

Normalized expression for solutions to linear fuzzy differential equations under combination of differences

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Abstract

We give a normalized expression for the solutions to the initial value problem for some linear fuzzy-interval differential equations by using a general notation which allows the combination of two types of differences. By switching between these types of differences, we derive several expressions for solutions corresponding to strongly generalized differentiability, providing a general formulation for these solutions to linear problems.

Keywords: Linear fuzzy differential equations, Strongly generalized differentiability, Operations with fuzzy intervals, Solutions

1. Introduction

The study of fuzzy differential equations has attracted the interest of many researchers thanks to the development of the concept of differentials of fuzzy functions [2, 3, 5, 11, 15, 16, 31, 38] and other useful techniques [17, 18, 25, 27, 34, 39], leading to continuous advances in the solvability of this type of equations since the publication of [19, 36, 20].

The analysis of uncertain systems has been developed from different points of view which include the use of several concepts of differences and derivatives, such as the Hukuhara difference and derivative [15, 20], strongly generalized derivative [2, 3, 4, 8, 21], generalized Hukuhara difference and differentiability [5, 11, 37, 38], but also from different approaches such as differential inclusions (starting in [17] and continuing, for instance, in [10, 18, 35]), the use of different representations of fuzzy numbers [39], the embedding of fuzzy sets on spaces with a linear structure [34], the detailed analysis of the implications of Zadeh's extension principle [7, 27, 33], the use of generalized metric spaces [30] or the study of the connections between different approaches [6, 8, 9] and the perspectives of the problem [1].

The study of the initial value problems for linear fuzzy differential equations has been accomplished by many authors by using the different types of derivatives [12, 13, 14, 21, 22, 24, 29, 36]. In the literature, we can also find some results on periodic

boundary value problems [32] or even functional fuzzy differential equations [23, 26]. Numerical approach has also been developed (see, for instance, [28]).

Here, we do not discuss the advantages of the different methods available to obtain the solution to a fuzzy differential equation. The approach of Zadeh's extension principle to get a fuzzy solution starting from a deterministic solution allows to obtain a reasonable solution to the problem of interest. This approach is not affected by the choice of a different formulation for the linear problem as long as the corresponding real equations are equivalent, in such a way that the existence of a unique solution for all the equivalent real differential equations allows to obtain a unique fuzzy solution by applying Zadeh's extension principle. However, if we use a fuzzy derivative to define the concept of solution, then several similar differential equations (which coincide in the real case) might not be equivalent and lead to solutions with very different properties.

It is considered (see [10]) that the approach following Zadeh's extension principle is adequate since it preserves some interesting properties of the classical solutions. Nevertheless, the computation using this point of view can be difficult under certain circumstances [10], so that other approaches are useful both as interesting methods by themselves and to get approximations of the problem.

Moreover, the analysis of fuzzy differential equations based on different concepts of fuzzy derivatives is also an interesting issue (see [4]), in addition to the achievements related to the treatment of the problem without the requirement of a fuzzy derivative concept. Therefore, this study is motivated by the interest arisen in the study of the properties of the solutions to fuzzy differential equations by using different types of derivatives, which are adapted to the special structure and particularities of fuzzy intervals, in such a way that the behavior of the solutions to fuzzy problems may differ substantially from the classical case.

The formulation used in this work is based on the use of strongly generalized differentiability, since the corresponding solutions show interesting behavior and properties which vary depending on the type of differentiability chosen, being adapted to diverse

situations. As mentioned above, there are other approaches independent of the particularities of a fuzzy derivative such as Zadeh's extension principle and differential inclusions' approach, which are also interesting.

For the essential concepts on fuzzy sets and fuzzy differential equations, we refer, for instance, to [15, 25]. Some additional notation needed for this contribution can be also found, for instance, in the references [3, 19].

2. Preliminaries

We consider here the space of fuzzy intervals $\mathbb{R}_{\mathcal{F}}$ (the set of mappings from \mathbb{R} to $[0, 1]$ which are normal, upper semi-continuous, fuzzy-convex and compact-supported). For $u \in \mathbb{R}_{\mathcal{F}}$ and $0 < \alpha \leq 1$, we can define the α -level set of u by the expression

$$[u]^{\alpha} = \{t \in \mathbb{R} : u(t) \geq \alpha\}$$

and the set $[u]^0 = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is called the support of u .

