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Gradual Complex Numbers

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Abstract

This paper aims to introduce the concept of **Gradual Complex Number (GCN)** based on the existing definition of gradual numbers and to show that the algebraic and polar forms, as well as some algebraic properties, of complex numbers can be directly extended from the crisp case. Furthermore, a kind of distance is established in order to deal with such numbers.

Keywords: Fuzzy Numbers, Gradual Numbers, Gradual Complex Numbers.

1. Introduction

In 1965, Lofti Zadeh [4] introduced the concept of fuzzy sets to quantify subjective linguistic terms such as: *"approximately"*, *"around"* etc. Fuzzy sets generalize classical sets, the classical characteristic functions are generalized to membership functions — i.e. the values of pertinence are generalized from $\{0,1\}$ to $[0,1]$.

A fuzzy subset under certain conditions is called a fuzzy number. Fuzzy numbers were introduced in order to extend usual numbers. Many authors purposed to equip such entities with arithmetics, which usually inherits the properties of the arithmetic on intervals and does not provide inverses for addition and multiplication. Considering this problem, Didier Dubois and Henri Prade [3] introduced two new concepts: (1) **Gradual Elements** and (2) **Real Gradual Numbers**.

This article introduces **Gradual Complex Numbers (GCN)** endowed with a full compatible arithmetic with usual complex numbers. It shows how is the algebraic and the polar form of such numbers. The paper is structured as follows: Sections 2 provide the required concepts to the development of this paper. Section 3 introduces the numbers proposed here together with its algebraic and polar forms. Section 4 provides the final remarks.

It shows how gradual complexes can be characterized in terms of a fuzzy complex number as well as how Zhang complex fuzzy numbers [5] can be characterized in terms of gradual complex numbers. Finally, in order to achieve the results, this paper proposes a notion of distance for GCNs.

2. Gradual Real Number(GCN)

Gradual real numbers were introduced by J. Fortin, Didier Dubois e H. Fargier in [4].

Definition 2.1 *A* ***gradual real number*** $\tilde{r}$ *is defined by a total assignment function* $A_{\tilde{r}} : (0,1] \to \mathbb{R}$ *such that* $A_{\tilde{r}}(1) = r$.

Consider $\tilde{\mathbb{R}}$ as the set of all gradual real numbers. A real number r can be seen as a gradual real number $\tilde{r}$ such that, for all $\alpha \in (0,1]$, $a_{\tilde{r}}(\alpha) = r$. The operations defined in $\mathbb{R}$ can be extended to the set $\tilde{\mathbb{R}}$. It is easy to see that the operations of addition and multiplication defined on $\tilde{\mathbb{R}}$ inherits the algebraic properties of real numbers.

2.1. Fuzzy number represented by gradual numbers

Observation 2.1 (Fortin et. al. [4]) *An ordered pair of gradual real numbers can be seen as a fuzzy real number. To describe a fuzzy number F as a pair of gradual real numbers* $(\tilde{r}_1, \tilde{r}_2)$*, it is necessary that their assignment function* $A_{\tilde{r}_1}$ *and* $A_{\tilde{r}_2}$ *be total, increasing and decreasing (respectively), and for all* $0 < \alpha \leq 1$, $A_{\tilde{r}_1}(\alpha) \leq A_{\tilde{r}_2}(\alpha)$.

thus, the membership function of F is defined as: [1]

$$\mu_F(x) = \begin{cases} Sup\{\alpha / A_{\tilde{r}_1}(\alpha) \leq x\}, \text{ if } x \in Im(A_{\tilde{r}_1}) \\ 1, \text{ if } A_{\tilde{r}_1}(1) \leq x \leq A_{\tilde{r}_2}(1) \\ Sup\{\alpha / A_{\tilde{r}_2}(\alpha) \geq x\}, \text{ if } x \in Im(A_{\tilde{r}_2}) \\ 0, \text{ otherwise} \end{cases} \tag{2}$$

The next section proposes a definition for Gradual Complex Numbers.

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[1]Note that when $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$ are invertible functions, the fuzzy membership function, μ_F, is composed by such inverses:

$$\mu_F(x) = \begin{cases} A_{\tilde{r}_1}^{-1}(x), \text{ if } x \in Im(A_{\tilde{r}_1}) \\ 1, \text{ if } A_{\tilde{r}_1}(1) \leq x \leq A_{\tilde{r}_2}(1) \\ A_{\tilde{r}_2}^{-1}(x), \text{ if } x \in Im(A_{\tilde{r}_2}) \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

