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On (OP)-polynomial implications

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Abstract

The disposal of many families of fuzzy implications is crucial to handle the great number of applications these operators have. In this work, the class of fuzzy (OP)-polynomial implications is introduced as those fuzzy implications satisfying the ordering property (OP) whose expression in the region where $x > y$ is given by a polynomial of two variables. Since not all polynomials are adequate to generate a fuzzy implication, some properties related to the values of the coefficients of the polynomial to accomplish this fact have been studied. The (OP)-polynomial implications with degree less than 2 are fully characterized. Among the implications obtained in these results, there are some well-known implications such as the Łukasiewicz implication, among many others. Finally, several construction methods of these implications from other well-known families of fuzzy implications are given.

Keywords: Fuzzy implication, (OP)-polynomial implication, R-implication, exchange principle.

1. Introduction

Fuzzy implications play an indispensable role in many fields such as in approximate reasoning, fuzzy control and fuzzy mathematical morphology, among many others, leading in being crucial for all the applications derived from these theories. The importance of these operators lies on the fact that they perform an analogous function to the classical implication in binary logic. Since restricted to $\{0,1\}^2$ they coincide with the classical implication, fuzzy implications generalize the binary ones. Nowadays, these operations are modelled through monotonic functions $I : [0,1]^2 \to [0,1]$ satisfying the aforementioned border conditions. Many researchers have focused their attention and efforts to the study of these logical connectives. Thus, the survey [7] and the books [2] and [3], entirely devoted to fuzzy implications, stand out. This peak of interest in fuzzy implications is induced from both the theoretical point of view and the applied one. Both approaches feed each other and most of the theoretical studies are useful to improve the applicability of these operators in a concrete field.

Today there exist a large bunch of different classes of implications to answer adequately to all the applications where fuzzy implications are used. In [12] the relevance of having many different classes of implications is pointed out. The main reason is that any "If-Then" rule can be modelled by a fuzzy implication and therefore, depending on the context and the proper behaviour of the rule, different implications can be suitable in each case. Moreover, since backward and forward inferences are usually performed by fuzzy implications, the inference rule, that the fuzzy implication is going to model, determines up to a certain degree the choice of the operator.

From this previous fact, the introduction of new classes of fuzzy implications has become an important topic in this field. In the literature, we can find two main strategies to obtain new classes. The first one is based on the use of aggregation functions (t-norms, t-conorms, uninorms or aggregation functions in general) and other logical connectives, such as fuzzy negations. Some examples of this strategy are R and (S,N)-implications, QL and D-operations, among many others (see [3]). The second one is based on the use of univalued generators, obtaining the well-known Yager's implications or the h-implications, among many others. An exhaustive compilation of the different classes of fuzzy implications can be found in [10].

However, as this search for novel fuzzy implications advances, the more complex are the expressions of these new operators. It is well-known that the implications obtained by means of the two previous strategies can have very different expressions that will depend on the expressions of the aggregation functions or the generators used in their construction. However, we may have into account that the final expression of the fuzzy implication is important for its use in any application. Having a tough and complex expression is one of the main reasons to discard a fuzzy implication for its use in applications. In this way, expressions of these characteristics induce a high computational cost to compute their values and they are more propitious to spread possible errors caused by numerical approximations of the inputs to the outputs. In fact, even some theoretical properties of the fuzzy implication could not hold in practice due to the numerical approximations of the values of the implication. This is of course a great problem which needs to be tackled adequately.

Some of the fuzzy implications which have been used more in practice, such as the Łukasiewicz or the Reichenbach implication, have polynomials in their expressions and therefore, they are more robust to numerical computational errors. The search for nice expressions of some operators is not new. Since the only polynomial t-norm is the product t-norm, in [1] and [6], all the rational Archimedean continuous t-norms were characterized resulting in the well-known Hamacher family of t-norms. Moreover, in [4], Fodor characterized all the rational uninorms. In this case, since there are no continuous uninorms in $[0,1]^2$, there are no polynomial uninorms. On the other hand, the family of all fuzzy implications does contain implications with polynomial expressions. In [8], fuzzy polynomial implications were introduced and fully characterized up to degree 3. However, fuzzy polynomial implications have an important drawback since they do not satisfy the ordering property (OP), an important additional property of fuzzy implications which is required in some applications. The question which arises from this fact is related to the existence of fuzzy implications satisfying this property and having in the rest of their domain a polynomial expression. Although it is evident that there exist such implications, just some particular instances are known and consequently, in this work we want to study this class of implications, which we will call (OP)-polynomial implications, study their properties and characterize some of them.

