

Aggregation of fuzzy α - C -equivalences

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Abstract

The article deals with notions of fuzzy equivalences, where transitivity is defined by the use of a fuzzy conjunction. In particular fuzzy C -equivalences and fuzzy α - C -equivalences are considered. Moreover, the problem of the preservation of their axioms by some aggregation functions is examined. This paper gives a contribution to the discussion of tolerance analysis in soft computing, decision making, approximate reasoning, information fusion problems, and fuzzy control.

Keywords: aggregation functions, fuzzy conjunctions, fuzzy equivalences, fuzzy C -equivalences, fuzzy α - C -equivalences, dominance

1. Introduction

Aggregation functions can be useful in a variety of information fusion problems [2, 12]. The problem of aggregation of diverse mathematical objects is rather well known. We may aggregate for example fuzzy relations and consider the problem of preservation of fuzzy relation properties during aggregation process (e.g. [6, 19, 24, 25]) or examine fuzzy connectives and preservation of their properties by aggregation [11].

Examination of preservation of axioms and properties of fuzzy connectives finds its applications in decision making, approximate reasoning and fuzzy control. Moreover, preservation in the aggregation process not only individual properties, but all axioms of a given kind of fuzzy connective shows the possible way of generating new fuzzy connectives from a given one. In this context, some of the fuzzy connectives (negation, conjunction, disjunction and implication) were examined in [11], a fuzzy implication was examined in [5].

In this paper we consider aggregation of a fuzzy equivalence as one of the fuzzy connectives. We take into account the type of the possible definition of a fuzzy equivalence which is based on the notion of fuzzy equivalence relation which is reflexive, symmetric and transitive. We discuss two types of transitivity properties and in this way we obtain two types of fuzzy equivalences. We take into account C -transitivity property and its weaker version α - C -transitivity, where C is a fuzzy conjunction and $\alpha \in [0, 1]$. The idea of α -transitivity appeared in [14] and it was developed later in [15].

In the paper [16] there were considered other C -equivalences, namely the ones based on definition of semi-transitivity and weak transitivity.

In Section 2, basic notions useful in the paper are presented. In Section 3, diverse types of fuzzy equivalences are described, and in Section 4, aggregation of fuzzy equivalences are discussed.

2. Preliminaries

Here we recall basic notions and their properties which will appear in the sequel. We consider aggregation functions, relation of dominance between operations and fuzzy conjunctions. We will also need in the sequel the following

Lemma 1 (cf. [8], Lemma 1.1).

$$\forall_{a,b \in [0,1]} \forall_{\alpha \in [0,1]} (a \leq \alpha \Rightarrow b \leq \alpha) \Leftrightarrow b \leq a, \quad (1)$$

$$\forall_{a,b \in [0,1]} \forall_{\alpha \in [0,1]} (a \geq \alpha \Rightarrow b \geq \alpha) \Leftrightarrow a \leq b. \quad (2)$$

2.1. Aggregation functions

Now we present useful information about aggregation functions.

Definition 1 (cf. [4], pp. 6-22, [21], pp. 216-218). Let $n \in \mathbb{N}$. A function $A : [0, 1]^n \rightarrow [0, 1]$ which is increasing, i.e. for $x_i, y_i \in [0, 1], x_i \leq y_i, i = 1, \dots, n$

$$A(x_1, \dots, x_n) \leq A(y_1, \dots, y_n)$$

is called an aggregation function if

$$A(0, \dots, 0) = 0, \quad A(1, \dots, 1) = 1. \quad (3)$$

Moreover, we call an aggregation function A a mean if it is idempotent, i.e.

$$A(x, \dots, x) = x, \quad x \in [0, 1]. \quad (4)$$

Definition 2. Let $n \in \mathbb{N}$. We say that a function $A : [0, 1]^n \rightarrow [0, 1]$:

- has a zero element $z \in [0, 1]$ (cf. [4], Definition 10) if for any $1 \leq k \leq n$ and any $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \in [0, 1]$

$$A(x_1, \dots, x_{k-1}, z, x_{k+1}, \dots, x_n) = z, \quad (5)$$

- fulfils strong 1-boundary condition if

$$\forall_{x_1, \dots, x_n \in [0,1]} (A(x_1, \dots, x_n) = 1 \Leftrightarrow (\bigwedge_{1 \leq k \leq n} x_k = 1)). \quad (6)$$

Let us notice that by putting $n = 2$ in the above definition we get the respective well-known conditions for binary operations. A description of other families of aggregation functions can be found in [4].