3. Gradual Complex Numbers

Definition 3.1 *Let $\mathbb{C}$ be the set of complex numbers , $\tilde{a}$ and $\tilde{b}$ gradual real numbers with assignment functions $A_{\tilde{a}}, A_{\tilde{b}} : (0,1] \to \mathbb{R}$, correspondingly. A* ***gradual complex number*** *$\bar{z} = (\tilde{a}, \tilde{b})$ is defined by an assignment function $A_{\bar{z}} : (0,1] \to \mathbb{C}$, s.t. $A_{\bar{z}}(\alpha) = (A_{\tilde{a}}(\alpha), A_{\tilde{b}}(\alpha))$. $A_{\tilde{a}}$ called* ***real part*** *of the complex gradual $\bar{z}$ and is denoted by $Re_{\bar{z}}$, $A_{\tilde{b}}$ called* ***imaginary part****, and is denoted by $Im_{\bar{z}}$. The* ***set of all gradual complex numbers*** *is denoted by $\bar{\mathbb{C}}$.*

Definition 3.2 *A* ***set of gradual complex numbers*** *$G = \{\bar{z}_i : i \in I\}$ is defined by its assignment function: $A_G : (0,1] \to \mathcal{P}(\mathbb{C})$, s.t. $A_G(\alpha) = \{a_{\bar{z}_i}(\alpha) : i \in I\}$.*

Proposition 3.1 *For all gradual complex number $\bar{z}$, $A_{\bar{z}}(1) \in \mathbb{C}$.*

Proof:
Given $\bar{z} \in \bar{\mathbb{C}}$, since $\bar{z} = (\tilde{a}, \tilde{b})$, $A_{\tilde{a}}(1) = a$, $A_{\tilde{b}}(1) = b \in \mathbb{R}$. Then, $A_{\bar{z}}(1) \overset{def}{=} (A_{\tilde{a}}(1), A_{\tilde{b}}(1)) = (a, b) \in \mathbb{C}$.

QED

Proposition 3.2 *All complex number $z = (a, b) \in \mathbb{C}$ can be seen as a gradual complex number $\bar{z}$ defined by the assignment function $A_{\bar{z}}(\alpha) = (a, b)$ for all $\alpha \in (0,1]$.*

3.1. Arithmetic on $\bar{\mathbb{C}}$

Definition 3.3 (equality) *Given $\bar{z}_1$ and $\bar{z}_2 \in \bar{\mathbb{C}}$, then $\bar{z}_1 = \bar{z}_2 \Leftrightarrow A_{\bar{z}_1}(\alpha) = A_{\bar{z}_2}(\alpha)$, $\forall \alpha \in (0,1]$.*

Definition 3.4 *Given $\bar{z}_1, \bar{z}_2$ $\bar{z}_3 \in \bar{\mathbb{C}}$, $\alpha \in (0,1]$ and assuming the complex arithmetic: "$+, -, \cdot$ and $/$", the operations defined on the set $\bar{\mathbb{C}}$ are given by:.*

- ***addition****: $(A_{\bar{z}_1} + A_{\bar{z}_2})(\alpha) \overset{def}{=} A_{\bar{z}_1+\bar{z}_2}(\alpha) = A_{\bar{z}_1}(\alpha) + A_{\bar{z}_2}(\alpha)$*
- ***subtraction****:$(A_{\bar{z}_1} - A_{\bar{z}_2})(\alpha) \overset{def}{=} A_{\bar{z}_1-\bar{z}_2}(\alpha) = A_{\bar{z}_1}(\alpha) - A_{\bar{z}_2}(\alpha)$*
- ***multiplication****: $(A_{\bar{z}_1} \cdot A_{\bar{z}_2})(\alpha) \overset{def}{=} A_{\bar{z}_1\cdot\bar{z}_2}(\alpha) = A_{\bar{z}_1}(\alpha) \cdot A_{\bar{z}_2}(\alpha)$*
- ***division****: Consider $A_{\bar{z}_2}(\alpha) \neq (0,0)$, $\forall \alpha \in (0,1]$.*
  $(A_{\bar{z}_1}/A_{\bar{z}_2})(\alpha) \overset{def}{=} A_{\bar{z}_1/\bar{z}_2}(\alpha) = \frac{A_{\bar{z}_1}(\alpha)}{A_{\bar{z}_2}(\alpha)}$

Proposition 3.3 *The structure $\langle \bar{\mathbb{C}}, +, -, \times, \div, 1, 0 \rangle$ is a field.*

