

# Analysis of the intermediate quantifier “Many” in fuzzy natural logic \*

Petra Murinová, Vilém Novák

Centre of Excellence IT4Innovations, Division of the University of Ostrava  
Institute for Research and Applications of Fuzzy Modeling  
30. dubna 22, 702 00 Ostrava, Czech Republic

## Abstract

In the previous papers, we introduced a general principle for an introduction of new intermediate quantifiers and proved that generalized square of opposition works with them accordingly. This paper is devoted to interpretation of an intermediate quantifier “Many” and to analysis of generalized 5-square of opposition in higher order fuzzy logic.

**Keywords:** Fuzzy type theory; Intermediate generalized quantifiers; Peterson’s square of opposition; Generalized 5-square of opposition

## 1. Introduction

In fuzzy set theory, the concept of “fuzzy quantifier” was introduced by Zadeh’s in [29]. His approach is characterized by the interpretation of fuzzy quantifiers as fuzzy numbers. This theory has been later complexly elaborated by Glöckner in [9]. The semantics properties have been studied by the same author in [10] and later elaborated by Holčapek in [11, 12]. Dubois et al. in [6] proposed a framework in which fuzzy quantifiers are represented as intervals (for example “*more than a half*”  $[0.5, 1]$ ). Mathematical models of some of these quantifiers were suggested by several authors (see in [23, 24]).

Intermediate quantifiers are expressions of natural language such as *most*, *many*, *almost all*, *a few*, *a large part of*, etc. They are called “intermediate” because their meanings lay between the limit cases *for all* ( $\forall$ ) and *exists* ( $\exists$ ). A deep analysis of them is contained in the book [25].

A systematic formalization of them in mathematical (higher-order) fuzzy logic was first given in [18] and further elaborated in several papers [13, 14, 15]. Its basic idea consists in the assumption that intermediate quantifiers are just classical quantifiers  $\forall$  or  $\exists$  but the universe of quantification is modified. This is achieved using the theory of evaluative linguistic expressions (see [17]) developed as a special theory of higher-order fuzzy logic.

In this paper, we address the quantifier “Many”, that was partially omitted in the cited papers because its behavior is not as straightforward as the

behavior of the other intermediate quantifiers. The previously published results show that the property “contrary” is characteristic for quantifiers that represent a set (or a fuzzy set) covering more than half of the universe (in the sense of the chosen measure). We argue that “Many” is more vague than the other quantifiers and so, it does not necessarily have the property of sub-contrary, as argued by Peterson in [25]. The quantifier “Many” was also semantically and complexly studied in [8, 21].

Consider the following example: let us have a set of 100 people who like riding a bike. Peterson gives arguments that “Many” should mean at least 25% or more. But then, if, e.g., 25 people like riding the bicycle and 25 not, we see that the statements “Many people like riding a bike” and “Many people do not like riding a bike” can be both valid at the same time. Now, suppose that there are altogether 5 children and we ask how many of them like riding a bicycle. The question is how many is “Many”? Is it 2 or 3? We intuitively feel that this number should be at least 3. But then, of course, both previous statements cannot be true. We conclude that there is a model where the intermediate quantifier “Many” behaves as sub-contrary, and also a model in which this quantifier behaves as contrary.

Having in mind the Peterson’s analysis requiring that “Many” should mean at least 25% of the universe or more, we come to the following definition introduced in [18]:

$$\text{“Many } Bs \text{ are } A \text{”} := (\exists z)((\Delta(z \subseteq B) \& (\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm \bar{\nu})((\mu B)z)). \quad (1)$$

This definition says that all elements  $x$  from the fuzzy set  $z$  (being a subset of the universe  $B$ ) have the property  $A$  and, at the same time, the size of  $z$  (measured by the measure  $\mu$ ) is *not small* (i.e.,  $\neg(Sm \bar{\nu})$ ).

In this paper, we will first briefly overview the main concepts of  $\mathbb{L}$ -FTT the basic definitions of the theory of intermediate quantifiers and the concept of contraries, contradictories, sub-contraries and sub-alterns. The main contribution are Section 2 and Section 3, where we will analyze properties of the intermediate quantifier “Many” defined in (1). Finally we will introduce the interpretation of generalized 5-square of opposition in the higher

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order fuzzy logic. By **5-square\*** of opposition we mean the square with five basic intermediate quantifiers (“All”, “Almost all”, “Most”, “Many” and “Some”) defined in higher order fuzzy logic and interpreted using the properties of contraries, contradictories, sub-contraries and subalterns in our theory.

## 2. The theory of intermediate quantifiers

### 2.1. Fuzzy type theory

The formal theory of intermediate quantifiers is developed within Łukasiewicz fuzzy type theory (Ł-FTT). The reader can find details in several papers [14, 16, 17]. The corresponding algebra of truth values is a linearly ordered  $MV$ -algebra extended by the delta operation and denoted by  $MV_{\Delta}$ -algebra (see [5, 20]).