Example 1 (cf. [4], pp. 44-56, [13]). A_0, A_1 are the least and the greatest aggregation functions, where

$$A_0(x_1, \dots, x_n) = \begin{cases} 1, & (x_1, \dots, x_n) = (1, \dots, 1) \\ 0, & (x_1, \dots, x_n) \neq (1, \dots, 1) \end{cases}$$

$$A_1(x_1, \dots, x_n) = \begin{cases} 0, & (x_1, \dots, x_n) = (0, \dots, 0) \\ 1, & (x_1, \dots, x_n) \neq (0, \dots, 0) \end{cases}$$

$x_1, \dots, x_n \in [0, 1]$. Simple examples of aggregation functions are given by standard means such as lattice operations \min, \max and

- projections

$$P_k(x_1, \dots, x_n) = x_k, \text{ for } k = 1, 2, \dots, n, \quad (7)$$

- geometric mean

$$G(x_1, \dots, x_n) = \sqrt[n]{x_1 \cdots x_n}, \quad (8)$$

- weighted arithmetic means

$$A_w(x_1, \dots, x_n) = \sum_{k=1}^n w_k x_k, \quad (9)$$

for $w_k > 0, \sum_{k=1}^n w_k = 1,$

- quasi-arithmetic means

$$M_\varphi(x_1, \dots, x_n) = \varphi^{-1}\left(\frac{1}{n} \sum_{k=1}^n \varphi(x_k)\right), \quad (10)$$

- quasi-linear means

$$F(x_1, \dots, x_n) = \varphi^{-1}\left(\sum_{k=1}^n w_k \varphi(x_k)\right), \quad (11)$$

where $w_k > 0, k = 1, \dots, n, \sum_{k=1}^n w_k = 1,$

$x_1, \dots, x_n \in [0, 1]$ and $\varphi : [0, 1] \rightarrow \mathbb{R}$ is a continuous, strictly increasing function.

Let us notice that geometric mean, weighted arithmetic means and quasi-arithmetic means are special cases of quasi-linear means.

2.2. Fuzzy conjunctions

Now, the definition and some properties of a fuzzy conjunction is presented.

Definition 3 ([11]). An operation $C : [0, 1]^2 \rightarrow [0, 1]$ is called a fuzzy conjunction if it is increasing and

$$C(1, 1) = 1, \quad C(0, 0) = C(0, 1) = C(1, 0) = 0.$$

Let us observe that fuzzy conjunctions are aggregation functions for $n = 2$. Directly from the definition we obtain a useful property of a fuzzy conjunction.

Corollary 1. A fuzzy conjunction has a zero element $z = 0$.

Conversely, if a binary aggregation function has the zero element $z = 0$ (as in the case of the geometric mean), then it is a fuzzy conjunction.

Corollary 2. If $F : [0, 1]^2 \rightarrow [0, 1]$ is increasing and has the neutral element 1, then it is a fuzzy conjunction fulfilling property $F \leq \min$.

In the literature we can find many subfamilies of fuzzy conjunctions which were distinguished due to their significance in applications.

Definition 4. An operation $C : [0, 1]^2 \rightarrow [0, 1]$ is called:

- a weak t-norm [18] if it is a fuzzy conjunction and $C(x, 1) \leq x, C(1, y) = y,$
- an overlap function [3] if it is a commutative, continuous fuzzy conjunction without zero divisors, fulfilling condition $C(x, y) = 1$ if and only if $xy = 1,$
- a t-seminorm [20] (a semicopula [1], a conjunctor [7]) if it is a fuzzy conjunction and has the neutral element 1,
- a pseudo-t-norm [17] if it is an associative t-seminorm,
- a t-norm [27] if it is a commutative pseudo-t-norm,
- a quasi-copula ([22], Section 6.2) if it is a 1-Lipschitz t-seminorm, i.e. for any $x, y, u, v \in [0, 1]$ the following condition holds

$$|C(x, y) - C(u, v)| \leq |x - u| + |y - v|.$$

Example 2. Consider the following family of fuzzy conjunctions for $a \in [0, 1]$

$$C^a(x, y) = \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \\ a & \text{otherwise} \end{cases} \quad (12)$$

Operations C^0 and C^1 are the least and the greatest fuzzy conjunction, respectively.

Example 3. Other examples of fuzzy conjunctions are listed below. Among them we recall the well-known t-norms: minimum, product, Łukasiewicz, drastic, which are denoted in the traditional way $T_M, T_P, T_L, T_D,$ respectively:

$$C_2(x, y) = \begin{cases} y, & \text{if } x = 1 \\ 0, & \text{if } x < 1 \end{cases},$$

$$C_3(x, y) = \begin{cases} x, & \text{if } y = 1 \\ 0, & \text{if } y < 1 \end{cases},$$

$$C_4(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1 \\ y, & \text{if } x + y > 1 \end{cases},$$

$$C_5(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1 \\ x, & \text{if } x + y > 1 \end{cases},$$

$$\begin{aligned}
C_6(x, y) &= \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0, \\ x & \text{otherwise} \end{cases} \\
C_7(x, y) &= \begin{cases} 1, & \text{if } x = y = 1 \\ 0, & \text{if } x = 0 \text{ or } y = 0, \\ y & \text{otherwise} \end{cases} \\
T_M(x, y) &= \min(x, y), \\
T_P(x, y) &= xy, \\
T_L(x, y) &= \max(x + y - 1, 0), \\
T_D(x, y) &= \begin{cases} x, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

Example 4. The condition $C \leq \min$, useful in the sequel, is fulfilled by the following fuzzy conjunctions from the last two examples: C^0 , C_2 , C_3 , T_M , T_P , T_L , T_D . Moreover, almost all families of fuzzy conjunctions from Definition 4 fulfil $C \leq \min$. The exception are overlap functions for which this inequality does not necessarily hold.