So, after recalling some definitions and results which will be used in this work, the main target is the introduction of (OP)-polynomial implications, those implications satisfying (OP) which have a polynomial of two variables as expression in their remaining domain. Some basic properties will be studied. After that, we will fully characterize all (OP)-polynomial implications of degree less than 2 and we will study which additional properties they fulfil. From the derived results, the relationship of these implications with other families of fuzzy implications will be established. Finally, we will give some construction methods of (OP)-polynomial implications from some well-known families. The paper ends with some conclusions and future work we want to develop.

2. Preliminaries

Let us recall some concepts and results that will be used throughout this paper. First, we give the definition of fuzzy negation.

Definition 1 (**[5, Definition 1.1]**)**.** A non-increasing function $N : [0,1] \to [0,1]$ is a *fuzzy negation*, if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is

(i) *strict*, if it is continuous and strictly decreasing.

(ii) *strong*, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$.

Next, we recall the definition of fuzzy implication.

Definition 2 (**[5, Definition 1.15]**)**.** A binary operator $I : [0,1]^2 \to [0,1]$ is called a *fuzzy implication*, if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

From the definition, we can deduce that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ while the symmetrical values $I(x,0)$ and $I(1,x)$ are not determined. Some additional properties of fuzzy implications which will be used in this work are:

- The *left neutrality principle*,

$$I(1,y) = y, \quad y \in [0,1]. \tag{\textbf{NP}}$$

- The *exchange principle*,

$$I(x,I(y,z)) = I(y,I(x,z)), \quad x,y,z \in [0,1]. \tag{\textbf{EP}}$$

- The *ordering property*,

$$x \leq y \Longleftrightarrow I(x,y) = 1, \quad x,y \in [0,1]. \tag{\textbf{OP}}$$

Now, let us define the natural negation of a fuzzy implication.

Definition 3 (**[3, Definition 1.4.15]**)**.** Let I be a fuzzy implication. The function N_I defined by $N_I(x) = I(x,0)$ for all $x \in [0,1]$ is called the *natural negation* induced by I.

Finally, we recall the definitions of (S,N) and R-implications.

Definition 4 (**[3, Definition 2.4.1]**)**.** A function $I : [0,1]^2 \to [0,1]$ is called an *(S,N)-implication* if there exist a t-conorm S and a fuzzy negation N such that

$$I_{S,N}(x,y) = S(N(x),y), \quad x,y \in [0,1].$$

Definition 5 (**[3, Definition 2.5.1]**)**.** A function $I : [0,1]^2 \to [0,1]$ is called an *R-implication* if there exists a t-norm T such that

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}, \ \ x,y \in [0,1].$$

3. (OP)-Polynomial Implications

In this section we will introduce the concept of (OP)-polynomial implication and we will determine some necessary conditions on the coefficients of the polynomial in order to obtain a fuzzy implication from this expression.

Remark 1. Note that as in [8] where the authors focused on polynomial implications, we will restrict ourselves to (OP)-polynomial implications. Although in the introduction, it is said that the characterizations of rational Archimedean continuous t-norms and rational uninorms (understanding a rational function as a quotient of two polynomials) are known, the analogous problem for fuzzy implications is richer and more complex. This is a direct consequence of the definition of a fuzzy implication. While uninorms and t-norms are associative functions and therefore, there exists a quite restrictive property in their definitions, fuzzy implications have a more flexible definition which allows the existence of a great number of fuzzy polynomial implications and therefore, their study is worthy in itself. In fact, as we will show, the problem to characterize (OP)-polynomial implications is even harder than the one to characterize polynomial implications and therefore, in this article we will only be able to fully characterize them for degrees less than 2.

Let us start introducing the formal definition of an (OP)-polynomial implication.

Definition 6. Consider $n \in \mathbb{N}$. A binary operator $I : [0,1]^2 \to [0,1]$ is called a *fuzzy (OP)-polynomial implication of degree* n if it is a fuzzy implication and its expression is given by

$$I(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \displaystyle\sum_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i y^j & \text{if } x > y, \end{cases} \tag{1}$$

for all $x, y \in [0,1]$ where $a_{ij} \in \mathbb{R}$ and there exist some $0 \leq i, j \leq n$ with $i + j = n$ such that $a_{ij} \neq 0$.