Proof

- **Commutativity**: For all $\alpha \in (0,1]$, $A_{\bar{z}_1+\bar{z}_2}(\alpha) \overset{def}{=} A_{\bar{z}_1}(\alpha) + A_{\bar{z}_2}(\alpha) = A_{\bar{z}_2}(\alpha) + A_{\bar{z}_1}(\alpha) = A_{\bar{z}_2+\bar{z}_1}(\alpha)$.
- **Identity**: The identity element for addition is a gradual complex number $\bar{e}$, such that for all $\alpha \in (0,1]$, $A_{\bar{z}_1+\bar{e}}(\alpha) = A_{\bar{z}_1}(\alpha)$. Consider $\bar{e}$ defined by the assignment function: $A_{\bar{e}}(\alpha) = (A_{\tilde{x}}(\alpha), A_{\tilde{y}}(\alpha))$, $A_{\bar{z}_1+\bar{e}}(\alpha) \overset{def}{=} A_{\bar{z}_1}(\alpha) + A_{\bar{e}}(\alpha) = A_{\bar{z}_1}(\alpha)$. Therefore, $A_{\bar{e}} = A_{\tilde{0}}$.

Therefore $A_{\tilde{0}}$ is the identity element for the sum of gradual complex numbers. Similarly the remaining properties of a field can be easily verified.

Observe that in contrast to fuzzy numbers, the assignment of uncertainty to the complex numbers does not result the loss of algebraic properties.

Observation 3.1 *Note that there is an algebraic injection $\tilde{\mathbb{R}}$ in $\bar{\mathbb{C}}$, given by the function $I : \tilde{\mathbb{R}} \to \bar{\mathbb{C}}$, $I(A_{\tilde{r}}) = (A_{\tilde{r}}, A_{\tilde{0}})$, where $A_{\tilde{0}} = 0$, $\forall \alpha \in (0,1]$.*

3.2. Algebraic form of a Gradual Complex Number

The **Gradual Imaginary Unit** is the the gradual complex number $\bar{i}$ characterized by the assignment function $A_{\bar{i}}(\alpha) = (A_{\tilde{0}}, A_{\tilde{1}})$, where $A_{\tilde{0}}(\alpha) = 0$ and $A_{\tilde{1}}(\alpha) = 1$, for all $\alpha \in (0,1]$.

Proposition 3.4 *All gradual complex number $\bar{z} = (\tilde{a}, \tilde{b})$, defined by the assignment function $A_{\bar{z}}(\alpha) = (A_{\tilde{a}}(\alpha), A_{\tilde{b}}(\alpha))$, can be written as $A_{\bar{z}}(\alpha) = A_{\tilde{a}+\tilde{b}\cdot\bar{i}}(\alpha)$. This representation is called* ***algebraic form*** *of a gradual complex number, it can also be seen as $\bar{z} = \tilde{a} + \tilde{b} \cdot \bar{i}$.*

Proof
If $A_{\bar{z}}(\alpha) = (A_{\tilde{a}}(\alpha), A_{\tilde{b}}(\alpha))$, then

$$\begin{aligned}
(A_{\tilde{a}}(\alpha), A_{\tilde{b}}(\alpha)) &= (A_{\tilde{a}}(\alpha), A_{\tilde{0}}(\alpha)) + (A_{\tilde{0}}(\alpha), A_{\tilde{b}}(\alpha)) \\
&= (A_{\tilde{a}}(\alpha), A_{\tilde{0}}(\alpha)) + (A_{\tilde{b}}(\alpha), A_{\tilde{0}}(\alpha)) \cdot A_{\bar{i}}(\alpha) \\
&= A_{\tilde{a}}(\alpha) + A_{\tilde{b}}(\alpha) \cdot A_{\bar{i}}(\alpha) \\
&= A_{\tilde{a}+\tilde{b}}(\alpha) \cdot A_{\bar{i}}(\alpha) \\
&= A_{\tilde{a}+\tilde{b}\cdot\bar{i}}(\alpha)
\end{aligned}$$

Definition 3.5 *Let $\bar{z} = \tilde{a} + \tilde{b}\bar{i}$ be a gradual complex number defined by the assignment function $A_{\bar{z}}(\alpha) = (A_{\tilde{a}} + A_{\tilde{b}} \cdot A_{\bar{i}})(\alpha)$. The* ***conjugate*** *of $\bar{z}$ is the gradual complex number $\bar{z'}$ defined by the assignment function $A_{\bar{z'}}(\alpha)$, where for all $\alpha \in (0,1]$,*

$$A_{\bar{z'}}(\alpha) = A_{\tilde{a}}(\alpha) - A_{\tilde{b}}(\alpha) \cdot A_{\bar{i}} \qquad (3)$$

It is not difficult to conclude that the proposed gradual complex numbers retrieves all the usual complex algebra, e.g. division may be obtained using the conjugate concept.