2.3. Dominance

Dominance is one of the interesting dependencies between operations which will be applied in Section 4. This is why we recall or prove some useful results which are related to the concept of dominance.

Definition 5 (cf. Saminger et al. [25], Definition 2.5; cf. [28], p. 209). Let $m, n \in \mathbb{N}$. An operation $F : P^m \rightarrow P$ dominates an operation $G : P^n \rightarrow P$ (shortly $F \gg G$), if for any matrix $[a_{i,k}] = A \in P^{m \times n}$ they fulfil the inequality

$$\begin{aligned}
F(G(a_{1,1}, \dots, a_{1,n}), \dots, G(a_{m,1}, \dots, a_{m,n})) &\geq \\
G(F(a_{1,1}, \dots, a_{m,1}), \dots, F(a_{1,n}, \dots, a_{m,n})). &\quad (13)
\end{aligned}$$

Let us consider an example of dominating operations.

Example 5. Let $n \in \mathbb{N}$. Each two projections

$$P_k(x_1, \dots, x_n) = x_k, \quad k \in \{1, \dots, n\}, \quad x_1, \dots, x_n \in [0, 1] \quad (14)$$

dominate each other.

In this paper we consider dominance between n -ary aggregation function A and binary fuzzy conjunction C . In this case the dominance $A \gg C$ means fulfilling the condition

$$\begin{aligned}
A(C(a_{1,1}, a_{1,2}), \dots, C(a_{n,1}, a_{n,2})) &\geq \\
\geq C(A(a_{1,1}, \dots, a_{n,1}), A(a_{1,2}, \dots, a_{n,2})). &\quad (15)
\end{aligned}$$

Corollary 3. *Minimum dominates any fuzzy conjunction. A weighted minimum dominates any t-seminorm.*

Proof. Since minimum dominates any increasing function (cf. [25]), it dominates also conjunctions. For the second property let us see that if C is an arbitrary t-seminorm, then for any $a, b, c \in [0, 1]$ one has superdistributivity of maximum over C , i.e.

$$C(\max(a, b), \max(a, c)) \leq \max(a, C(b, c)). \quad (16)$$

Let $s_k, t_k, w_k \in [0, 1]$, $k = 1, \dots, n$. We have to show that

$$\min_{1 \leq k \leq n} \max(1 - w_k, C(s_k, t_k)) \geq \quad (17)$$

$$C(\min_{1 \leq k \leq n} \max(1 - w_k, s_k), \min_{1 \leq k \leq n} \max(1 - w_k, t_k)).$$

We will apply the following inequality

$$\min_{1 \leq k \leq n} C(s_k, t_k) \geq C(\min_{1 \leq k \leq n} s_k, \min_{1 \leq k \leq n} t_k), \quad (18)$$

which may be obtained by induction with respect to $n \in \mathbb{N}$ and follows from the fact that minimum dominates any fuzzy conjunction. As a result, by (16), (18), and by the fact that minimum is increasing, we get

$$\begin{aligned}
C(\min_{1 \leq k \leq n} \max(1 - w_k, s_k), \min_{1 \leq k \leq n} \max(1 - w_k, t_k)) \\
\leq \min_{1 \leq k \leq n} C(\max(1 - w_k, s_k), \max(1 - w_k, t_k)) \leq \\
\min_{1 \leq k \leq n} \max(1 - w_k, C(s_k, t_k)),
\end{aligned}$$

which proves the inequality (17). \square

We may also characterize aggregation functions which dominate a given fuzzy conjunction. We recall here characterization theorem of all aggregation functions which dominate T_D .

Theorem 1 (cf. [25], Proposition 5.2). *Let $A : [0, 1]^n \rightarrow [0, 1]$ be aggregation function. Then $A \gg T_D$ if and only if there exists a non-empty subset $I = \{k_1, \dots, k_m\} \subset \{1, \dots, n\}$, $k_1 < \dots < k_m$, and an increasing mapping $B : [0, 1]^m \rightarrow [0, 1]$ satisfying the following conditions: $B(0, \dots, 0) = 0$ and $B(u_1, \dots, u_m) = 1 \Leftrightarrow u_1 = \dots = u_m = 1$, such that $A(x_1, \dots, x_n) = B(x_{k_1}, \dots, x_{k_m})$.*

Observe that function B in Theorem 1 is an aggregation function and concerning t-norms T we have $T(x_1, x_2) = 1$ if and only if $x_1 = x_2 = 1$ and thus $I = \{1, 2\}$, so $B = T$ and $T \gg T_D$ ([25], p. 32). Moreover, we deduce that the following holds true.

Example 6. Quasi-arithmetic means A dominate T_D and t-seminorms A dominate T_D . This is due to the fact that both types of the aggregation functions fulfil strong 1-boundary condition (6).