The following example shows the existence of infinite fuzzy (OP)-polynomial implications of any degree $n \in \mathbb{N}$ with $n \geq 2$.

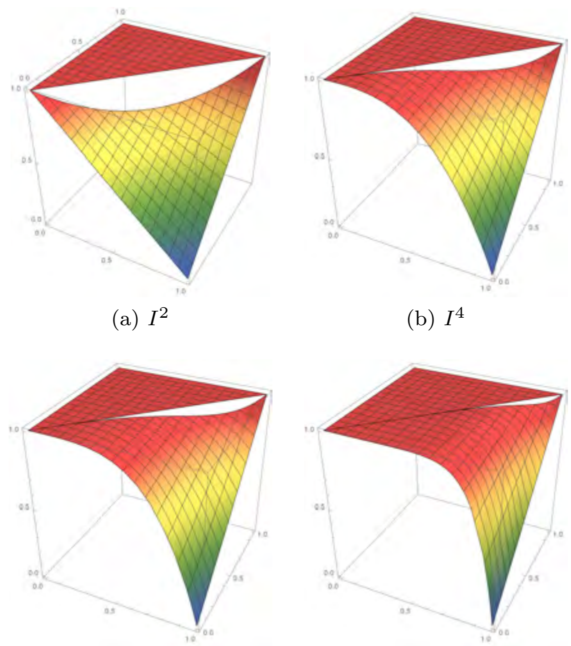
Example 1. Let us consider the parametrized family of fuzzy negations given by $N_n(x) = 1 - x^n$ for all $x \in [0,1]$ and $n \in \mathbb{Z}^+$, and the probabilistic sum t-conorm, whose expression is $S_P(x,y) = x + y - xy$ for all $x, y \in [0,1]$. It is straightforward to check that the probabilistic sum belongs to the family of Hamacher t-conorms (the dual t-conorms of the Hamacher t-norms) and moreover, it is the unique polynomial t-conorm. Then, if we consider these two operators, we can construct the following parametrized family of (S,N)-implications

$$I_{S_P, N_{n-1}}(x,y) = S_P(N_{n-1}(x), y) = 1 - x^{n-1} + x^{n-1} y$$

for all $x, y \in [0,1]$ and $n \geq 2$. As it can be easily observed, they are also Yager's f-generated implications with $f(x) = \sqrt[n-1]{1-x}$ (see [9]). Taking into account this family, we can construct a parametrized family of (OP)-polynomial implications as follows:

$$I^n(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x^{n-1} + x^{n-1} y & \text{if } x > y. \end{cases}$$

In Figure 1, some instances of this family are displayed. Note the evolution of the fuzzy implications to the greatest fuzzy implication as the value of n increases.



(a) I^2 (b) I^4

(c) I^5 (d) I^{10}

Figure 1: Some fuzzy implications of the parametrized family of (OP)-polynomial implications I^n.

A first property which can be derived straightforwardly from the definition of these implications is the satisfaction of (OP).

Proposition 1. *All fuzzy (OP)-polynomial implications satisfy (OP).*

Remark 2. In Definition 6 we require a polynomial of two variables in the region $x > y$ of the fuzzy implication. These polynomials do not have a constant value in a non-trivial region unless their degree is 0, i.e., those polynomials whose expression is given by $p(x,y) = a_{00}$ where $a_{00} \in \mathbb{R}$. These are the only polynomials with a constant value in a non-trivial region that could be suitable to generate an (OP)-polynomial implication. As we will show later, only one of these polynomials generates a fuzzy implication. The remaining polynomials do not have a constant value in a non-trivial region and thus, some well-known implications are out of our definition since they do not satisfy the requirements of Definition 6:

$$\begin{aligned} I_{\mathrm{WB}}(x,y) &= \begin{cases} 1 & \text{if } x \neq 1, \\ y & \text{if } x = 1, \end{cases} \\ I_{Gt}(x,y) &= \begin{cases} 1 & \text{if } (x,y) \neq (1,0), \\ 0 & \text{if } (x,y) = (1,0). \end{cases} \end{aligned}$$

Note that the previous two fuzzy implications use piecewise polynomials in the region $x > y$.

While polynomial implications are always continuous, this is not the case of (OP)-polynomial implications. The continuity of these implications rely on a property of the underlying polynomial.