3.3. Trigonometric form of a gradual complex number

Consider $\bar{z} = \tilde{a} + \bar{i}\tilde{b}$ defined by the assignment function $A_{\bar{z}}(\alpha) = A_{\tilde{a}}(\alpha) + A_{\bar{i}}(\alpha) \cdot A_{\tilde{b}}(\alpha)$. Note that for each degree α, the assignment function $A_{\bar{z}}$ associates one complex number (a, b) and hence a point $P_\alpha = (a, b)$ in the Argand-Gauss plane. Thus, each α establishes a module ρ_α and an angle θ_α. So we have functions: $\theta, \rho : (0, 1] \to \mathbb{R}$, where $\rho(\alpha) = \rho_\alpha = \sqrt{A^2_{\tilde{a}}(\alpha) + A^2_{\tilde{b}}(\alpha)}$. Thus it can be perceived that there is a kind of distance between gradual complex numbers which is reflected here.

Definition 3.6 *Given two gradual complex numbers $\bar{z}_1, \bar{z}_2 : (0, 1] \to \mathbb{C}$ and a metric $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, the* ***distance between gradual complex numbers*** *$\bar{z}_1$* ***and*** *$\bar{z}_2$* ***based on*** *d is the function $\bar{d} : \bar{\mathbb{C}} \times \bar{\mathbb{C}} \to \tilde{\mathbb{R}}$, where $\bar{d}(\bar{z}_1, \bar{z}_2)(\alpha) = d(\bar{z}_1(\alpha), \bar{z}_2(\alpha))$.*

Proposition 3.5 *: For all $\bar{z}, \bar{z}_1, \bar{z}_2 \in \bar{\mathbb{C}}$,*

1. *$\bar{d}(\bar{z}, \bar{z}) = \tilde{0}$;*
2. *$\bar{d}(\bar{z}_1, \bar{z}_2) = \bar{d}(\bar{z}_2, \bar{z}_1)$;*
3. *$\bar{d}(\bar{z}_1, \bar{z}_3) \leq \bar{d}(\bar{z}_1, \bar{z}_2) + \bar{d}(\bar{z}_2, \bar{z}_3)$*

Proof

1. $\bar{d}(\bar{z}, \bar{z})(\alpha) = d(\bar{z}(\alpha), \bar{z}(\alpha)) = 0 = \tilde{0}(\alpha)$.
2. $\bar{d}(\bar{z}_1, \bar{z}_2)(\alpha) = d(\bar{z}_1(\alpha), \bar{z}_2(\alpha)) = d(\bar{z}_2(\alpha), \bar{z}_1(\alpha)) \overset{def}{=} \bar{d}(\bar{z}_2, \bar{z}_1)(\alpha)$
3. $\bar{d}(\bar{z}_1, \bar{z}_3)(\alpha) = d(\bar{z}_1(\alpha), \bar{z}_3(\alpha)) \leq d(\bar{z}_1(\alpha), \bar{z}_2(\alpha)) + d(\bar{z}_2(\alpha), \bar{z}_3(\alpha)) \overset{def}{=} \bar{d}(\bar{z}_1, \bar{z}_2)(\alpha) + \bar{d}(\bar{z}_2, \bar{z}_3)(\alpha)$

Observe that a distance between two gradual complex numbers is a gradual real number and it has the same properties of a metric, except that its value is a function.

In what follows, we will see the module definition ρ_α depending on the distance $\bar{d}$.

Definition 3.7 [Modulus, Angle, Sine and Cosine] *Let $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be the Euclidean metric, $d[(x, y), (u, v)] = \sqrt{(x - u)^2 + (y - v)^2}$, the* ***modulus of a gradual complex number*** *$\bar{z} = (\tilde{a}, \tilde{b})$ is the function $\rho : \bar{\mathbb{C}} \to \tilde{\mathbb{R}}$, s.t. $\rho(\bar{z}) = \bar{d}(\bar{z}, \bar{0})$. The* ***argument or phase of a gradual complex number*** *$\bar{z} = \tilde{a} + \tilde{b} \cdot \bar{i}$ is the function $\theta : \bar{\mathbb{C}} \to \tilde{\mathbb{R}}$ such that: $\theta(\bar{z})(\alpha) = \arccos(\frac{A_{\tilde{a}}(\alpha)}{\rho(\alpha)})$.*

The ***cosine of the angle*** *θ* ***of a gradual complex number*** *$\bar{z}$ is the function $\overline{\cos} : (0, 1] \to [-1, 1]$ such that $\overline{\cos}(\alpha) = \frac{A_{\tilde{a}}(\alpha)}{\rho(\alpha)}$. Similarly, one can obtain the function $\overline{\sin} : (0, 1] \to [-1, 1]$, such that $\overline{\sin}(\alpha) = \frac{A_{\tilde{b}}(\alpha)}{\rho(\alpha)}$.*