We put also characterization of aggregation functions A which dominate $C = \min$.

Theorem 2 (cf. [25], Proposition 5.1). *An aggregation function A dominates minimum if and only if for any $x_1, \dots, x_n \in [0, 1]$*

$$A(x_1, \dots, x_n) = \min(f_1(x_1), \dots, f_n(x_n)), \quad (19)$$

where $f_k : [0, 1] \rightarrow [0, 1]$ are increasing, $k = 1, \dots, n$.

Example 7. Here are examples of functions which fulfil (19):

if $f_k(x) = x$, $k = 1, \dots, n$, then $A = \min$,
 if for some $k \in \{1, \dots, n\}$, $f_k(x) = x$, $f_i(x) = 1$ for $i \neq k$, then $A = P_k$,
 if $f_k(x) = \max(1 - v_k, x)$, $v_k \in [0, 1]$, $k = 1, \dots, n$,
 $\max_{1 \leq k \leq n} v_k = 1$, then A is weighted minimum

$$A(x_1, \dots, x_n) = \min_{1 \leq k \leq n} \max(1 - v_k, x_k), \quad (20)$$

where $v = (v_1, \dots, v_n) \in [0, 1]^n$, $\max_{1 \leq k \leq n} v_k = 1$.

Next, we present some examples of dominance between functions.

Example 8 (cf. [10, 25]). A weighted geometric mean dominates t-norm T_P . A weighted arithmetic mean dominates t-norm T_L . The aggregation function

$$A(t_1, \dots, t_n) = \frac{p}{n} \sum_{k=1}^n t_k + (1 - p) \min_{1 \leq k \leq n} t_k \quad (21)$$

dominates T_L , where $p \in (0, 1)$. A weighted minimum dominates every t-norm T (see [10]). Let us consider projections P_k . Then $F \gg P_k$ and $P_k \gg F$ for any function $F : [0, 1]^n \rightarrow [0, 1]$ (see [9]).

We also recall here some well known results on dominance in the family of t-norms and t-conorms.

Theorem 3. (cf. [25], p.16) *Let T be a t-norm. Then*

$$\begin{aligned} T \gg T, \quad T_M \gg T \gg T_D, \\ T_D \gg T \Leftrightarrow T = T_D, \quad T \gg T_M \Leftrightarrow T = T_M. \end{aligned}$$

The following results concern dominance between fuzzy conjunctions C fulfilling inequality $C \leq \min$.

Theorem 4 ([26]). *If a t-seminorm dominates T_L then it is a quasi-copula. If a t-seminorm dominates T_P then it is a quasi-copula.*

3. Fuzzy equivalences

In the literature one can meet various definitions of a fuzzy equivalence. A trivial case used in many contributions, for example those concerning generalized logical laws, is an equality, that is the function $E : [0, 1]^2 \rightarrow [0, 1]$ given by the formula (relation of identity)

$$E(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}. \quad (22)$$

Usually it is expected that such notion of a fuzzy equivalence is a generalization of the equivalence of classical propositional calculus, that is the function $E : [0, 1]^2 \rightarrow [0, 1]$ that fulfils conditions $E(0, 1) = E(1, 0) = 0$, $E(0, 0) = E(1, 1) = 1$. We will apply the approach in the proposed definitions of a fuzzy equivalence. Moreover, the definitions

follow from the notion of a fuzzy equivalence relation, namely relation which is reflexive, symmetric and transitive.

Definition 6 (cf. [23], p. 33). Let C be a fuzzy conjunction. A fuzzy C -equivalence is a function $E : [0, 1]^2 \rightarrow [0, 1]$ fulfilling the following conditions

$$E(0, 1) = 0 \quad (\text{boundary property}), \quad (23)$$

$$E(x, x) = 1, \quad x \in [0, 1] \quad (\text{reflexivity}), \quad (24)$$

$$E(x, y) = E(y, x), \quad x, y \in [0, 1] \quad (\text{symmetry}), \quad (25)$$

$$C(E(x, y), E(y, z)) \leq E(x, z), \quad x, y, z \in [0, 1] \quad (26)$$

(transitivity).

In the cited monograph [23] property (23) is omitted. However, in this case the constant function $E(x, y) = 1$, $x, y \in [0, 1]$ fulfils the definition of a fuzzy equivalence although it is not a generalization of crisp equivalence. This is why we add this assumption to the definition.

We may weaken conditions given in the previous definition by replacing the transitivity property with the appropriate weaker transitivity condition (for weaker transitivity conditions for fuzzy relations see [14], [15]). Such considerations have already been presented in [16]. Here we propose another kind of weaker transitivity property of fuzzy equivalence.

Definition 7. Let $\alpha \in [0, 1]$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. Function $E : [0, 1]^2 \rightarrow [0, 1]$ fulfilling conditions (23) - (25) and

$$\forall_{x, y, z \in X} C(E(x, y), E(y, z)) \geq 1 - \alpha \Rightarrow \quad (27)$$

$$C(E(x, y), E(y, z)) \leq E(x, z)$$

will be called a fuzzy α - C -equivalence. The condition (27) will be called α - C -transitivity.