By the properties of fuzzy intervals, the α -level set of $u \in \mathbb{R}_{\mathcal{F}}$ is a compact interval, for every α . For the solvability of differential equations over the space of fuzzy intervals, it is very useful the notation $[u]^{\alpha} = [\underline{u}^{\alpha}, \bar{u}^{\alpha}]$, $\alpha \in [0, 1]$, for the level sets. Besides, in the space $\mathbb{R}_{\mathcal{F}}$, the operations of sum $u + v$ and multiplication by a real scalar $\lambda \cdot u$ are defined levelwise:

$$[u + v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}, [\lambda \cdot u]^{\alpha} = \lambda[u]^{\alpha}, \forall \alpha \in [0, 1].$$

The space $\mathbb{R}_{\mathcal{F}}$ is a complete metric space [15] with the distance D defined as

$$D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|\underline{u}^{\alpha} - \underline{v}^{\alpha}|, |\bar{u}^{\alpha} - \bar{v}^{\alpha}|\},$$

for $u, v \in \mathbb{R}_{\mathcal{F}}$. We use the concept of integration for fuzzy-valued functions as given in [3].

Concerning the different approaches in the solvability of fuzzy differential equations, we consider here strongly generalized differentiability and we use a simple notation which allows the combination of two types of differences, with the aim of finding a common framework to compare the different expressions known. As mentioned, other approaches are, for instance, those in [4, 8, 27, 29, 30, 38, 39].

Definition 1 Given $u, v \in \mathbb{R}_{\mathcal{F}}$, if there exists $z \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + z$, then we say that z is the H -difference of u and v (denoted $u \ominus v$).

Note that this concept of difference is not equivalent to the addition of the opposite element.

Next, we recall the concept of generalized differentiability as introduced in [3] (see also [4, 8, 23, 28]) for fuzzy functions defined on $I \subseteq \mathbb{R}$ an open interval.

Definition 2 ([3]) Let $F : I \rightarrow \mathbb{R}_{\mathcal{F}}$ and fix $t_0 \in I$. We say that F is differentiable at t_0 if there exists an element $F'(t_0) \in \mathbb{R}_{\mathcal{F}}$ such that either:

- (i) for all $h > 0$ sufficiently close to 0, the H -differences $F(t_0 + h) \ominus F(t_0), F(t_0) \ominus F(t_0 - h)$ exist and the limits (in the metric D)

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \\ = \lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0), \end{aligned}$$

or

- (ii) for all $h > 0$ sufficiently close to 0, the H -differences $F(t_0) \ominus F(t_0 + h), F(t_0 - h) \ominus F(t_0)$ exist and the limits (in the metric D)

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} \\ = \lim_{h \rightarrow 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0). \end{aligned}$$

In the references [3, 8], the fundamental properties of this derivative can be found.

Definition 3 Given u, v two fuzzy intervals, by $u \hat{\ominus} v$ we denote either the H -difference $u \ominus v$ (provided it exists) or the addition of the opposite element $u + (-1)v$.

In Definition 3, the notation $\hat{\ominus}$ represents the possibility of choosing one of the two options, in the sense that both of them are possible in many cases. Depending on the particular choice, different situations may appear.

In this note, the use of this general notation $\hat{\ominus}$ allows to give a uniform expression for the solutions to some linear fuzzy differential equations depending on the particular meaning we give to $u \hat{\ominus} v$. Of course, the interpretation of $u \hat{\ominus} v$ as $u \ominus v$ is subject to its existence, which has to be checked appropriately but, here, we are mainly interested in the usefulness of this notation to give uniform expressions for the solutions to linear fuzzy differential equations.

3. Solving fuzzy linear differential equations

The initial value problem

$$\begin{cases} y'(t) = a(t)y(t) + b(t), & t \in I, \\ y(0) = y_0, \end{cases} \quad (1)$$

was considered in [4], where $y_0 \in \mathbb{R}_{\mathcal{F}}$, $a : I \rightarrow \mathbb{R}$ and $b : I \rightarrow \mathbb{R}_{\mathcal{F}}$ and this problem is not equivalent to any of the following:

$$\begin{cases} y'(t) + (-a(t))y(t) = b(t), \\ y(0) = y_0 \end{cases} \quad (2)$$

or

$$\begin{cases} y'(t) + (-b(t)) = a(t)y(t), \\ y(0) = y_0, \end{cases} \quad (3)$$

which are not equivalent either (see [4]). Moreover, if y is simultaneously a solution to (1) and (2), then it is necessarily real. It is also well-known [4], that there exists no fuzzy (non-real) solution to problem

$$\begin{cases} y'(t) + (-b(t)) + (-a(t))y(t) = \tilde{0}, \\ y(0) = y_0. \end{cases} \quad (4)$$

In this note, we provide a normalized expression for the solutions to problems (1)–(3) from the point of view of the use of the notation $\hat{\ominus}$, which allows to formulate the strongly generalized differentiable solutions under a common style.