Proposition 3.6 *All gradual complex number $\bar{z}$ defined by the assignment function $A_{\bar{z}}(\alpha) = (A_{\tilde{a}}(\alpha), A_{\tilde{b}}(\alpha))$, where for all $\alpha \in (0, 1]$, $A_{\tilde{a}}(\alpha) \neq 0$ or $A_{\tilde{b}}(\alpha) \neq 0$, can be represented by the assignment function*

$$A_{\bar{z}}(\alpha) = \rho(\alpha) \cdot (\overline{\cos}(\alpha) + \bar{i} \cdot \overline{\sin}(\alpha)) \quad (4)$$

Called ***trigonometric form ou polar form*** *of a gradual complex number.*

Proof: Straightforward.

Proposition 3.7 *For all gradual number $\bar{z}$ defined by an assignment function $A_{\bar{z}}(\alpha) = \rho(\alpha) \cdot (\overline{\cos}(\alpha) + \bar{i} \cdot \overline{\sin}(\alpha))$, can get their algebraic form.*

Proof:
Given $\bar{z}$ with its assignment function form $A_{\bar{z}}(\alpha) = \rho(\alpha) \cdot (\overline{\cos}(\alpha) + \bar{i} \cdot \overline{\sin}(\alpha))$, where, for all $\alpha \in (0, 1]$, $\overline{\cos}(\alpha)$ and $\overline{\sin}(\alpha)$ are numbers of the form $\frac{A_{\tilde{a}}(\alpha)}{\rho(\alpha)}$ and $\frac{A_{\tilde{b}}(\alpha)}{\rho(\alpha)}$, respectively. Then, $A_{\bar{z}}(\alpha) = \rho(\alpha) \cdot (\overline{\cos}(\alpha) + \bar{i} \cdot \overline{\sin}(\alpha)) = \rho(\alpha) \cdot (\frac{A_{\tilde{a}}(\alpha)}{\rho(\alpha)} + \bar{i} \cdot \frac{A_{\tilde{b}}(\alpha)}{\rho(\alpha)}) = A_{\tilde{a}}(\alpha) + \bar{i} \cdot A_{\tilde{b}}(\alpha) = (A_{\tilde{a}} + \bar{i} \cdot A_{\tilde{b}})(\alpha)$.

QED

Example 3.1 *Given $\tilde{a} = \{0.3/1, 0.5/2, 0.6/-1, 0.8/2, 1/2\}$ and $\tilde{b} = \{0.3/1,\ 0.5/1,\ 0.6/\sqrt{3},\ 0.8/2,\ 1/0\}$, one gradual complex $\bar{z} = (\tilde{a}, \tilde{b})$ it will be $\bar{z} = \{0.3/(1, 1),\ 0.5/(2, 1),\ 0.6/(-1, \sqrt{3}),\ 0.8/(2, 2),\ 1/(2, 0)\}$ can be represented algebraically as $\bar{z} = \{0.3/1 + \boldsymbol{i}, 0.5/2 + \boldsymbol{i}, 0.6/-1 + \sqrt{3}\boldsymbol{i}, 0.8/2 + 2\boldsymbol{i}, 1/2\}$ and the polar representation could be as follows: For $\alpha = 0.3$,*

$$\begin{aligned} \rho(0.3) &= \sqrt{A_{\tilde{a}^2}(0.3) + A_{\tilde{b}^2}(0.3)} = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \bar{\cos}(0.3) &= \frac{A_{\tilde{a}}(0.3)}{\rho(0.3)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ \bar{\sin}(0.3) &= \frac{A_{\tilde{b}}(0.3)}{\rho(0.3)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \end{aligned} \quad (5)$$

Finally, to the extent 0.3 the polar representation of $\bar{z}$ it would be $A_{\bar{z}}(0.3) = \rho(0.3) \cdot (\cos\theta(0.3) + \boldsymbol{i}\sin\theta(0.3)) = \sqrt{2} \cdot (\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4})$. Similarly, one can obtain the triginomÃ©trica representation for all α.

3.4. Fuzzy number represented by gradual complex numbers

In Fortin et. al. [4] it is shown how an ordered pair of real numbers can be seen as a fuzzy number. However, as a pair of gradual real is nothing more than a gradual complex number, this representation can be seen the following definition.

