}) = A^{(n)}(\boldsymbol{x}).$$

Definition 5 *A function $A : [0,1]^n \longrightarrow [0,1]$ is called an n-ary* aggregation function *if it is monotonic and satisfies $A(\boldsymbol{1}) = 1$ and $A(\boldsymbol{0}) = 0$. An aggregation function is said to be* strict *if it is strictly monotonic.*

For the sake of simplicity, the n-arity is omitted whenever it is clear from the context.

It is easy to see that every idempotent aggregation function is compensative, and viceversa.

3.1. Dual decomposition of aggregation functions

We now briefly recall the so-called *dual decomposition* of an aggregation function into its self-dual core and associated anti-self-dual remainder, due to García-Lapresta and Marques Pereira [17]. First we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation function, establishing which properties are inherited in each case from the original aggregation function. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

Definition 6 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The aggregation function $A^* : [0,1]^n \longrightarrow [0,1]$ defined as*

$$A^*(\boldsymbol{x}) = 1 - A(\mathbf{1} - \boldsymbol{x})$$

is known as the dual *of the aggregation function A.*

Clearly, $(A^*)^* = A$. Thus, an aggregation function A is self-dual if and only if $A^* = A$. According to García-Lapresta *et al.* [15, Prop. 2], the dual A^* inherits from the aggregation function A the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, self-duality, stability for translations and invariance for replications, whenever A has these properties. In addition, A^* is Schur-convex (resp. Schur-concave) whenever A is Schur-concave (resp. Schur-convex).

3.1.1. *The self-dual core of an aggregation function*

Aggregation functions are not in general self-dual. However, a self-dual aggregation function can be associated with any aggregation function in a simple manner.

Definition 7 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The function $\widehat{A} : [0,1]^n \longrightarrow [0,1]$ defined as*

$$\widehat{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) - A(\mathbf{1} - \boldsymbol{x}) + 1}{2}$$

is called the core *of the aggregation function A.*

Since $\widehat{A}$ is self-dual, we say that $\widehat{A}$ is the *self-dual core* of the aggregation function A. Notice that $\widehat{A}$ is clearly an aggregation function.

The following results[3] can be found in García-Lapresta and Marques Pereira [17].

[3] Excepting that invariance for replications is inherited by the core (the proof is immediate).

Proposition 1 *An aggregation function $A : [0,1]^n \longrightarrow [0,1]$ is self-dual if and only if $\widehat{A}(\boldsymbol{x}) = A(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$.*

Proposition 2 *The self-dual core $\widehat{A}$ inherits from the aggregation function A the properties of continuity, idempotency (hence, compensativeness), symmetry, strict monotonicity, stability for translations, and invariance for replications, whenever A has these properties.*

3.1.2. *The anti-self-dual remainder of an aggregation function*

We now introduce the *anti-self-dual remainder* $\widetilde{A}$, which is simply the difference between the original aggregation function A and its self-dual core $\widehat{A}$.

Definition 8 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The function $\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ defined as $\widetilde{A}(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}(\boldsymbol{x})$, that is,*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) - A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) - 1}{2},$$

is called the remainder *of the aggregation function A.*

Since $\widetilde{A}$ is anti-self-dual, we say that $\widetilde{A}$ is the *anti-self-dual remainder* of the aggregation function A. Clearly, $\widetilde{A}$ is not an aggregation function. In particular, $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$ violates idempotency and implies that $\widetilde{A}$ is either non monotonic or everywhere null.

The following results[4] can be found in García-Lapresta and Marques Pereira [17].

Proposition 3 *An aggregation function $A : [0,1]^n \longrightarrow [0,1]$ is self-dual if and only if $\widetilde{A}(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in [0,1]^n$.*

Proposition 4 *The anti-self-dual remainder $\widetilde{A}$ inherits from the aggregation function A the properties of continuity, symmetry, invariance for replications, plus also Schur-convexity and Schur-concavity, whenever A has these properties.*

Summarizing, every aggregation function A decomposes additively $A = \widehat{A} + \widetilde{A}$ in two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation function.

The following result concerns two more properties of the anti-self-dual remainder based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core (see García-Lapresta and Marques Pereira [17]).

Proposition 5 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function.*

[4] Excepting that invariance for replications is inherited by the remainder (the proof is immediate).

1. *If A is idempotent, then $\widetilde{A}(x \cdot \mathbf{1}) = 0$ for every $x \in [0,1]$.*
2. *If A is stable for translations, then $\widetilde{A}$ is invariant for translations.*

These properties of the anti-self-dual remainder are suggestive. The first statement establishes that anti-self-dual remainders of idempotent aggregation functions are null on the main diagonal. The second statement applies to the subclass of stable aggregation functions. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation function A is stable for translations, the value $\widetilde{A}(\boldsymbol{x})$ does not depend on the average value of the $\boldsymbol{x}$ coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder $\widetilde{A}$ suggest that it may give some indication on the dispersion of the $\boldsymbol{x}$ coordinates.

3.2. Exponential means

Quasiarithmetic means are the only aggregation functions satisfying continuity, idempotency, symmetry, strict monotonicity and decomposability (see Kolmogoroff [21], Nagumo [25] and Fodor and Roubens [12, pp. 112-114]).