In what follows, the coefficients a and b in the above-mentioned problems are continuous functions in an open interval $I \subseteq \mathbb{R}$ ($a : I \rightarrow \mathbb{R}$, $b : I \rightarrow \mathbb{R}_{\mathcal{F}}$) and the initial condition y_0 is a fuzzy interval.

As a background for our study, the expressions of the solutions to some equations of types (1)–(3) are provided in [4]. Besides, in [22], the general expression of the solutions to (1) are studied in terms of the sign of the coefficient a . We include these expressions here for completeness.

In what follows, for simplicity, we also use the notation

$$\varphi(t) = \int_0^t a(u)du, \quad t \in I.$$

Theorem 1 ([22]) For $a < 0$, the (i)-solution to problem (1) is given by

$$\begin{aligned} y_1(t) = & \cosh(\varphi(t)) \left(y_0 + \int_0^t \psi(s)ds \right) \\ & + \sinh(\varphi(t)) \left(y_0 + \int_0^t \psi(s)ds \right), \end{aligned}$$

where

$$\psi(s) = b(s) \cosh(\varphi(s)) \ominus b(s) \sinh(\varphi(s)),$$

provided that the above H -differences exist.

On the other hand, for $a < 0$, the (ii)-solution to problem (1) is given by

$$y_2(t) = e^{\varphi(t)} \left(y_0 \ominus \int_0^t (-b(s))e^{-\varphi(s)}ds \right),$$

provided that the H -differences exist.

Proof: See [4] for the expression of the (ii)-solution and [22] for the expression of the (i)-solution. \square

Theorem 2 ([22]) For $a > 0$, the (i)-solution to problem (1) is given by

$$y_3(t) = e^{\varphi(t)} \left(y_0 + \int_0^t b(s)e^{-\varphi(s)}ds \right).$$

On the other hand, for $a > 0$, the (ii)-solution to problem (1) is given by

$$\begin{aligned} y_4(t) = & \cosh(\varphi(t)) \left(y_0 \ominus \int_0^t \phi(s)ds \right) \\ & \ominus -\sinh(\varphi(t)) \left(y_0 \ominus \int_0^t \phi(s)ds \right), \end{aligned}$$

provided the previous H -differences exist, where

$$\phi(s) = b(s) \sinh(\varphi(s)) - b(s) \cosh(\varphi(s)).$$

Proof: The case of the (i)-solution can be seen in [4], while the (ii)-solution is considered in [22]. \square

On the other hand, the solutions to problems (2) and (3) are provided in [4, 24].

Theorem 3 ([24]) Let $a < 0$ and consider

$$y_5(t) = e^{\varphi(t)} \left(y_0 + \int_0^t b(s)e^{-\varphi(s)}ds \right) \quad (5)$$

and

$$\begin{aligned} y_6(t) = & \cosh(\varphi(t)) \left(y_0 \ominus (-1) \int_0^t \psi(s)ds \right) \\ & + \sinh(\varphi(t)) \left(y_0 \ominus (-1) \int_0^t \psi(s)ds \right), \end{aligned} \quad (6)$$

where ψ is defined in Theorem 1, provided that the H -differences in (6) and the expression of ψ exist.

Then the following assertions are valid:

- y_5 is a (i)-solution to problem (2) on I (if it is (i)-differentiable on I).
- y_6 is a (ii)-solution to problem (2) on I (if it is (ii)-differentiable on I).
- y_5 is a (ii)-solution to problem (3) on I (if it is (ii)-differentiable on I).
- y_6 is a (i)-solution to problem (3) on I (if it is (i)-differentiable on I).

Theorem 4 ([24]) Let $a > 0$ and consider

$$\begin{aligned} y_7(t) = & \cosh(\varphi(t)) \left(y_0 - \int_0^t \phi(s)ds \right) \\ & \ominus -\sinh(\varphi(t)) \left(y_0 - \int_0^t \phi(s)ds \right), \end{aligned} \quad (7)$$

provided that the levelwise H -differences define, for each $t \in I$, a fuzzy interval $y_7(t)$ (ϕ is given in Theorem 2) and

$$y_8(t) = e^{\varphi(t)} \left(y_0 \ominus \int_0^t (-b(s))e^{-\varphi(s)}ds \right), \quad (8)$$

provided that the previous H -differences exist.