Corollary 4. *Let $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. Function $E : [0, 1]^2 \rightarrow [0, 1]$ is a fuzzy 1- C -equivalence if and only if E is a fuzzy C -equivalence.*

Example 9. For any fuzzy conjunction C and $\alpha \in [0, 1]$, the function (22) is a fuzzy α - C -equivalence and a fuzzy C -equivalence.

Theorem 5. *Let $\beta \in [0, 1]$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If $E : [0, 1]^2 \rightarrow [0, 1]$ is a fuzzy β - C -equivalence, then it is α - C -equivalence for any $\alpha \in [0, \beta]$.*

Proof. Let $x, y \in [0, 1]$, $\alpha, \beta \in [0, 1]$ and $\alpha \leq \beta$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If E is a fuzzy β - C -equivalence and $C(E(x, y), E(y, z)) \geq 1 - \alpha$, then $C(E(x, y), E(y, z)) \geq 1 - \alpha \geq 1 - \beta$ and by assumption that E is a fuzzy β - C -equivalence we get $E(x, z) \geq C(E(x, y), E(y, z))$. As a result E is a fuzzy α - C -equivalence for arbitrary $\alpha \in [0, \beta]$. \square

Corollary 5. Let $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If $E : [0, 1]^2 \rightarrow [0, 1]$ is a fuzzy C -equivalence, then E is a fuzzy α - C -equivalence for any $\alpha \in [0, 1]$.

Theorem 6. Let $\alpha \in [0, 1]$, $C_1, C_2 : [0, 1]^2 \rightarrow [0, 1]$ be fuzzy conjunctions, $C_1 \leq C_2$. If $E : [0, 1]^2 \rightarrow [0, 1]$ is a fuzzy α - C_2 -equivalence, then it is a fuzzy α - C_1 -equivalence.

Proof. Let $\alpha \in [0, 1]$, $C_1, C_2 : [0, 1]^2 \rightarrow [0, 1]$ be fuzzy conjunctions, $C_1 \leq C_2$. If E is α - C_2 -equivalence, then

$$C_1(E(x, y), E(y, z)) \geq 1 - \alpha \Rightarrow$$

$$C_2(E(x, y), E(y, z)) \geq 1 - \alpha \Rightarrow$$

$$E(x, z) \geq C_2(E(x, y), E(y, z)) \Rightarrow$$

$$E(x, z) \geq C_1(E(x, y), E(y, z)),$$

so E is a fuzzy α - C_1 -equivalence. \square

Corollary 6. Let $\alpha \in [0, 1]$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction, $C \leq \min$. If $E : [0, 1]^2 \rightarrow [0, 1]$ is a fuzzy min-equivalence, then it is a fuzzy α - C -equivalence.

So, by the above corollaries we see that a fuzzy α - C -transitivity, based on the parameter α , is a weaker property than a fuzzy C -transitivity.

The following two theorems indicate examples of fuzzy C -equivalences and fuzzy α - C -equivalences.

Theorem 7. The function

$$E(x, y) = \begin{cases} 1, & \text{if } x = y \\ \min(x, y), & \text{otherwise} \end{cases} \quad (28)$$

is a fuzzy C -equivalence if and only if $C \leq \min$.

Proof. Obviously, conditions (23) - (25) are fulfilled. Let us examine the property of C -transitivity (26). (\Rightarrow) Let $x, z \in [0, 1]$. If $x \neq z$ then for $y = 1$ one has

$$C(x, z) = C(E(x, 1), E(1, z)) \leq E(x, z) = \min(x, z).$$

On the other hand if $x = z$ then for $x = 1$ one has $C(x, x) = C(1, 1) = 1 \leq 1 = x$, and for $x \neq 1$ there exists $t \geq x$ and then $C(x, x) \leq C(x, t) \leq \min(x, t) = x$.

(\Leftarrow) Let $C \leq \min$, $x, y, z \in [0, 1]$. If $x \neq y$ and $y \neq z$ then

$$\begin{aligned} C(E(x, y), E(y, z)) &= C(\min(x, y), \min(y, z)) \leq \\ &\leq \min(\min(x, y), \min(y, z)) \leq \min(x, z) \leq E(x, z). \end{aligned}$$

For $x = y$ one obtains

$$\begin{aligned} C(E(x, y), E(y, z)) &= C(1, \min(y, z)) \leq \min(y, z) \\ &= \min(x, z) \leq E(x, z). \end{aligned}$$

Similarly, for $y = z$ one has

$$\begin{aligned} C(E(x, y), E(y, z)) &= C(\min(x, y), 1) \leq \min(x, y) \\ &= \min(x, z) \leq E(x, z). \end{aligned}$$

\square

Theorem 8. Let $a \in (0, 1)$, $\alpha \in [0, 1]$. The function E given by (28) is a fuzzy α - C^a -equivalence if and only if $a < 1 - \alpha$.