The basic syntactical objects of Ł-FTT are classical (cf. [2]), namely the concepts of type and formula. The atomic types are  $\epsilon$  (elements) and  $o$  (truth values). General types are denoted by Greek letters  $\alpha, \beta, \dots$ . We will omit the type whenever it is clear from the context. The set of all types is denoted by  $Types$ . The *language* of Ł-FTT denoted by  $J$ , consists of variables  $x_{\alpha}, \dots$ , special constants  $c_{\alpha}, \dots$  ( $\alpha \in Types$ ), the symbol  $\lambda$ , and brackets.

Interpretation of formulas is realized in a general *frame* that is a tuple

$$\mathcal{M} = \langle (M_{\alpha}, =_{\alpha})_{\alpha \in Types}, \mathcal{L}_{\Delta} \rangle \quad (2)$$

so that the following holds:

- (i) The  $\mathcal{L}_{\Delta}$  is an algebra of truth values (i.e., the  $MV_{\Delta}$ -algebra). We put  $M_o = L$  and assume that each set  $M_{oo} \cup M_{(oo)o}$  contains all the operations from  $\mathcal{L}_{\Delta}$ .
- (ii)  $=_{\alpha}$  is a fuzzy equality on  $M_{\alpha}$  and  $=_{\alpha} \in M_{(o\alpha)\alpha}$  for every  $\alpha \in Types$ .
- (iii) For all types  $\beta\alpha$ ,  $M_{\beta\alpha} \subseteq M_{\beta}^{M_{\alpha}}$ .

Interpretation of a formula  $A_{\alpha}$  is an element  $\mathcal{M}(A_{\alpha}) \in M_{\alpha}$ . Moreover, let the type  $\alpha$  be  $\alpha = \gamma\beta$ . Then  $\mathcal{M}(A_{\gamma\beta}) = f \in M_{\gamma\beta}$  where  $f$  is a function  $f : M_{\beta} \rightarrow M_{\gamma}$ . If, moreover,  $B_{\beta}$  is a formula of type  $\beta$  having an interpretation  $\mathcal{M}(B_{\beta}) = b \in M_{\beta}$  then interpretation  $\mathcal{M}(A_{\gamma\beta}B_{\beta})$  of the formula  $A_{\gamma\beta}B_{\beta}$  is a functional value  $f(b) \in M_{\gamma}$  of the function  $f$  at point  $b$ . Therefore, we also define:  $=_o := \leftrightarrow$ ,  $=_{\epsilon}$  is given explicitly, and  $[f =_{\beta\alpha} f'] = \bigwedge_{m, m' \in M_{\alpha}} [f(m) =_{\beta} f'(m')]$  where  $f, f' \in M_{\beta\alpha}$ . Special formulas are  $\perp, \dagger, \top$  interpreted as  $\mathcal{M}(\perp) = 0, \mathcal{M}(\dagger) = 0.5, \mathcal{M}(\top) = 1$ . Recall that FTT is syntactical-semantically complete w.r.t. general models (cf. [2, 18]).

\*) In the previous paper we introduced generalized complete square of opposition without the interpretation of the quantifier “Many”. Because the properties are not the same as in Peterson’s square then we will use the new notion.

A general model  $\mathcal{M}$  is a *model of a theory*  $T$ ,  $\mathcal{M} \models T$ , if

$$\mathcal{M}(A_o) = 1$$

holds for all axioms of  $T$ . A formula  $A_o$  is *true* in the degree  $a \in L$  in  $T$ , if

$$a = \bigwedge \{ \mathcal{M}_p(A_o) \mid \mathcal{M} \models T, p \in Asg(\mathcal{M}) \}. \quad (3)$$

In this case, will write  $T \models_a A_o$ . If  $a = 1$  then we omit the subscript and write simply  $T \models A_o$ .

The following completeness theorem can be proved (the proof is analogous to the proof of completeness given in [16]).

#### Theorem 1 (completeness)

- A theory  $T$  is consistent iff it has a general model  $\mathcal{M}$ .
- For every theory  $T$  and a formula  $A_o$

$$T \vdash A_o \quad \text{iff} \quad T \models A_o.$$

The following special formulas play a role in our theory below:

$$\Upsilon_{oo} \equiv \lambda z_o \cdot \neg \Delta(\neg z_o), \quad (\text{nonzero truth value})$$

$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg \Delta(z_o \vee \neg z_o). \quad (\text{general truth value})$$

Thus,  $\mathcal{M}(\Upsilon(A_o)) = 1$  iff  $\mathcal{M}(A_o) > 0$ , and  $\mathcal{M}(\hat{\Upsilon}(A_o)) = 1$  iff  $\mathcal{M}(A_o) \in (0, 1)$  holds in any model  $\mathcal{M}$ .