Definition 3.8 *let $\bar{z}$ be a gradual complex number. To describe a fuzzy number $F_{\bar{z}}$ as a gradual complex number $\bar{z}$, it is necessary that*

1. *$Re_{\bar{z}}$ and $Im_{\bar{z}}$ be continuous function;*
2. *$Re_{\bar{z}}$ and $Im_{\bar{z}}$ be increasing and decreasing, respectively;*
3. *$Re_{\bar{z}}(\alpha) \leq Im_{\bar{z}}(\alpha)$, for all $0 < \alpha \leq 1$.*

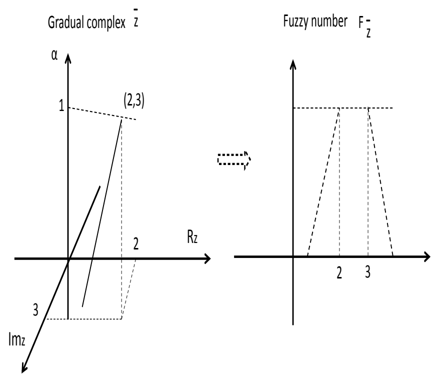

Figure 1: Fuzzy real built from gradual complex

thus, the membership function of $F_{\bar{z}}$ *is defined as:*

$$\mu_{F_{\bar{z}}}(x) = \begin{cases} Sup\{\alpha/R_{\bar{z}}(\alpha) \leq x\}, \ \textit{if } x \in ran(R_{\bar{z}})^2 \\ 1, \ \textit{if } R_{\bar{z}}(1) \leq x \leq Im_{\bar{z}}(1) \\ Sup\{\alpha/Im_{\bar{z}}(\alpha) \geq x\}, \ \textit{if } x \in ran(Im_{\bar{z}}) \\ 0, \ \textit{otherwise} \end{cases} \tag{6}$$

Remember that, graphically, a gradual complex number is a straight line in the plane $\mathbb{R}^3$. For numbers which satisfy the definition 3.8, the real and imaginary part of gradual complex can be see as the lower and upper limit of the fuzzy number represented. In other words, for the fuzzy number $F_{\bar{z}}$, saying that $A_{\bar{z}}(0.5) = (2, 3)$ means the same that $[2, 3]$ being a closed interval containing all the elements of $\mathbb{R}$ having degrees of membership larger or equal than 0.5, so $A_{\bar{z}}(\alpha) = (x, y)$, whereupon $[x, y]^\alpha$ is called the interval α-curt of $F_{\bar{z}}$. Also note that, if $A_{\bar{z}}(1) = (x, x)$, this means that the interval α-curt of $F_{\bar{z}}$, whenever $\alpha = 1$, is the interval $[x, x]$ and the fuzzy number represented by $\bar{z}$ will be a triangular fuzzy number. Otherwise, the fuzzy number represented will be a trapezoidal fuzzy number. In figure 1 this representation can be observed.

In the next section we will see how gradual complexes can be characterized in terms of a fuzzy complex number as well as how Zhang fuzzy complex numbers can be characterized in terms of gradual complex numbers.

3.5. Gradual complex numbers and fuzzy complex numbers

The fuzzy complex numbers were introduced by Buckley [1, 2]. However, Dong et. al. in [9] points out some errors in Lemma and theorem founds in the early work produced by Buckley. Posteriorly, new approaches have emerged in literature, as can be seen in [9], [6], [2], [8] and [5] to redefine a fuzzy complex number.

Most of cited works define a fuzzy complex number as a membership function where each element of the set of complex number $\mathbb{C}$ is mapped to a value in the interval $[0, 1]$. However, according to Vladik et. al., in [8], a fuzzy complex number may be characterized not only as a membership function of real value, but by a family of membership functions which map the set of real numbers in the interval $[0, 1]$ and where each function describes different characteristics of these numbers, such as its real part, its imaginary part, its absolute value, etc. Also thinking about representing a fuzzy complex number through membership functions which map the set of real numbers in the interval $[0, 1]$, Zhang et. al. in [5], show that as a complex number is a pair of real numbers, we can represent a fuzzy complex number as an ordered pair of fuzzy numbers. Therefore, this new approach doesn't need to stablish validation criteria of the membership function which defines a fuzzy complex number, since the definition of a fuzzy complex number is sustained by the definition of the fuzzy number. In accordance with this, the study about complex numbers held here will be fully based on this approach.

According to Zang et. al. [5] a fuzzy complex number is defined as follows:

Considering $\tilde{\mathbb{R}}$ the set of all fuzzy numbers.