Exponential means are the only quasiarithmetic means satisfying stability for translations.

Given $\alpha \neq 0$, the exponential mean A_α is the aggregation function defined as

$$A_\alpha(\boldsymbol{x}) = \frac{1}{\alpha} \ln \frac{e^{\alpha x_1} + \cdots + e^{\alpha x_n}}{n}.$$

We now show the dual decomposition of exponential means (see García-Lapresta and Marques Pereira [17, Sect. 6] for more details).

Given $\alpha \neq 0$, the self-dual core of A_α is the aggregation function $\widehat{A}_\alpha$ defined as

$$\widehat{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \frac{e^{\alpha x_1} + \cdots + e^{\alpha x_n}}{e^{-\alpha x_1} + \cdots + e^{-\alpha x_n}}.$$

For every $\alpha \neq 0$, $\widehat{A}_\alpha$ is continuous, idempotent, symmetric, strictly monotonic, compensative, stable for translations, self-dual and invariant for replications.

Given $\alpha \neq 0$, the anti-self-dual remainder of A_α is the mapping $\widetilde{A}_\alpha$ defined as

$$\widetilde{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \frac{(e^{\alpha x_1} + \cdots + e^{\alpha x_n})(e^{-\alpha x_1} + \cdots + e^{-\alpha x_n})}{n^2}.$$

For every $\alpha \neq 0$, $\widetilde{A}_\alpha(\boldsymbol{x}) = 0$ if and only if $x_1 = \cdots = x_n$. Moreover, $\widetilde{A}_\alpha$ is continuous, symmetric, anti-self-dual, invariant for translations and invariant for replications.

The following result presents the parameter limits of the exponential mean and those of the associated self-dual core, and anti-self-dual remainder (see García-Lapresta and Marques Pereira [17, Prop. 35]).

Proposition 6 *For every $\boldsymbol{x} \in [0,\infty)^n$, the following statements hold:*

1. $\lim\limits_{\alpha\to\infty} A_\alpha(\boldsymbol{x}) = x_{[1]}$.
2. $\lim\limits_{\alpha\to-\infty} A_\alpha(\boldsymbol{x}) = x_{[n]}$.
3. $\lim\limits_{\alpha\to 0} A_\alpha(\boldsymbol{x}) = \lim\limits_{\alpha\to 0} \widehat{A}_\alpha(\boldsymbol{x}) = \mu(\boldsymbol{x})$.
4. $\lim\limits_{\alpha\to\infty} \widehat{A}_\alpha(\boldsymbol{x}) = \lim\limits_{\alpha\to-\infty} \widehat{A}_\alpha(\boldsymbol{x}) = \dfrac{x_{[1]} + x_{[n]}}{2}$.
5. $\lim\limits_{\alpha\to\infty} \widetilde{A}_\alpha(\boldsymbol{x}) = \dfrac{x_{[1]} - x_{[n]}}{2}$.
6. $\lim\limits_{\alpha\to-\infty} \widetilde{A}_\alpha(\boldsymbol{x}) = -\dfrac{x_{[1]} - x_{[n]}}{2}$.
7. $\lim\limits_{\alpha\to 0} \widetilde{A}_\alpha(\boldsymbol{x}) = 0$.

4. Fuzzy exponential poverty measures

Let $Z = \{\boldsymbol{z} = (z_1, z_2) \in (0,\infty)^2 \mid z_1 \leq z_2\}$ be the set of admisible pairs of poverty lines.

The (crisp) set of poor people is defined as

$$\begin{aligned} Q(\boldsymbol{x},\boldsymbol{z}) &= \{i \in \{1,\ldots,n\} \mid \mu_{\boldsymbol{z}}(x_i) > 0\} \\ &= \{i \in \{1,\ldots,n\} \mid x_i < z_2\} \end{aligned}$$

and $q = q(\boldsymbol{x},\boldsymbol{z}) = \#Q(\boldsymbol{x},\boldsymbol{z})$ denotes the number of the poor.

The headcount ratio $H : [0,\infty)^n \times Z \longrightarrow [0,1]$ is defined as

$$H(\boldsymbol{x},\boldsymbol{z}) = \frac{q(\boldsymbol{x},\boldsymbol{z})}{n}$$

that measures the percentage of poor people in the society.

Definition 9 *Given $\alpha > 0$, the* poverty measure associated with A_α *is the function*

$$P_\alpha : [0,\infty)^n \times Z \longrightarrow [0,1]$$

defined as

$$P_\alpha(\boldsymbol{x},\boldsymbol{z}) = \begin{cases} H(\boldsymbol{x},\boldsymbol{z}) \cdot A_\alpha(g_1,\ldots,g_q), & \text{if } q \neq 0 \\ 0, & \text{if } q = 0, \end{cases}$$

where $g_1,\ldots,g_q$ are the non-null elements of $g_{\boldsymbol{z}}(x_1),\ldots,g_{\boldsymbol{z}}(x_n)$.