### 2.2. Theory of evaluative linguistic expressions

A constituent of the formal theory of intermediate quantifiers is also the formal theory of *evaluative linguistic expressions*  $T^{Ev}$  introduced in [17]. Recall that the latter are expressions of natural language such as *small, medium, big, about fourteen, very short, more or less deep, quite roughly strong*, etc. The language  $J^{Ev}$  of  $T^{Ev}$  contains the following symbols:

- The standard constants  $\top, \perp$  (truth and falsity), also a constant  $\dagger \in Form_o$ , which represents a middle truth value (in the standard Łukasiewicz  $MV_{\Delta}$ -algebra, it is interpreted by 0.5).
- Special constant is  $\sim \in Form_{(oo)o}$  for an additional fuzzy equality on the set of truth values  $L$ .
- Three special formulas  $LH, MH, RH \in Form_{oo}$  for the left, right and middle horizon, respectively.
- A special constant  $\bar{\nu} \in Form_{oo}$  for the standard (i.e. empty) hedge and further special constants  $Ex, Si, Ve, ML, Ro, QR, VR$  for specific hedges.
- Special constants  $\mathbf{a}_{\nu}, \mathbf{b}_{\nu}, \mathbf{c}_{\nu}$  associated with each hedge  $\nu \in \{Ex, Si, Ve, ML, Ro, QR, VR\}$ .

By the context in  $T^{\text{Ev}}$ , we understand a formula  $w_{\alpha o}$  whose interpretation is a function  $w : L \rightarrow M_{\alpha}$ . Hence, the context determines in  $M_{\alpha}$  a triple of elements  $\langle v_L, v_S, v_R \rangle$  where  $v_L, v_S, v_R \in M_{\alpha}$  and  $v_L = \mathcal{M}_p(w \perp)$ ,  $v_S = \mathcal{M}_p(w \dagger)$ ,  $v_R = \mathcal{M}_p(w \top)$ . In this paper we will need only standard which is the set of truth values  $M_o$  with the left bound  $v_L = 0$ , the central points  $v_S = 0.5$  and the right bound  $v_R$ .

In this paper, we will deal with simple abstract evaluative expressions only, i.e., with expressions such as “small, very big”, etc. in which not objects are specified. Therefore, their intensions are defined with respect to the standard context only and so, they coincide with extensions. The following formulas represent the intensions:

- $Sm := \lambda \nu \lambda z \cdot \nu(LH z)$ ,
- $Me := \lambda \nu \lambda z \cdot \nu(MH z)$ ,
- $Bi := \lambda \nu \lambda z \cdot \nu(RH z)$ .

Note that the structure of these formulas represents construction of the corresponding extensions, whose interpretation in a model is schematically explained in [17].

### 2.3. General definition of intermediate quantifiers

Let  $\mathcal{S} \subset \text{Types}^{\dagger}$  be a set of selected types. The theory of intermediate quantifiers is a special formal theory  $T^{\text{IQ}}[\mathcal{S}]$  of  $\mathbb{L}$ -FTT extending  $T^{\text{Ev}}$ . The detailed structure of  $T^{\text{IQ}}[\mathcal{S}]$  and precise definitions can be found in [13, 14, 18].

The following definition introduces special formulas that will be taken as mathematical model of intermediate quantifiers.

#### Definition 1

Let  $T^{\text{IQ}}[\mathcal{S}]$  be a theory of intermediate quantifiers and  $Ev \in \text{Form}_{oo}$  be an intension of some evaluative expression. Furthermore, let  $z \in \text{Form}_{o\alpha}$ ,  $x \in \text{Form}_{\alpha}$  be variables and  $A, B \in \text{Form}_{o\alpha}$  be formulas representing measurable fuzzy sets. An intermediate generalized quantifier interpreting the sentence

“ $\langle \text{Quantifier} \rangle$   $B$ ’s are  $A$ ”

is one of the following formulas:

$$(Q_{Ev}^{\forall} x)(B, A) := (\exists z)((\Delta(z \subseteq B) \& (\forall x)(zx \Rightarrow Ax)) \wedge Ev((\mu B)z)), \quad (4)$$

$$(Q_{Ev}^{\exists} x)(B, A) := (\exists z)((\Delta(z \subseteq B) \& (\exists x)(zx \wedge Ax)) \wedge Ev((\mu B)z)). \quad (5)$$

In some cases we also must consider *presupposition*<sup>‡</sup>.

<sup>†</sup>The set  $\mathcal{S}$  of distinguished types must be considered to avoid possible difficulties with interpretation of a formula  $\mu$  representing measure.

<sup>‡</sup>The intermediate generalized quantifier with presupposition is the formula  $(*Q_{Ev}^{\forall} x)(B, A) \equiv (\exists z)((\Delta(z \subseteq B) \& (\exists x)zx \& (\forall x)(zx \Rightarrow Ax)) \wedge Ev((\mu B)z))$ .