Then the following assertions are valid:

- y_7 is a (i)-solution to problem (2) on I (if it is (i)-differentiable on I).
- y_8 is a (ii)-solution to problem (2) on I (if it is (ii)-differentiable on I).
- y_7 is a (ii)-solution to problem (3) on I (if it is (ii)-differentiable on I).
- y_8 is a (i)-solution to problem (3) on I (if it is (i)-differentiable on I).

Remark 1 From the previous results, it is obvious that the expressions y_2 and y_8 are equal (provided they exist) and the same for y_3 and y_5 . However, note that the use of their respective expressions strongly depends on the sign of a .

Next, we give a uniform expression for the solutions obtained in Theorems 1–4, by the introduction of the general notation of the difference $\widehat{\ominus}$. In the following, we represent with the symbol $(*)$ the cases where the choice $\widehat{\ominus} = \ominus$ is used, so that additional restrictions have to be imposed in order to guarantee the existence of the corresponding H-differences.

First, we provide the expression which generalizes y_2 (or y_8) and y_3 (or y_5) under a common frame.

Theorem 5 *The function*

$$y(t) = e^{\varphi(t)} \left(y_0 \widehat{\ominus} (-1) \int_0^t b(s) e^{-\varphi(s)} ds \right) \quad (9)$$

represents, for $a < 0$:

- (ii)-solution to problem (1) $(*)$;
- (i)-solution to problem (2), provided it is (i)-differentiable;
- (ii)-solution to problem (3), provided it is (ii)-differentiable

and, for $a > 0$:

- (i)-solution to problem (1);
- (ii)-solution to problem (2), if it is (ii)-differentiable $(*)$;
- (i)-solution to problem (3), if it is (i)-differentiable $(*)$.

Proof: Indeed, if we interpret $\widehat{\ominus}$ as \ominus in (9), we obtain the functions y_2 and y_8 . On the other hand, if $u \widehat{\ominus} v$ is understood as $u + (-1)v$ in (9), then we obtain the functions y_3 and y_5 . The conclusion follows from Theorems 1–4. \square

Concerning expressions y_1 , y_4 , y_6 and y_7 , we consider the following function (provided it is well-defined):

$$y(t) = \cosh(\varphi(t)) \left(y_0 \widehat{\ominus} (-1) \int_0^t \gamma(s) ds \right) \\ \widehat{\ominus} - \sinh(\varphi(t)) \left(y_0 \widehat{\ominus} (-1) \int_0^t \gamma(s) ds \right),$$

where

$$\gamma(s) = b(s) \cosh(\varphi(s)) \widehat{\ominus} b(s) \sinh(\varphi(s)).$$

In order to represent that any of the $\widehat{\ominus}$ differences can be chosen differently (if the corresponding function exists), we rewrite this expression by using the differences $\widehat{\ominus}_1$, $\widehat{\ominus}_2$ and $\widehat{\ominus}_3$, in such a way that the three of them represent $\widehat{\ominus}$ but they can be interpreted in the same or different sense, by choosing one of the interpretations given in Definition 3. This way, we work with the function

$$y(t) = \cosh(\varphi(t)) \left(y_0 \widehat{\ominus}_1 (-1) \int_0^t \gamma(s) ds \right) \\ \widehat{\ominus}_2 - \sinh(\varphi(t)) \left(y_0 \widehat{\ominus}_1 (-1) \int_0^t \gamma(s) ds \right) \quad (10)$$

and

$$\gamma(s) = b(s) \cosh(\varphi(s)) \widehat{\ominus}_3 b(s) \sinh(\varphi(s)), \quad (11)$$

where the differences can be switched, provided the resulting function is well-defined. Taking different combinations of the differences $\widehat{\ominus}_1$, $\widehat{\ominus}_2$ and $\widehat{\ominus}_3$, we prove that we can derive the different types of solutions, so that (10)–(11) will represent a normalized expression for all the solutions given in Theorems 1–4. We stress the fact that (10) gives uniformity to the various expressions, more than paying attention to the conditions guaranteeing the good definition of the function itself.

To simplify the proof of the following result, we define \ominus_1 and \ominus_2 , respectively, by $u \ominus_1 v = u \ominus v$ and $u \ominus_2 v = u + (-1)v$.