Proposition 2. *Let I be an (OP)-polynomial implication of degree n with polynomial coefficients (a_{ij}) with $1 \leq i, j \leq n$ and $i + j \leq n$. Then I is continuous if, and only if, $a_{00} = 1$ and for all $1 \leq k \leq n$ we have that*

$$\sum_{\substack{0 \leq i,j \leq k \\ i+j=k}} a_{ij} = 0.$$

The following result gives us the first relationship of these implications with another family of fuzzy implications, that is R-implications.

Proposition 3. *Let I be an (OP)-polynomial implication of degree n. If I satisfies (EP), then I is an R-implication generated by the left continuous t-norm T given by*

$$T(x, y) = \min\{t \in [0, 1] | I(x, t) \geq y\}$$

for all $x, y \in [0, 1]$.

To end with these first basic properties, let us give the expression of the natural negation of these implications.

Proposition 4. *Let I be an (OP)-polynomial implication of degree n with polynomial coefficients (a_{ij}) with $1 \leq i, j \leq n$ and $i + j \leq n$. Then the natural negation of I is given by*

$$N_I(x) = \begin{cases} 1 & \text{if } x = 0, \\ \displaystyle\sum_{i=0}^{n} a_{i0} x^i & \text{otherwise.} \end{cases}$$

In addition, this natural negation will be continuous if, and only if, $a_{00} = 1$.

However, the question of which polynomials can generate (OP)-polynomial implications remains still unanswered. The problem relies on to characterize which coefficients $a_{ij} \in \mathbb{R}$ have to be chosen in order to generate a polynomial $p(x, y)$ suitable to construct a fuzzy implication satisfying the conditions of Definition 2. We will partially answer this question in general for polynomials of degree n. First of all, the next result determines the necessary and sufficient conditions a polynomial must satisfy in order to be the expression in $x > y$ of a fuzzy (OP)-polynomial implication.

Theorem 5. *Let p be a polynomial of degree n of two variables given by $p(x, y) = \sum_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i y^j$. Then the function given by the Expression (1) is an (OP)-polynomial implication if, and only if, the following properties hold:*

(i) $p(1, 0) = 0$.

(ii) $\frac{\partial p(x,y)}{\partial x} \leq 0$ *for all* $0 \leq y < x \leq 1$.

(iii) $\frac{\partial p(x,y)}{\partial y} \geq 0$ *for all* $0 \leq y < x \leq 1$.

(iv) $0 \leq p(1, y), p(x, 0) \leq 1$ *for all* $x, y \in [0, 1]$.

(v) $0 \leq p(y, y) \leq 1$ *for all* $y \in [0, 1]$.

This theorem is the counterpart for (OP)-polynomial implications to Theorem 1 in [8] for polynomial implications. As we can see, the conditions given in the case of fuzzy polinomial implications are more restrictive since the polynomial acts also in the 1-horizontal section and the 0-vertical section of the fuzzy implication. Consequently, more restrictions to the parameters are available. In the case of (OP)-polynomial implications, just the first property provides directly some conditions on the coefficients a_{ij} of the polynomial $p(x, y)$.

Proposition 6. *Let $p(x, y) = \sum_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i y^j$ be a polynomial of degree n. Then we have the following equivalence:*

$$p(1, 0) = 0 \text{ if, and only if, } \sum_{i=0}^{n} a_{i0} = 0.$$

Thus, in this case it is not possible to achieve a result as Corollary 1 in [8] making harder a characterization of (OP)-polynomial implications in terms of the coefficients of the polynomial in order to know which polynomial coefficients are suitable to obtain a fuzzy implication according to Definition 2. Consequently, and with the aim of characterizing some (OP)-polynomial implications, from now on we will restrict the study to (OP)-polynomial implications of degree less than 2.

3.1. Degree Zero

First, we are going to study the existence of (OP)-polynomial implications of degree 0, i.e., fuzzy implications given by the following expression

$$I(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ a_{00} & \text{if } x > y. \end{cases}$$

It is easy to check that the only (OP)-polynomial implication of degree 0 is the Rescher implication given by

$$I_{\mathrm{RS}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x > y, \end{cases}$$

and which is depicted in Figure 2.

Proposition 7. *There is only one (OP)-polynomial implication of degree 0, that is the Rescher implication I_{RS}.*

Let us recall that there exist (OP)-piecewise polynomial implications of degree 0. A well-known example of these implications is the greatest fuzzy implication I_{Gt} which has been recalled in Remark 2.