By setting specific evaluative linguistic expression in the formulas (4) or (5), we obtain concrete intermediate quantifiers. Definitions of the five basic intermediate quantifiers can be found in [14]. Below we recall the definition of the quantifier “Many” that will be needed later.

#### Definition 2

Let  $A, B \in \text{Form}_{o\alpha}$  be formulas,  $z \in \text{Form}_{o\alpha}$  and  $x \in \text{Form}_{\alpha}$  be variables. The intermediate quantifier “Many” can be introduced as follows:

$$\begin{aligned} \mathbf{K}: \text{Many } B \text{ are } A &:= (Q_{\neg(Sm \bar{\nu})}^{\forall} x)(B, A) \equiv \\ &(\exists z)((\Delta(z \subseteq B) \& (\forall x)(zx \Rightarrow Ax)) \wedge \\ &\neg(Sm \bar{\nu})((\mu B)z)), \end{aligned}$$

$$\begin{aligned} \mathbf{G}: \text{Many } B \text{ are not } A &:= (Q_{\neg(Sm \bar{\nu})}^{\forall} x)(B, \neg A) \equiv \\ &(\exists z)((\Delta(z \subseteq B) \& (\forall x)(zx \Rightarrow \neg Ax)) \wedge \\ &\neg(Sm \bar{\nu})((\mu B)z)). \end{aligned}$$

### 2.4. Relations between quantifiers

#### Definition 3

Let  $T$  be a consistent theory of  $\mathbb{L}$ -FTT and  $P_1, P_2 \in \text{Form}_o$  be formulas of type  $o$ .

- (i)  $P_1$  and  $P_2$  are contraries if  $T \vdash P_1 \& P_2 \equiv \perp$ .  
By completeness, this is equivalent with<sup>§</sup>)

$$\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = 0$$

in every model  $\mathcal{M} \models T$ .

- (ii)  $P_1$  and  $P_2$  are sub-contraries if  $T \vdash (P_1 \nabla P_2) \neq \perp$ . This is equivalent with

$$\mathcal{M}(P_1) \oplus \mathcal{M}(P_2) \neq 0$$

for every model  $\mathcal{M} \models T$ .

- (iii)  $P_1$  and  $P_2$  are contradictories if both  $T \vdash \Delta P_1 \& \Delta P_2 \equiv \perp$  as well as  $T \vdash \Delta P_1 \nabla \Delta P_2$ . By completeness,

$$\mathcal{M}(\Delta P_1) \otimes \mathcal{M}(\Delta P_2) = 0,$$

as well as

$$\mathcal{M}(\Delta P_1) \oplus \mathcal{M}(\Delta P_2) = 1$$

in every model  $\mathcal{M} \models T$ .

- (iv)  $P_2$  is a subaltern of  $P_1$  in  $T$  if  $T \vdash P_1 \Rightarrow P_2$ . Then  $P_1$  is superaltern of  $P_2$ . Alternatively we can say that  $P_2$  is a subaltern of  $P_1$  and  $P_1$  is a superaltern of  $P_2$  if the inequality

$$\mathcal{M}(P_1) \leq \mathcal{M}(P_2)$$

holds true in every model  $\mathcal{M} \models T$ .

<sup>§</sup>) Recall that  $a \otimes b = 0 \vee (a + b - 1)$  is Łukasiewicz conjunction and  $a \oplus b = 1 \wedge (a + b)$  is Łukasiewicz disjunction.

### 3. Interpretation of “Many”

In this section, we focus on the main task of this paper, which is to study and analyze the intermediate quantifier “Many” in the generalized complete square of opposition. Note that this topic, in classical logic, was first studied by Thompson in [27]. Recall that the classical Aristotelian square works with two quantifiers only: the *universal* and the *existential*. Then, we will extend the square by four (vague) intermediate quantifiers *almost all*, *few*, *most* and *many* to obtain the *generalized complete square of opposition*. The generalized square and its properties are formally modeled inside  $\mathbb{L}$ -FTT and will be introduced and explained in Section 4..

There are many publications that are related to this area. Recall the work of Peterson [25] and many others (see [3]). In [22], the author presents classical and modern squares of opposition, where the problem with presupposition is discussed<sup>¶</sup>. In [4], the author introduced generalized quantifiers and the classical square of opposition based on first-order classical predicate logic. We can find here definitions of generalized quantifiers as well as definitions of many other relations between generalized quantifiers (equivalence, anti-subalternation, anti-superalternation) that are defined in classical logic. The semantical properties of generalized quantifiers (isomorphism, extension and conservativity) based on classical logic are studied in [1]. All of the cited papers consider classical logic only.