Proof. Obviously, conditions (23) - (25) are fulfilled. Let us examine the property of a fuzzy α - C^a -transitivity (27).

(\Rightarrow) For the proof by contradiction let us assume that $a \geq 1 - \alpha$ and let us consider $x \in (0, 1)$ such that $x < a$ and $y = z = 1$. We have

$$\begin{aligned} C^a(E(x, y), E(y, z)) &= C^a(E(x, 1), E(1, 1)) = \\ &= C^a(x, 1) = a \geq 1 - \alpha \end{aligned}$$

and $E(x, z) = E(x, 1) = x < a$. Hence $C^a(E(x, y), E(y, z)) > E(x, z)$ and the function E is not α - C^a -transitive.

(\Leftarrow) Let us assume that for some $x, y, z \in [0, 1]$ we have $C^a(E(x, y), E(y, z)) \geq 1 - \alpha$. By the assumption that $a < 1 - \alpha$ and the formula of a fuzzy conjunction C^a (12) we obtain that $C^a(E(x, y), E(y, z)) = 1$. Again by the definition of C^a and E we have $E(x, y) = E(y, z) = 1$ and $x = y = z$. Thus, we obtain $E(x, z) = 1$, what shows that inequality $C^a(E(x, y), E(y, z)) \leq E(x, z)$ holds and the function E is α - C^a -transitive. \square

The following corollary and example show that the converse statement to the one in Corollary 5 does not hold and consequently present examples of fuzzy α - C -equivalences.

Corollary 7. Let us consider the function E given by (28) and a fuzzy conjunction C^a for $a \in (0, 1)$. As C^a does not fulfil the condition $C^a \leq \min$ thus, by Theorem 7, E is not a fuzzy C^a -equivalence. On the other hand, by Theorem 8, it is a fuzzy α - C^a -equivalence for any $\alpha < 1 - a$.

Example 10. Let us consider a function

$$E(x, y) = \begin{cases} 1, & \text{if } x = y \\ \frac{\min(x, y)}{\max(x, y)}, & \text{if } x \neq y \end{cases} \quad (29)$$

and conjunction $C = \min$. Obviously E fulfils conditions (23) - (25). We shall show that it does not fulfil (26). For $x = 0.2$, $y = 0.4$, $z = 0.8$ one has

$$\begin{aligned} C(E(x, y), E(y, z)) &= \min\left(\frac{\min(x, y)}{\max(x, y)}, \frac{\min(y, z)}{\max(y, z)}\right) \\ &= \min\left(\frac{0.2}{0.4}, \frac{0.4}{0.8}\right) = 0.5. \end{aligned}$$

Moreover, $E(x, z) = \frac{\min(x, z)}{\max(x, z)} = \frac{0.2}{0.8} = 0.25$. Thus, $C(E(x, y), E(y, z)) > E(x, z)$ and the function (29) is not min-equivalence. Let us consider a condition $\min(E(x, y), E(y, z)) \geq 1$. From this it follows that both $E(x, y) = 1$ and $E(y, z) = 1$ and it means that $x = y$ and $y = z$. Thus, $x = z$ and $E(x, z) = 1$. This means that the function (29) is 0-min-equivalence.

4. Preservation of fuzzy C -equivalences in aggregation process

Fundamental properties of aggregation for fuzzy relations were gathered by J.C. Fodor and M. Roubens [19]. We may aggregate diverse objects: fuzzy relations, fuzzy connectives etc. Here we consider aggregation of fuzzy equivalences defined in the previous section.

Definition 8 (cf. [19], p. 14). Let $n \in \mathbb{N}$ and A be an arbitrary aggregation function. For given fuzzy equivalences E_1, \dots, E_n , we consider a binary operation for all $x, y \in [0, 1]$

$$E(x, y) = A(E_1(x, y), \dots, E_n(x, y)). \quad (30)$$

We say that an aggregation function A preserves a property of the given fuzzy equivalences if the operation E from (30) has such a property for arbitrary E_1, \dots, E_n fulfilling this property. A class of fuzzy equivalences is closed under an aggregation function A if the result of aggregation belongs to this class for arbitrary fuzzy equivalences from the class.

Example 11. For example, any projection (7) preserves fuzzy C -equivalence and fuzzy α - C -equivalence (for any $\alpha \in [0, 1]$) because in the formula (30) with $A = P_k$ we get $E = E_k$ for $k \in \{1, \dots, n\}$.

From condition (3) we get

Lemma 2 (cf. [11]). *Any aggregation function preserves binary truth tables of aggregated equivalences.*

Lemma 3 (cf. [11]). *Any aggregation function preserves symmetry of aggregated equivalences.*

Lemma 4. *Any aggregation function preserves the property (24) of the aggregated operations.*

Proof. Let E_1, \dots, E_n be operations fulfilling the conditions $E_1(x, x) = 1, \dots, E_n(x, x) = 1$ for $x \in [0, 1]$. According to (30), for any aggregation function A one has

$$E(x, x) = A(E_1(x, x), \dots, E_n(x, x)) = A(1, \dots, 1) = 1,$$

where $x \in [0, 1]$. □

From Lemmas 2-4 it follows that any aggregation function preserves boundary property, symmetry and reflexivity of a fuzzy equivalence. Thus, for aggregation of fuzzy equivalences it is enough to consider only transitivity conditions. We will do it in the sequel.