Definition 3.9 *([5]): let* $\tilde{A}$ *and* $\tilde{B}$ *be two fuzzy numbers belonging to* $\tilde{\mathbb{R}}$*. A* ***fuzzy complex number*** $\hat{C}$ *is a fuzzy set defined by the following function*

$$\begin{aligned} \rho_{\hat{C}}(x) : \mathbb{R} \times \mathbb{R} &\rightarrow [0, 1] \\ (a, b) &\rightarrow min(\mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)) \end{aligned} \tag{7}$$

where $\mu_{\tilde{A}}$ *and* $\mu_{\tilde{B}}$ *are membership functions which represent two fuzzy numbers* $\tilde{A}$ *and* $\tilde{B}$*, respectively.* $\tilde{A}$ *is the real part of* $\tilde{C}$*, denoted by* $Re(\tilde{C}) = \tilde{A}$*, and* $\tilde{B}$ *is the imaginary part of* $\tilde{C}$*, denoted by* $Im(\tilde{C}) = \tilde{B}$*. Thus, we can represent a fuzzy complex number* $\hat{C}$ *by a ordered pair of fuzzy numbers* $(\tilde{A}, \tilde{B})$*.*

In the following it will be shown how a fuzzy complex number can be obtained starting from gradual complex numbers.

Before proceeding, note that the definition 3.8 has been seen that it is possible get a fuzzy number starting from gradual complex number. As a fuzzy complex number is a pair of fuzzy real numbers, in order to get them starting from gradual, it will be required a pair of gradual complex numbers, since a gradual complex number describes a fuzzy number. Following this reasoning, the definition following shows how to obtain the membership function of a fuzzy complex number built starting from one pair of gradual complex numbers.

Definition 3.10 *let* $\bar{z}$ *and* $\bar{w}$ *be gradual complex numbers that satisfy the definition 3.8. The membership function* $\mu_{\hat{C}_{(\bar{z},\bar{w})}}$ *of fuzzy complex number*

$\hat{C}_{(\bar{z},\bar{w})}$ *built from a pair* $(\bar{z},\bar{w})$ *is defined by function* $\mu_{\hat{C}_{(\bar{z},\bar{w})}}(x,y) =$

$$\begin{cases} min(Sup^{-}(Re_{\bar{z}}(x)), Sup^{-}(Re_{\bar{w}}(y))), if\ x \in ran(Re_{\bar{z}})\ and\ y \in ran(\\ min(Sup^{-}(Re_{\bar{z}}(x)), Sup^{+}(Im_{\bar{w}}(y))), if\ x \in ran(Re_{\bar{z}})\ and\ y \in ran(\\ min(Sup^{+}(Im_{\bar{z}}(y)), Sup^{-}(Re_{\bar{w}}(y))), if\ x \in ran(Im_{\bar{z}})\ and\ y \in ran \\ min(Sup^{+}(Im_{\bar{z}}(x), Sup^{+}(Im_{\bar{w}}(y))), if\ x \in ran(Im_{\bar{z}})\ and\ y \in ran(\\ 1,\ if\ Re_{\bar{z}}(1), \leq x \leq Im_{\bar{z}}(1)\ and\ Re_{\bar{w}}(1) \leq y \leq Im_{\bar{w}}(1) \\ Sup^{-}(Re_{\bar{z}}(x)), if\ x \in ran(Re_{\bar{z}})\ and\ Re_{\bar{w}}(1) \leq y \leq Im_{\bar{w}}(1) \\ Sup^{+}(Im_{\bar{z}}(y)), if\ x \in ran(Im_{\bar{z}})\ and\ Re_{\bar{w}}(1) \leq y \leq Im_{\bar{w}}(1) \\ Sup^{-}(Re_{\bar{w}}(y)), if\ Re_{\bar{z}}(1) \leq x \leq Im_{\bar{z}}(1)\ and\ y \in ran(Re_{\bar{w}}) \\ Sup^{+}(Im_{\bar{w}}(y)), if\ Re_{\bar{z}}(1), \leq x \leq Im_{\bar{z}}(1)\ and\ y \in ran(Im_{\bar{w}}) \\ 0,\ otherwise \end{cases} \quad (8)$$

Considering $Sup^{-}(Re_{\bar{z}}(x)) = Sup\{\alpha, Re_{\bar{z}}(\alpha) \leq x\}$ *and* $Sup^{+}(Im_{\bar{z}}(x)) = Sup\{\alpha, Im_{\bar{z}}(\alpha) \geq x\}$ [3] *the same is true for* $Sup^{-}(Re_{\bar{w}})$ *and* $Sup^{+}(Im_{\bar{w}})$ [4].