Theorem 6 *The function defined by (10)–(11) represents, for $a < 0$:*

- (i)-solution to problem (1);
- (ii)-solution to problem (2), if it is (ii)-differentiable $(*)$;
- (i)-solution to problem (3), if it is (i)-differentiable $(*)$

and, for $a > 0$:

- (ii)-solution to problem (1) $(*)$.
- (i)-solution to problem (2), if it is (i)-differentiable $(*)$;
- (ii)-solution to problem (3), if it is (ii)-differentiable $(*)$.

Proof: In (10)–(11), if we take $\widehat{\ominus}_1 = \widehat{\ominus}_3 = \ominus_1 = \ominus$ and $\widehat{\ominus}_2 = \ominus_2$, then $\gamma \equiv \psi$ and we obtain function y_6 , so that, in case of existence and differentiability and for $a < 0$, we have a (ii)-solution to problem (2) or a (i)-solution to problem (3) (see Theorem 3).

On the other hand, if we take $\widehat{\ominus}_1 = \widehat{\ominus}_2 = \ominus_2$ and $\widehat{\ominus}_3 = \ominus_1 = \ominus$, then $\gamma \equiv \psi$ again and we obtain function y_1 , which, from Theorem 1 is, for $a < 0$, the (i)-solution to problem (1).

If we select $\widehat{\ominus}_1 = \widehat{\ominus}_2 = \ominus_1 = \ominus$ and $\widehat{\ominus}_3 = \ominus_2$, then

$$y(t) = \cosh(\varphi(t)) \left(y_0 \ominus (-1) \int_0^t \gamma(s) ds \right) \\ \ominus - \sinh(\varphi(t)) \left(y_0 \ominus (-1) \int_0^t \gamma(s) ds \right)$$

and

$$-\gamma(s) = - (b(s) \cosh(\varphi(s)) - b(s) \sinh(\varphi(s))) \\ = \phi(s),$$

so that we obtain function y_4 , which is, from Theorem 2 and for $a > 0$, the (ii)-solution to problem (1).

Finally, if $\widehat{\ominus}_1 = \widehat{\ominus}_3 = \ominus_2$ and $\widehat{\ominus}_2 = \ominus_1 = \ominus$, then

$$y(t) = \cosh(\varphi(t)) \left(y_0 + \int_0^t \gamma(s) ds \right) \\ \ominus - \sinh(\varphi(t)) \left(y_0 + \int_0^t \gamma(s) ds \right),$$

where $\gamma \equiv -\phi$, so that we obtain function y_7 , which is, from Theorem 4 and for $a > 0$, a (i)-solution to problem (2) or a (ii)-solution to problem (3). \square

Moreover, expressions (10)–(11) are also a generalization of (9). This way, (10)–(11) gives a uniform expression for the different solutions obtained.

Theorem 7 *The function defined by (10)–(11) generalizes the expression of all the solutions y_1, \dots, y_8 given in Theorems 1–4.*

Proof: This comes from the fact that the function defined by (10)–(11) includes the expression (9) and the proof of Theorems 5–6. Indeed, denote $C(t) := \left(y_0 \hat{\ominus}_1 (-1) \int_0^t \gamma(s) ds \right)$ (understanding that it is well-defined).

In the case $a > 0$, we have $\varphi(t) > 0$ and $\cosh(\varphi(t)) > \sinh(\varphi(t)) > 0$, thus, if we consider $\hat{\ominus}_2 = \ominus_2$ in (10), we get

$$\begin{aligned} y(t) &= \cosh(\varphi(t))C(t) + \sinh(\varphi(t))C(t) \\ &= (\cosh(\varphi(t)) + \sinh(\varphi(t)))C(t) = e^{\varphi(t)}C(t). \end{aligned}$$

Besides, if $\hat{\ominus}_3 = \ominus_1 = \ominus$, since $a > 0$,

$$\begin{aligned} \gamma(s) &= b(s) \cosh(\varphi(s)) \ominus b(s) \sinh(\varphi(s)) \\ &= b(s) (\cosh(\varphi(s)) - \sinh(\varphi(s))) \\ &= b(s)e^{-\varphi(s)}, \end{aligned}$$

so that we have

$$y(t) = e^{\varphi(t)} \left(y_0 \hat{\ominus}_1 (-1) \int_0^t b(s)e^{-\varphi(s)} ds \right)$$

and it coincides with (9).