In [14], we analyzed the generalized Aristotle’s square of opposition in higher order fuzzy logic. Our analysis followed Peterson’s square in which, however, we omitted the intermediate quantifier “Many” because it turned out that this quantifier is ambivalent and depends on the given situation. Consequently, it may lead to two different squares that will be called *generalized 5-square of opposition*.

Recall that properties of the intermediate quantifiers do not hold in general logic but only in a special theory. Below is definition of such theory.

#### Definition 4 ([14])

Let  $\mathcal{S} = \{\beta \mid \beta \in \text{Types}, \beta \text{ does not contain } o\}$ . Let  $\underline{\mathbf{B}} = \{\{B_{o\alpha}, B'_{o\alpha}\} \mid B_{o\alpha}, B'_{o\alpha} \in \text{Form}_{o\alpha}, \alpha \in \mathcal{S}\}$ . A special theory  $T[\underline{\mathbf{B}}]$  is a consistent extension of  $T^{IQ}$  such that for all couples  $\{B_{o\alpha}, B'_{o\alpha}\} \in \underline{\mathbf{B}}$  the following is provable:

- (B1)  $T[\underline{\mathbf{B}}] \vdash B \equiv B'$ ,
- (B2)  $T[\underline{\mathbf{B}}] \vdash (\exists x_\alpha) \Delta Bx$ .

It is trivial to show that  $T[\underline{\mathbf{B}}] \vdash (\exists x_\alpha) \Delta B'x$  where  $B'$  is the second member of the couple  $\{B_{o\alpha}, B'_{o\alpha}\} \in \underline{\mathbf{B}}$ .

The formulas  $B_{o\alpha}, B'_{o\alpha}$  represent universes of quantification for different quantifiers. However,

<sup>¶</sup>For the detail analysis of the presupposition in  $\mathbb{L}$ -FTT we refer the readers to see the Section 4 in [14].

since in syllogisms we relate them, axiom (B1) assures that the considered intermediate quantifiers act on equal universes. Since the universes are, in general, fuzzy sets, we also require their normality that is assured by (B2).

The following theorem tells us that if we work with the quantifiers  $\mathbf{K}$  (“Many  $B$  are  $A$ ”) and  $\mathbf{G}$  (“Many  $B$  are not  $A$ ”) and they represent a set (or a fuzzy set) covering more than a half of the measure then they are contraries in  $\mathbb{L}$ -FTT.

#### Theorem 2 ([14])

Let  $T[\underline{\mathbf{B}}]$  be a theory from Definition 4,  $z, z' \in \text{Form}_{o\alpha}$  be variables. Then the following is provable for every  $\{B_{o\alpha}, B'_{o\alpha}\} \in \underline{\mathbf{B}}$ :

(a)

$$\begin{aligned} T[\underline{\mathbf{B}}] \vdash & (\exists z)(\exists z') \Delta((z \subseteq B) \& (z' \subseteq B')) \& \\ & \neg(Sm(\bar{\nu}))((\mu B)z) \& \neg(Sm(\bar{\nu}))((\mu B')z')) \& \\ & (\exists x)(zx \& z'x). \end{aligned}$$

(b)  $T[\underline{\mathbf{B}}] \vdash \mathbf{K} \& \mathbf{G} \equiv \perp$  i.e., the quantifiers  $\mathbf{K}$  and  $\mathbf{G}$  are contraries in  $T[\underline{\mathbf{B}}]$ .

To realize that the intermediate quantifiers  $\mathbf{K}$  and  $\mathbf{G}$  can also be sub-contraries, let us extend the theory  $T[B, B']$  by special axioms that characterize specific situation.

#### Lemma 1

Let  $T[\underline{\mathbf{B}}]$  be a theory from Definition 4. Let  $z, z', A, B \in \text{Form}_{o\alpha}, \alpha \in \mathcal{S}$ . Let  $T$  be an extension of  $T[\underline{\mathbf{B}}]$  such that

$$\begin{aligned} T = T[\underline{\mathbf{B}}] \cup & \{(\exists z)(\exists z') \Delta((z \subset B) \& (z' \subset B')) \& \\ & \Upsilon(\neg(Sm \bar{\nu}))((\mu B)z) \& \Upsilon(\neg(Sm \bar{\nu}))((\mu B')z')) \& \\ & \neg(\exists x)(zx \& \neg Ax) \& \neg(\exists x)(z'x \& Ax))\} \quad (6) \end{aligned}$$

be a theory. Then there exists a model  $\mathcal{M} \models T$ , i.e.,  $T$  is consistent.