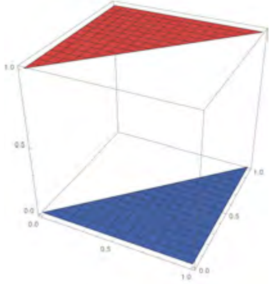


Figure 2: Rescher implication.

3.2. Degree 1

Now we deal with the characterization of all (OP)-polynomial implications of degree 1, i.e., those whose expressions are given by

$$I(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ a_{00} + a_{10}x + a_{01}y & \text{if } x > y, \end{cases}$$

with $a_{10}^2 + a_{01}^2 \neq 0$. First of all, using Proposition 6, we obtain that $a_{10} = -a_{00}$. However, this condition is not sufficient to guarantee that the operation is a fuzzy implication. Next theorem fully characterizes which polynomial coefficients must be chosen in order to get an (OP)-polynomial implication of degree 1.

Theorem 8. *Let I be a function given by*

$$I(x,y) = \begin{cases} 1 & \textit{if } x \le y, \\ a_{00} + a_{10}x + a_{01}y & \textit{if } x > y. \end{cases}$$

Then I is an (OP)-polynomial implication of degree 1 if, and only if, $0 \le a_{00}, a_{01} \le 1$ and $a_{10} = -a_{00}$ with $a_{10}^2 + a_{01}^2 \neq 0$.

Consequently, we can denote these implications as

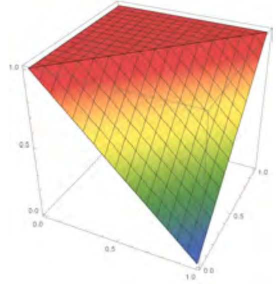
$$I^1_{\alpha,\beta}(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ \alpha - \alpha x + \beta y & \text{if } x > y, \end{cases} \tag{2}$$

with $0 \le \alpha, \beta \le 1$ and $\alpha^2 + \beta^2 > 0$. In Figure 3, some of these implications are displayed. Note that there are some well-known implications such as the Łukasiewicz and the Gödel implications.

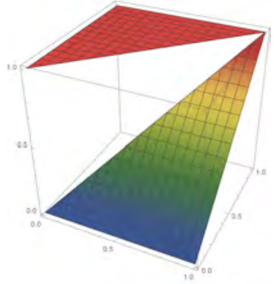
At this stage, let us study some properties of these implications in order to determine after that, the class of fuzzy implications which these operations belong to. First of all, it is easy to determine which $I^1_{\alpha,\beta}$ satisfy (NP).

Proposition 9. *Let $I^1_{\alpha,\beta}$ be an (OP)-polynomial implication. Then $I^1_{\alpha,\beta}$ satisfies (NP) if, and only if, $\beta = 1$.*

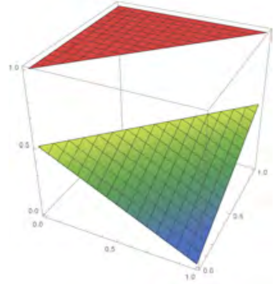
Focusing now in (EP), the following result fully characterizes which of these implications satisfy this property. Moreover, from Proposition 3, all the fuzzy implications obtained in the next result are R-implications.



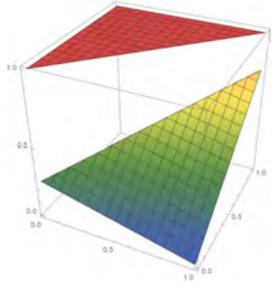
(a) $I^1_{1,1} = I_{\mathrm{LK}}$



(b) $I^1_{0,1} = I_{\mathrm{GD}}$



(c) $I^1_{\frac{1}{2},\frac{1}{2}}$



(d) $I^1_{\frac{1}{4},\frac{3}{4}}$

Figure 3: Some (OP)-polynomial implications of degree 1.

Proposition 10. *Let $I^1_{\alpha,\beta}$ be an (OP)-polynomial implication of degree 1. Then the following statements are equivalent:*

1. *$I^1_{\alpha,\beta}$ satisfies (EP).*
2. *$I^1_{\alpha,\beta}$ is an R-implication obtained from a left-continuous t-norm.*
3. *One of the following cases holds:*
   - *$\alpha = 0$ and $0 < \beta < 1$.*
   - *$\beta = 1$.*

Note that as well-known classes of R-implications, the first subcase of the third item includes the Gödel implication while the Łukasiewicz implication belongs to the second subcase. The second of these implications is the only (OP)-polynomial implication which is continuous. The result is an immediate consequence of Proposition 2.