Next, we put the followings results concerning fuzzy C -equivalences which are continuation and extension of the studies presented in [16].

Theorem 9. *Let C be a fuzzy conjunction. An aggregation function $A : [0, 1]^n \rightarrow [0, 1]$ preserves condition (26) of the aggregated fuzzy equivalences E_1, \dots, E_n if and only if A dominates C ($A \gg C$).*

By Theorems 6 and 9 and Corollary 4 we get

Corollary 8. *Let C_1, C_2 be fuzzy conjunctions, $C_1 \leq C_2$. If aggregation function A dominates C_2 and we have fuzzy C_2 -equivalences E_1, \dots, E_n , then $A(E_1, \dots, E_n)$ is a fuzzy C_1 -equivalence.*

From Theorem 9 and Lemmas 2-4 one obtains

Theorem 10. *The family of all fuzzy C -equivalences is closed under aggregation functions A that dominate C ($A \gg C$).*

By Corollary 3 we get

Theorem 11. *Minimum preserves fuzzy C -equivalence for any fuzzy conjunction C . Weighted minimum preserves fuzzy C -equivalence for any t -seminorm C .*

By Examples 6-8 and Theorem 3 we get

Corollary 9. *Quasi-arithmetic means A preserve fuzzy T_D -equivalence and t -seminorms A preserve fuzzy T_D -equivalence. Weighted minimum (20) preserves fuzzy C -equivalence for $C = \min$. Weighted arithmetic means preserve fuzzy T_L -equivalence. Geometric mean preserves fuzzy T_P -equivalence. The aggregation function A described by the formula (21) preserves fuzzy T_L -equivalence. Any t -norm preserves fuzzy C -equivalence for C being a t -norm. The only t -norm that preserves fuzzy C -equivalence for $C = \min$ is T_M .*

Now we want to pay attention to preservation of fuzzy α - C -equivalence. The results which will be considered here were presented for fuzzy relations in [15].

Lemma 5 ([14]). *Let $\alpha \in [0, 1]$, $t = (t_1, \dots, t_n) \in [0, 1]^n$. $F : [0, 1]^n \rightarrow [0, 1]$ fulfils*

$$F|_{[0, 1]^n \setminus [1-\alpha, 1]^n} < 1 - \alpha \quad (31)$$

if and only if

$$\forall_{t \in [0, 1]^n} F(t_1, \dots, t_n) \geq 1 - \alpha \Rightarrow \min_{1 \leq k \leq n} t_k \geq 1 - \alpha. \quad (32)$$

Theorems 12 and 13 will be presented for a function F (not necessarily an aggregation function) to underline the required assumptions for preservation of fuzzy α - C -equivalence.

Theorem 12. *Let $\alpha \in [0, 1]$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If $F : [0, 1]^n \rightarrow [0, 1]$, which is increasing, fulfils (31) and $F \gg C$, i.e. for any $(s_1, \dots, s_n), (t_1, \dots, t_n) \in [0, 1]^n$*

$$F(C(s_1, t_1), \dots, C(s_n, t_n)) \geq \quad (33)$$

$$C(F(s_1, \dots, s_n), F(t_1, \dots, t_n)),$$

then it preserves a fuzzy α - C -equivalence.

Proof. Let $\alpha \in [0, 1]$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction, $x, y, z \in [0, 1]$. If E_1, \dots, E_n are fuzzy α - C -equivalences, F fulfils (31) and (33) then using notations

$$s_k = E_k(x, y), \quad t_k = E_k(y, z), \quad k = 1, \dots, n \quad (34)$$

and

$$E(x, y) = F(E_1(x, y), \dots, E_n(x, y)), \quad x, y \in [0, 1], \quad (35)$$

and applying Lemma 5 one obtains

$$\begin{aligned} C(E(x, y), E(y, z)) &\geq 1 - \alpha \Leftrightarrow \\ C(F(E_1(x, y), \dots, E_n(x, y)), F(E_1(y, z), \dots, E_n(y, z))) &\geq 1 - \alpha \Leftrightarrow \\ C(F(s_1, \dots, s_n), F(t_1, \dots, t_n)) &\geq 1 - \alpha \Rightarrow \\ F(C(s_1, t_1), \dots, C(s_n, t_n)) &\geq 1 - \alpha \Rightarrow \\ \min_{1 \leq k \leq n} C(s_k, t_k) \geq 1 - \alpha &\Rightarrow \bigvee_{1 \leq k \leq n} C(s_k, t_k) \geq 1 - \alpha \\ \Leftrightarrow \bigvee_{1 \leq k \leq n} C(E_k(x, y), E_k(y, z)) &\geq 1 - \alpha. \end{aligned}$$