PROOF: We will define the following frame:

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in \text{Types}}, \mathcal{L}_\Delta \rangle$$

where  $M_o = [0, 1]$  is support of the standard Łukasiewicz  $MV_\Delta$ -algebra. The fuzzy equality  $=_o$  is the Łukasiewicz biresiduation  $\leftrightarrow$ . Furthermore,  $M_\epsilon = \{u_1, \dots, u_r\}$  is a finite set with a fixed numbering of its elements and  $=_\epsilon$  is a fuzzy equality with truth value defined by

$$[u_i =_\epsilon u_j] = (1 - \min\left(1, \frac{|i - j|}{s}\right))$$

for some fixed natural number  $s \leq r$ . This a separated fuzzy equality w.r.t. the Łukasiewicz conjunction  $\otimes$ . It can be verified that all the logical axioms of  $\mathbb{L}$ -FTT are true in the degree 1 in  $\mathcal{M}$  (all the considered functions are weakly extensional w.r.t.  $\mathcal{M}(\equiv)$ ). Moreover,  $\mathcal{M}$  is nontrivial

because  $1 - \frac{|i-j|}{s} \in (0, 1)$  implies  $\frac{|i-j|}{s} \in (0, 1)$  and thus, taking the assignment  $p$  such that  $p(x_\epsilon) = u_i$ ,  $p(y_\epsilon) = u_j$  and considering  $A_o := x_\epsilon \equiv y_\epsilon$ , we obtain  $\mathcal{M}_p(A_o \vee \neg A_o) \in (0, 1)$ .

To make  $\mathcal{M}$  a model of  $T^{\text{Ev}}$  and  $T^{\text{IQ}}$ , we define interpretation of  $\sim$  by  $\mathcal{M}(\sim) = \leftrightarrow^2$ ,  $\mathcal{M}(\dagger) = 0.5$  and put  $\mathcal{M}(\nu)$  equal to a function  $\nu_{\alpha, b, c}$  which is a simple partially quadratic function given in [17]. It can be verified that  $\mathcal{M} \models T^{\text{Ev}}$ .

Let  $A \subseteq M_\alpha$ ,  $\alpha \in \mathcal{S}$ . Because its support is finite, we can put

$$|A| = \sum_{u \in \text{Supp}(A)} A(u), \quad u \in M_\alpha, \alpha \in \mathcal{S} \quad (7)$$

and define for all fuzzy sets  $A, B \subseteq M_\alpha$ ,  $\alpha \in \mathcal{S}$

$$F_R(B)(A) = \begin{cases} 1 & \text{if } B = \emptyset \text{ or } A = B, \\ \frac{|A|}{|B|} & \text{if } B \neq \emptyset \text{ and } A \subseteq B, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Interpretation of the constants  $R \in \text{Form}_{o(o\alpha)(o\alpha)}$ ,  $\alpha \in \mathcal{S}$  is  $\mathcal{M}(R) = F_R$ . It can be verified that axioms of  $T^{\text{IQ}}$  are true in the degree 1 in  $\mathcal{M}$  and so,  $\mathcal{M} \models T[\mathbf{B}]$ .

To verify that  $\mathcal{M}$  is a model of  $T$  means to find the greatest fuzzy sets  $\mathcal{M}(z_{o\alpha})$  and  $\mathcal{M}(z'_{o\alpha})$  such that the axioms in (6) are fulfilled.

(a) Let for simplicity  $U_1 = \{u_1, \dots, u_N\} \subset M_\epsilon$  be a support of all the fuzzy sets  $\mathcal{M}(B_{o\epsilon})$ ,  $\mathcal{M}(z_{o\epsilon})$ ,  $\mathcal{M}(z'_{o\epsilon})$ ,  $\mathcal{M}(A_{o\epsilon}) \subseteq U_1$  and let them be defined in such a way that they have the following shapes:

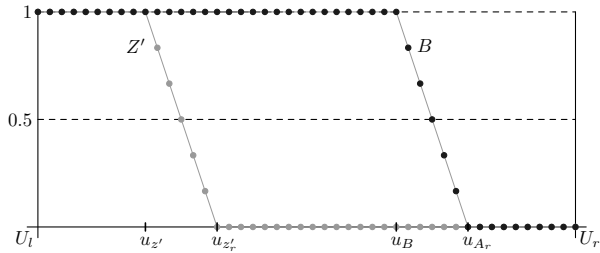


Figure 1: Shapes of the fuzzy sets  $Z'$  and  $B$ .

Let  $u_A, u_B, u_z, u_{A_r}, u_{A_l} \in M_\epsilon$  be parameters such that  $u_1 \leq u_z \leq u_{A_l} \leq u_A \leq u_B \leq u_{A_r} \leq u_N$ . For example, if we put  $N = 45$  then we can define these parameters as follows:

- (a)  $u_1 = u_l$  for some  $l$  and  $u_N = u_r$ ,
- (b)  $u_{z'} = u_{\frac{1}{5}N}$  and  $u_{A_l} = u_{z_r} = u_{\frac{1}{3}N}$ ,
- (c)  $u_A = u_{\frac{7}{15}N}$ ,  $u_B = u_{\frac{2}{3}N}$  and  $u_{A_r} = u_{\frac{4}{15}N}$ .