Proposition 11. *The only continuous (OP)-polynomial implication of degree 1 is the Łukasiewicz implication $I^1_{1,1} = I_{LK}$.*

4. Construction methods of (OP)-polynomial implications

Since the full characterization of (OP)-polynomial implications of degree greater than or equal to 2 is difficult due to the few restrictions that can be imposed in the polynomial coefficients, in this section we will present several construction methods of fuzzy (OP)-polynomial implications from other families of implications or operators.

4.1. Construction from polynomial implications

Polynomial implications were introduced in [8] as those implications whose expressions are a polynomial of two variables. As it is clear, polynomial and (OP)-polynomial implications are related and they only differ in the region where the polynomial acts. However, reducing the domain where the polynomial is defined is in fact the main reason, as we have already commented, why the characterization of (OP)-polynomial implications is harder. Let us recall the definition of polynomial implications.

Definition 7 ([8, Definition 5]). Let $n \in \mathbb{N}$. A binary operator $I : [0,1]^2 \to [0,1]$ is called a *fuzzy polynomial implication of degree* n if it is a fuzzy implication and its expression is given by

$$I(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} a_{ij} x^i y^j$$

for all $x, y \in [0,1]$ where $a_{ij} \in \mathbb{R}$ and there exist some $0 \le i,j \le n$ with $i+j=n$ such that $a_{ij} \neq 0$.

In fact, polynomial implications can be used to generate (OP)-polynomial implications in a straightforward way.

Proposition 12. *Let* $I^*(x,y) = \sum_{\substack{0 \le i,j \le n \\ i+j \le 3}} a_{ij} x^i y^j$ *be a fuzzy polynomial implication of degree* n*. Then the function* $I : [0,1]^2 \to [0,1]$ *given by*

$$I(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ I^*(x,y) & \text{if } x > y, \end{cases}$$

is a fuzzy (OP)-polynomial of degree n*.*

From the previous result and using the characterization of polynomial implications of degree 3 given in Theorem 2 in [8], we can define a parametrized family of fuzzy (OP)-polynomial implications of degree 3 given by

$$I^3_{\alpha,\beta} = \begin{cases} 1 & \text{if } x \le y, \\ I^*(x,y) & \text{if } x > y, \end{cases}$$

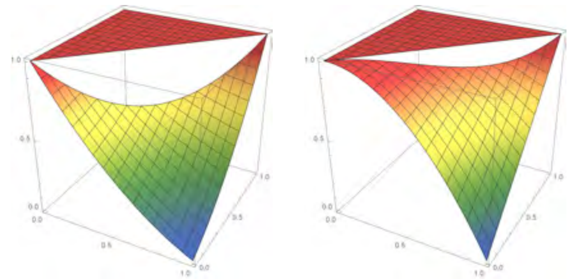
with $I^*(x,y) =$

$$= 1 + \alpha x + (-1-\alpha)x^2 + \beta xy + (1+\alpha)x^2 y + (-\alpha-\beta)xy^2$$

and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq -1$, $\alpha \neq -\beta$, and one of these cases hold:

- $-2 \le \alpha \le -1$ and $-1-\alpha \le \beta \le 2$.
- $-1 < \alpha < 0$ and $0 \le \beta \le -2\alpha$.
- $\alpha = \beta = 0$.

In Figure 4, we have depicted two fuzzy implications of this family.



(a) $I^3_{-\frac{3}{2},1}$ (b) $I^3_{0,0}$

Figure 4: Some (OP)-polynomial implications of degree 3 generated from polynomial implications.

4.2. Construction from I^N-implications

Another way to construct (OP)-polynomial implications is from I^N implications. Let us recall the definition of these implications which are generated from only a fuzzy negation N.

Definition 8 ([11]). Let N be a fuzzy negation. Then the function $I^N : [0,1]^2 \to [0,1]$ defined by

$$I^N(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ \dfrac{(1-N(x))y}{x} + N(x) & \text{if } x > y, \end{cases}$$

is a fuzzy implication.