On the other hand, if $a < 0$, we have $\varphi(t) < 0$ and $\cosh(\varphi(t)) > -\sinh(\varphi(t)) > 0$, thus, if we consider $\hat{\ominus}_2 = \ominus_1 = \ominus$ in (10), we obtain

$$\begin{aligned} y(t) &= \cosh(\varphi(t))C(t) \ominus -\sinh(\varphi(t))C(t) \\ &= (\cosh(\varphi(t)) + \sinh(\varphi(t)))C(t) = e^{\varphi(t)}C(t), \end{aligned}$$

where we have used that $\lambda u \ominus \mu u = (\lambda - \mu)u$, for $\lambda > \mu > 0$. Furthermore, taking $\hat{\ominus}_3 = \ominus_2$, since $a < 0$, we have

$$\begin{aligned} \gamma(s) &= b(s) \cosh(\varphi(s)) - b(s) \sinh(\varphi(s)) \\ &= b(s) \cosh(\varphi(s)) + b(s)(-\sinh(\varphi(s))) \\ &= b(s) (\cosh(\varphi(s)) - \sinh(\varphi(s))) \\ &= b(s)e^{-\varphi(s)} \end{aligned}$$

and we deduce again the expression (9).

Therefore, as in the proof of Theorem 5, the choice of $\hat{\ominus}_1$ determines if $y = y_2 = y_8$ ($\hat{\ominus}_1 = \ominus_1 = \ominus$) or $y = y_3 = y_5$ ($\hat{\ominus}_1 = \ominus_2$). \square

Remark 2 *For the case $a \equiv 0$, each of the problems (1)–(3) are reduced to*

$$y'(t) = b(t), \quad t \in I, \quad y(0) = y_0.$$

In this case,

$$y_1(t) = y_3(t) = y_5(t) = y_7(t) = y_0 + \int_0^t b(s) ds$$

and

$$y_2(t) = y_4(t) = y_6(t) = y_8(t) = y_0 \ominus \int_0^t (-b(s)) ds$$

(subject to its existence). It is clear that, for $a \equiv 0$, (11) gives $\gamma \equiv b$ and (10) is reduced to

$$y(t) = y_0 \hat{\ominus} (-1) \int_0^t b(s) ds,$$

while (9) produces the same expression. It is obvious that the previous function coincides with y_{2k} , for $k = 1, \dots, 4$, if $\hat{\ominus} = \ominus$, and it is exactly y_{2k+1} , for $k = 0, \dots, 3$, if $\hat{\ominus} = \ominus_2$.

To conclude, in Table 1, we summarize the solutions obtained as particular cases of (10)–(11) depending on the choices of the differences $\hat{\ominus}_1$, $\hat{\ominus}_2$ and $\hat{\ominus}_3$, for a non null. There, we indicate with the symbol (*) the requirement of the existence of the corresponding H-differences and with (d) the requirement of the differentiability of the function involved.

	$\hat{\ominus}_2 = \ominus$	$\hat{\ominus}_2 = \ominus_2$
$\hat{\ominus}_1 = \ominus$	$\hat{\ominus}_3 = \ominus_2,$ $a < 0$ Function y_2 (*) (ii)-sol. to (1)	$\hat{\ominus}_3 = \ominus,$ $a < 0$ Function y_6 (*) (ii)-sol. to (2) (d) (i)-sol. to (3) (d)
	$\hat{\ominus}_3 = \ominus_2,$ $a > 0$ Function y_4 (*) (ii)-sol. to (1)	$\hat{\ominus}_3 = \ominus,$ $a > 0$ Function y_8 (*) (ii)-sol. to (2) (d) (i)-sol. to (3) (d)
$\hat{\ominus}_1 = \ominus_2$	$\hat{\ominus}_3 = \ominus_2,$ $a < 0$ Function y_5 (i)-sol. to (2) (d) (ii)-sol. to (3) (d)	$\hat{\ominus}_3 = \ominus,$ $a < 0$ Function y_1 (*) (i)-sol. to (1)
	$\hat{\ominus}_3 = \ominus_2,$ $a > 0$ Function y_7 (*) (i)-sol. to (2) (d) (ii)-sol. to (3) (d)	$\hat{\ominus}_3 = \ominus,$ $a > 0$ Function y_3 (i)-sol. to (1)

Table 1: Particular solutions obtained from (10)–(11) depending on $\hat{\ominus}_1$, $\hat{\ominus}_2$ and $\hat{\ominus}_3$.

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