Then we obtain

$$F_R(B)(Z') = \frac{|Z'|}{|B|} = \frac{11.5}{32.5} = 0.35, \quad (9)$$

$$F_R(B)(Z) = \frac{|Z|}{|B|} = \frac{14}{32.5} = 0.43. \quad (10)$$

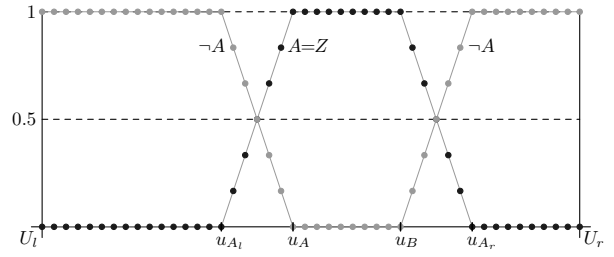


Figure 2: Shapes of the fuzzy sets  $A, Z, \neg A$ .

(b) Let  $U_2 = \{F_1, \dots, F_N\}$  be a finite support consisting of functions\*). Let the truth degrees of all  $F_j$ ,  $j = 1, \dots, N$ , belonging to the fuzzy sets  $Z, Z', B, A, \neg A \subseteq U_2$  are the same as in step (a). Then using the definition of measure we get

$$|Z| = \sum_{F_j \in \text{Supp}(Z)} Z(F_j), \text{ for all } j = 1, \dots, N.$$

Analogously we continue for the other fuzzy sets. Then we obtain

$$F_R(B)(Z') = \frac{\frac{N}{5} + 2.5}{\frac{2N}{3} + 2.5} \quad (11)$$

$$F_R(B)(Z) = \frac{\frac{N}{5} + 5}{\frac{2N}{3} + 2.5}. \quad (12)$$

By putting  $N \geq 29$  we obtain in (11) that  $0.1 < F_R(B)(Z) \leq 0.5$  and  $0.1 < F_R(B)(Z') \leq 0.5$ .

We continue with the verification of all the axioms for formulas of the type  $o\alpha$ . Let

$$\mathcal{M}(\neg(\text{Sm } \bar{\nu}))((\mu B_{o\alpha})(z'_{o\alpha})) = \neg(\overline{\text{Sm } \bar{\nu}})(F_R(B)(Z')) \in [0, 1]$$

where the fuzzy set representing extension of the evaluative linguistic expression  $\text{Sm } \bar{\nu}$  is denoted by  $\overline{\text{Sm } \bar{\nu}} \subseteq [0, 1]$  and its shape is clear from Figure 3.

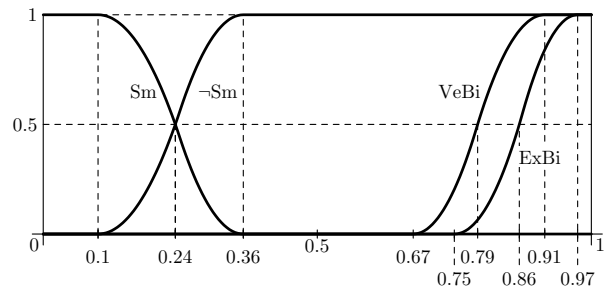


Figure 3: Shapes of the extensions of evaluative expressions in the context  $[0, 1]$ .

Then we obtain that  $\neg(\overline{\text{Sm } \bar{\nu}})(F_R(B)(Z)) > 0$  and  $\neg(\overline{\text{Sm } \bar{\nu}})(F_R(B)(Z')) > 0$ . This means that

\*) Recall that we work with the functions  $F_j : M_\beta \rightarrow M_\gamma$  for all  $j = 1, \dots, N$  where  $\beta, \gamma \in \mathcal{S}$ .

$$\mathcal{M}(\Upsilon(\neg(Sm\bar{\nu})(\mu(B_{o\alpha})(z_{o\alpha}))) = 1 \text{ and } \mathcal{M}(\Upsilon(\neg(Sm\bar{\nu})(\mu(B_{o\alpha})(z'_{o\alpha}))) = 1^\dagger).$$

The last two axioms can be verified as follows. Using the quantifier properties we obtain that

$$\begin{aligned} \mathcal{M}(\neg(\exists x_\alpha)(z'_{o\alpha}(x_\alpha) \& A_{o\alpha}(x_\alpha))) = \\ \mathcal{M}((\forall x_\alpha)(z'_{o\alpha}(x_\alpha) \Rightarrow \neg A_{o\alpha}(x_\alpha))) = \\ \bigwedge_{m \in M_\alpha} (Z'(m) \rightarrow \neg A(m)) = 1. \end{aligned} \quad (13)$$