Remark 1 Under the mentioned assumption of h is linear, the poverty level does not change when the units in which income is measured change: for all $\boldsymbol{x} \in [0,\infty)^n$, $\boldsymbol{z} \in Z$ and $\lambda > 0$, it holds $P_\alpha(\lambda\boldsymbol{x},\lambda\boldsymbol{z}) = P_\alpha(\boldsymbol{x},\boldsymbol{z})$. All the poverty indices referred to as "relative" satisfy this property.

In the following result we establish interesting properties of the proposed class of fuzzy poverty measures.

Theorem 1 *For every $\alpha > 0$, the poverty measure P_α satisfies the following properties:*

1. Normalization. *For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $\boldsymbol{z} \in Z$, $P_\alpha(\boldsymbol{x},\boldsymbol{z}) = 0$ if and only if $Q(\boldsymbol{x},\boldsymbol{z}) = \emptyset$, that is $x_i \geq z_2$ for every $i \in \{1,\dots,n\}$.*
2. Symmetry. *For all $\boldsymbol{x} \in [0,\infty)^n$, $\boldsymbol{z} \in Z$, and permutation σ on $\{1,\dots,n\}$, it holds that $P_\alpha(\boldsymbol{x}_\sigma,\boldsymbol{z}) = P_\alpha(\boldsymbol{x},\boldsymbol{z})$.*
3. Poverty Focus. *For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $\boldsymbol{z} \in Z$, if $Q(\boldsymbol{x},z) = Q(\boldsymbol{y},\boldsymbol{z}) = Q$ and $x_i = y_i$ for every $i \in Q$, then $P_\alpha(\boldsymbol{x},\boldsymbol{z}) = P_\alpha(\boldsymbol{y},\boldsymbol{z})$.*
4. Poverty Monotonicity. *For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $\boldsymbol{z} \in Z$, if $Q(\boldsymbol{x},\boldsymbol{z}) = Q(\boldsymbol{y},\boldsymbol{z}) = Q$ and $\boldsymbol{x} = \boldsymbol{y}$ except for $x_i > y_i$ with $i \in Q$, then $P_\alpha(\boldsymbol{x},\boldsymbol{z}) < P_\alpha(\boldsymbol{y},\boldsymbol{z})$.*
5. Transfer Sensitivity. *For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $\boldsymbol{z} \in Z$, if $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a progressive transfer among the poor, then $P_\alpha(\boldsymbol{y},\boldsymbol{z}) < P_\alpha(\boldsymbol{x},\boldsymbol{z})$.*
6. Replication Invariance. *For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $\boldsymbol{z} \in Z$, if $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a replication, that is $\boldsymbol{y} = (\overbrace{\boldsymbol{x},\dots,\boldsymbol{x}}^{m})$ for some $m \in \mathbb{N}$, then $P_\alpha(\boldsymbol{y},\boldsymbol{z}) = P_\alpha(\boldsymbol{x},\boldsymbol{z})$.*
7. Diminishing Transfer Sensitivity. *For all $\boldsymbol{x}, \boldsymbol{y} \in [0,\infty)^n$ and $\boldsymbol{z} \in Z$, if $Q(\boldsymbol{x},\boldsymbol{z}) = Q(\boldsymbol{y},\boldsymbol{z})$ and $\boldsymbol{y}$ is obtained from $\boldsymbol{x}$ by a progressive transfer from the poor person with income $x_i + c$ to the poor person with income x_i, for some $c > 0$, then the magnitude of decrease in poverty $P_\alpha(\boldsymbol{x},\boldsymbol{z}) - P_\alpha(\boldsymbol{y},\boldsymbol{z})$ is higher the lower x_i.*
   In addition, the poverty reduction effect increases as the parameter value alpha increases

Normalization requires that if all the individuals are non-poor, then the poverty index is equal to 0.

Symmetry requires that all the individuals are treated in the same way.

Poverty Focus requires that the poverty index should not depend on the income of the non-poor people, i.e., the poverty level should not vary if the non-poor incomes change, as long as the set of poor people remains unchanged.

Poverty Monotonicity demands that poverty should increase if the income of a poor person decreases.

Transfer Sensitivity goes beyond Poverty Monotonicity and demands that greater weight should be placed on the poorer person and that poverty should decrease if inequality among the poor decreases.

Replication Invariance requires that if society is replicated a number of times, then the poverty does not change; it allows comparing populations of different sizes.

The Diminishing Transfer Sensitivity principle requires that the poverty reduction effect of a "poor to poorer" progressive transfer should decrease as the income of the poorer person increases.

5. Conclusions and further research

In this paper, we have considered a poverty membership function that links between one and zero and decreases linearly with income through a straight-line, as in Cerioli and Zani [6]. A natural generalization of this approach consists of using other kind of poverty membership functions.

The proposed poverty measures have been defined as the headcount ratio multiplied by the outcome given by an exponential mean to the normalized gaps of poor people. Since the headcount ratio is not fuzzy, it could be interesting to analyze other poverty measures that avoid that crisp headcount ratio. For instance, by defining the poverty measure as the outcome given by an exponential mean to the normalized gaps of poor and non-poor people. This proposal has been preliminary considered in the classical setting by García-Lapresta *et al.* [14].

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