As it can be seen, these implications satisfy always (OP). So, we need only to guarantee that the expression of the implication when $x > y$ is a polynomial. The next theorem states a sufficient condition to achieve this fact.

Proposition 13. *Let* N *be a polynomial fuzzy negation. Then the binary operator* $I_p^N : [0,1]^2 \to [0,1]$ *given by*

$$I_p^N(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ I^N(x,y) & \text{if } x > y, \end{cases}$$

is an (OP)-polynomial implication.

Thus, from a polynomial negation N, we can obtain (OP)-polynomial implications using the previous construction method. In Example 1 we have already recalled one family of polynomial negations, N_n. It is easy to check that the family $I_p^{N_n}$ coincides with the one obtained in Example 2. However, we can consider more polynomial fuzzy negations than N_n such as for instance $N(x) = 1 - \frac{x+x^2}{2}$. In Figure 5, we can see the resulting (OP)-polynomial implication.

4.3. Construction using the N-reciprocality method

The final method we will present in this work is based on the N-reciprocal of a given implication. Recall that given a fuzzy negation N and a fuzzy

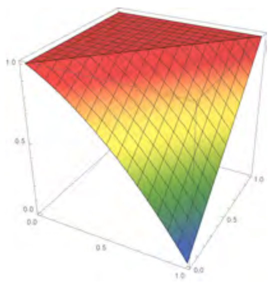


Figure 5: The (OP)-polynomial implication I_p^N obtained from $N(x) = 1 - \frac{x+x^2}{2}$.

implication N, the N-reciprocal implication is defined as

$$I_N(x,y) = I(N(y), N(x)), \quad x, y \in [0,1].$$

The next result states that the family of (OP)-polynomial implications is closed under the N-reciprocation method when the fuzzy negation N is a polynomial.

Proposition 14. *Let N be a polynomial fuzzy negation N and I an (OP)-polynomial implication. Then I_N is also an (OP)-polynomial implication.*

Example 2. Let us consider $N_C(x) = 1 - x$ the classical fuzzy negation and I_{LK} and I_{GD} the Łukasiewicz and Gödel implications. Applying the N-reciprocation method, while the N_C-reciprocal of I_{LK} is the same implication, the N_C-reciprocal of I_{GD} is given by

$$I_{N_C}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - x & \text{if } x > y, \end{cases}$$

which is an (OP)-polynomial of the same degree than I_{GD}.

Note that although the N-reciprocation method maintains the resulting implication into the family of (OP)-polynomial implications, it can change the degree of the resulting implication as the following example shows.

Example 3. Let us consider $N(x) = 1 - 2x^3 + x^5$ and I_{LK} again. While the degree of I_{LK} is 1, the degree of I_N is 5 since it is given by

$$I_N(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 1 - 2x^3 + 2y^3 + x^5 - y^5 & \text{if } x > y. \end{cases}$$

5. Conclusions and Future Work

In this paper, we have continued the study of fuzzy implications according to their final expression instead of the usual study on the construction methods of these operators using aggregation functions or generators. Concretely, we have studied the fuzzy (OP)-polynomial implications, presenting some general results for (OP)-polynomial implications of any degree and characterizing all fuzzy (OP)-polynomial implications of degree less than 2. This family of fuzzy implications is quite related with residual implications, although there are also fuzzy (OP)-polynomial implications which do not satisfy (EP) and consequently, they do not belong to any of the most usual families of implications. From the obtained results, some questions remain unanswered and must be tackled as future work. First,

Problem 1. Characterize all fuzzy (OP)-polynomial implications of any degree.

This problem is even more complex than Problem 1 in [8] since as we have already mentioned, there are fewer restrictions for the coefficients of the polynomial.

Another interesting problem is to study the properties that satisfy the (OP)-polynomial implications obtained through the three methods presented in Section 4. It would be also interesting to check the advantages of using (OP)-polynomial fuzzy implications instead of other implications in a concrete application in terms of computational cost saving and the reduction of the spreading of possible errors caused by numerical approximations of the inputs.

In addition, it will be worthy to study the class of rational and (OP)-rational implications, those whose expressions are given by $I(x,y) = \dfrac{p(x,y)}{q(x,y)}$ and

$$I(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \dfrac{p(x,y)}{q(x,y)} & \text{if } x > y, \end{cases}$$

respectively, where $p(x,y)$ and $q(x,y)$ are polynomials.

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