From  $Z' \subseteq \neg A$  is (13) fulfilled. From the definitions of the interpretations of the fuzzy sets  $A$  and  $Z$  ( $Z = A$ ) it follows that  $\mathcal{M}(\neg(\exists x)(zx \& \neg A(x))) = 1$ . Thus  $\mathcal{M} \models T$ .  $\square$

### Corollary 1

Let  $T$  be the theory considered in Lemma 1. Then

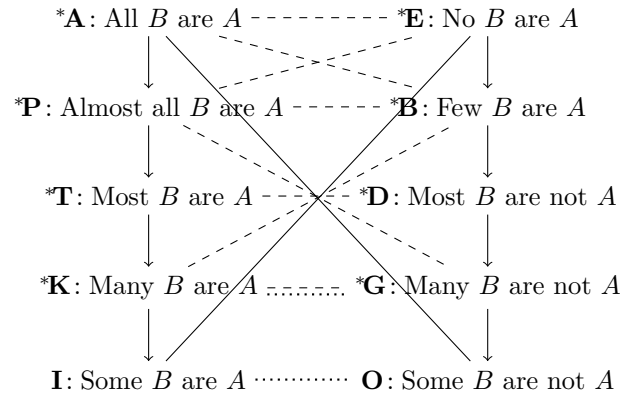
$$T \vdash (Q_{\neg(Sm\bar{\nu})}^\forall x)(B, A) \nabla (Q_{\neg(Sm\bar{\nu})}^\forall x)(B, \neg A) \not\equiv \perp, \quad (14)$$

i.e., the quantifiers **K** and **G** are sub-contraries in the theory  $T$ .

The following can be concluded from Theorem 2 and Corollary 1. The quantifiers **K** and **G** can both be contrary as well as sub-contrary depending on (interpretations of) the fuzzy sets  $z$  and  $z'$  that are considered inside them. If they are sufficiently big (in the sense of the measure  $\mu$ ) then both quantifiers are contrary. If they are rather small (but not too much so that the properties  $\neg Sm\bar{\nu}(\mu B)z$ ,  $\neg Sm\bar{\nu}(\mu B)z'$ ) have still high truth degree then the quantifiers are sub-contrary. The precise form of the property that  $z, z'$  must fulfill is given by the special axiom (6).

## 4. Generalized 5-square of opposition

The relations among intermediate quantifiers can now be demonstrated in the form of 5-square of opposition. Recall that the property of contraries and sub-contraries for the intermediate quantifier “Many” were proved in the previous section. The other properties of the generalized quantifiers presented below were proved in [14]. In the scheme, the straight lines mark contradictories, the dashed lines contraries and the dotted lines sub-contraries. The arrows indicate the relation superaltern–subaltern. In some cases, the given relation holds only if we consider presupposition (see [13]). This is denoted by the asterisk.



The scheme above shows that, for example, the quantifiers “Almost all” and “Many” are contraries in every model. On the other hand there is a model where the intermediate quantifiers “Many” behaves as sub-contrary and also a model in which the latter quantifier behaves as contrary.

The interpretation of the general 5-square of opposition very relates with the verification of the strong validity<sup>†</sup>) or the invalidity of the generalized syllogisms in Ł-FTT. The property of contraries between two intermediate quantifiers leads to the strong validity of generalized syllogisms of Figure-III with *particular* conclusion in Ł-FTT. For example, the contraries between the intermediate quantifiers “Almost all  $B$  are  $A$ ” and “Most  $B$  are not  $A$ ” leads to the strong validity of syllogisms

$$\frac{\text{Almost all old people are ill} \\ \text{Almost all old people have gray hair}}{\text{Some people with gray hair are ill}}$$

which was syntactically proved in [13]. On the other hand the syllogism

$$\frac{\text{Many people on earth eat meat} \\ \text{Many people on earth are women}}{\text{Some women eat meat}}$$

can be valid and also invalid in Ł-FTT. The detail interpretation of these syllogisms is prepared in the full paper.

## 5. Conclusion

This paper continues research in the theory of intermediate quantifiers in Łukasiewicz higher-order fuzzy logic started in [14]. Our main goal was to analyze the quantifier “Many”. We have shown that there are two kinds of models of the interpretation of the quantifier “Many”, namely a model where it behaves as sub-contrary, and also a model in which it behaves as contrary.

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<sup>†</sup>) Recall that if  $\mathcal{M}$  be an interpretation and  $p$  an assignment then  $\mathcal{M}(\Upsilon(z_o)) = 1$  iff  $p(z_o) > 0$

<sup>‡</sup>) We say that the syllogism  $\langle P_1, P_2, C \rangle$  is *strongly valid* if  $T \vdash P_1 \& P_2 \Rightarrow C$ , or equivalently, if  $T \vdash P_1 \Rightarrow (P_2 \Rightarrow C)$ .

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