So if E_k are fuzzy α - C -equivalences we see that $E_k(x, z) \geq C(E_k(x, y), E_k(y, z))$, for each $k = 1, \dots, n$. Thus by the monotonicity of F and (35), (34), (33) one has

$$\begin{aligned} C(E(x, y), E(y, z)) &= \\ C(F(E_1(x, y), \dots, E_n(x, y)), F(E_1(y, z), \dots, E_n(y, z))) &= \\ = C(F(s_1, \dots, s_n), F(t_1, \dots, t_n)) &\leq \\ F(C(s_1, t_1), \dots, C(s_n, t_n)) &= \\ F(C(E_1(x, y), E_1(y, z)), \dots, C(E_n(x, y), E_n(y, z))) &\leq \\ \leq F(E_1(x, z), \dots, E_n(x, z)) = E(x, z) & \end{aligned}$$

which proves that E is a fuzzy α - C -equivalence. \square

The following corollary is presented for aggregation functions.

Corollary 10. *Let $\alpha \in [0, 1]$, $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If an aggregation function $A : [0, 1]^n \rightarrow [0, 1]$ fulfils (31) and $A \gg C$, then it preserves a fuzzy α - C -equivalence.*

Theorem 13. *Let $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If a function $F : [0, 1]^n \rightarrow [0, 1]$, is increasing, $F \gg C$ and $F \leq \min$, then it preserves a fuzzy α - C -equivalence for any $\alpha \in [0, 1]$.*

Proof. Let F be increasing and fulfil (33). By Lemma 5 a function F fulfils (31) for arbitrary $\alpha \in [0, 1]$ if and only if

$$\begin{aligned} \bigvee_{\alpha \in [0, 1]} \bigvee_{t \in [0, 1]^n} F(t_1, \dots, t_n) &\geq 1 - \alpha \\ \Rightarrow \min_{1 \leq k \leq n} t_k &\geq 1 - \alpha. \end{aligned}$$

So by (2) this is equivalent to the fact that $F \leq \min$. By Theorem 12 a function F preserves a fuzzy α - C -equivalence for any $\alpha \in [0, 1]$. \square

Corollary 11. *Let $C : [0, 1]^2 \rightarrow [0, 1]$ be a fuzzy conjunction. If an aggregation function $A : [0, 1]^n \rightarrow [0, 1]$, fulfils $A \gg C$ and $A \leq \min$, then it preserves a fuzzy α - C -equivalence for any $\alpha \in [0, 1]$.*

Example 12. Theorem 13 gives only sufficient conditions for preservation of a fuzzy α - C -equivalence for any $\alpha \in [0, 1]$ and fuzzy conjunction $C : [0, 1]^2 \rightarrow [0, 1]$. Let us consider a projection P_k , $k \in \mathbb{N}$. We know that P_k preserves a fuzzy α - C -equivalence for any $\alpha \in [0, 1]$ (see Example 11), it dominates any operation $C : [0, 1]^2 \rightarrow [0, 1]$ ([9], Corollary 1), so also a fuzzy conjunction C , but it is not true that $P_k \leq \min$.

For our further considerations we need the following statement

Lemma 6 ([15]). *If $F : [0, 1]^n \rightarrow [0, 1]$ is increasing and has a neutral element $e = 1$, i.e.*

$$\bigvee_{t \in [0, 1]} \bigvee_{1 \leq k \leq n} F(1, \dots, 1, t, 1, \dots, 1) = t, \quad (36)$$

where t is in the k -th position, then $F \leq \min$.

Lemma 6 shows the way of finding increasing functions $F : [0, 1]^n \rightarrow [0, 1]$ which fulfil the condition $F \leq \min$. Typical examples of such binary operations are t -norms and fuzzy conjunctions fulfilling condition $C \leq \min$ (cf. Corollary 2, Example 4). Moreover, we have many results concerning relation of dominance in the family of t -norms (see Theorem 3) and Corollary 3. Applying these facts the following results hold true

Corollary 12. *Let T be an arbitrary t -norm. Minimum preserves a fuzzy α - T -equivalence for any $\alpha \in [0, 1]$. T preserves a fuzzy α - T_D -equivalence for any $\alpha \in [0, 1]$. T preserves a fuzzy α - T -equivalence for any $\alpha \in [0, 1]$. Minimum preserves a fuzzy α - C -equivalence for any fuzzy conjunction C and any $\alpha \in [0, 1]$.*

By Theorem 4 we obtain

Corollary 13. *If a quasi-copula dominates T_L (T_P), then it preserves a fuzzy α - T_L -equivalence (a fuzzy α - T_P -equivalence) for any $\alpha \in [0, 1]$.*

5. Conclusions

Aggregation functions preserving properties of aggregated connectives such as fuzzy equivalences were presented. There were considered two types of such connectives. In our further work we would like to discuss other equivalence notions (e.g. [19], p. 33) and preservation of their properties by aggregation functions.

Acknowledgements

This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge in Rzeszów, through Project Number RPPK.01.03.00-18-001/10.

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