Note that the above definition requires that the pair of gradual complex numbers $(\bar{z},\bar{w})$ satisfy the definition 3.8. In other words, the definition requires that it is possible to build two fuzzy real numbers $\tilde{F}_{\bar{z}}$ and $\tilde{F}_{\bar{w}}$ from $\bar{z}$ and $\bar{w}$, respectively. Therefore, the membership function given above is, actually, a combination of membership functions $\mu_{\tilde{F}_{\bar{z}}}$ and $\mu_{\tilde{F}_{\bar{w}}}$ obtained from the equation 6, in which the minimum operator in each of these combinations is applied. This means that the fuzzy complex number generated through this membership function is a complex of Zang, as it can be seen in the following proposition

Proposition 3.8 *Consider a pair of gradual complex number* $(\bar{z},\bar{w})$*, that satisfy the definition above, and* $\mu_{\hat{C}_{(\bar{z},\bar{w})}}$ *the membership function of a fuzzy complex number built from* $(\bar{z},\bar{w})$*. All fuzzy complex numbers generated through of* $\mu_{\hat{C}_{(\bar{z},\bar{w})}}$ *is a fuzzy complex number of Zang* $\mu_{\hat{Z}}(x,y)$ *and can be seen as the following form.*

$$\mu_{\tilde{C}_{(\bar{z},\bar{w})}} = \min(\mu_{\tilde{F}_{\bar{z}}}(x), \mu_{\tilde{F}_{\bar{w}}}(y)) = \mu_{\hat{Z}}(x,y)$$

where $\mu_{\tilde{F}_{\bar{z}}}(x)$ and $\mu_{\tilde{F}_{\bar{w}}}(y)$ are membership functions of the fuzzy real numbers $\tilde{F}_{\bar{z}}$ and $\tilde{F}_{\bar{w}}$ given in the equation 6.

proof

Given a pair of gradual complex numbers $(\bar{z},\bar{w})$, where each one satisfies the definition 3.8. According to definition 3.10, it is possible to obtain a fuzzy complex number $\hat{C}_{(\bar{z},\bar{w})}$ through of the membership function $\mu_{\hat{C}_{(\bar{z},\bar{w})}}$ given in the definition. However, note that if $(\bar{z},\bar{w})$ satisfies the definition 3.8 it is possible to obtain the fuzzy real numbers $\tilde{F}_{\bar{z}}$ and $\tilde{F}_{\bar{w}}$ through the functions $\mu_{\tilde{F}_{\bar{z}}}$ and $\mu_{\tilde{F}_{\bar{z}}}$, respectively.

According to definition 3.9 , given a pair of fuzzy real numbers $(\tilde{A},\tilde{B})$, with their respective membership functions $(\mu_{\tilde{A}},\mu_{\tilde{B}})$, a fuzzy complex number $\hat{Z}$ is defined as: $\mu_{\hat{Z}}(x,y) = \min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(y))$. Now, consider $\tilde{A} = \tilde{F}_{\bar{z}}$ and $\tilde{B} = \tilde{F}_{\bar{w}}$, thus, by definition, we have that $\mu_{\hat{Z}}(x,y) = \min(\mu_{\tilde{F}_{\bar{z}}}(x),\mu_{\tilde{F}_{\bar{w}}}(y))$.

However, obseerve that $\min(\mu_{\tilde{F}_{\bar{z}}}(x),\mu_{\tilde{F}_{\bar{w}}}(y))$ is given by the function 8.

Thus, $\min(\mu_{\tilde{F}_{\bar{z}}}(x),\mu_{\tilde{F}_{\bar{w}}}(y)) = \mu_{\hat{C}_{(\bar{z},\bar{w})}}(x,y) = \mu_{\hat{Z}}(x,y)$ and, therefore, the fuzzy complex described by the membership function $\mu_{\hat{C}_{(\bar{z},\bar{w})}}(x,y)$ given in the definition 3.10 is a fuzzy complex number of Zang and can be represented of the form $\mu_{\hat{C}_{(\bar{z},\bar{w})}}(x,y) = \min(\mu_{\tilde{F}_{\bar{z}}}(x),\mu_{\tilde{F}_{\bar{w}}}(y)) = \mu_{\hat{Z}}(x,y) =$.

QED

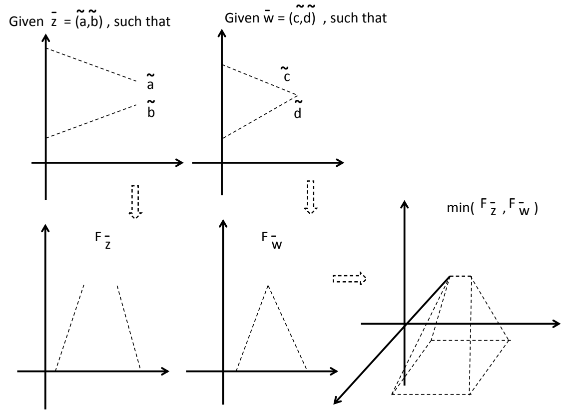

Figure 2: Fuzzy complex built from gradual complex

In the figure 2, it can be seen how to obtain a fuzzy complex number of Zang from a pair of gradual complex numbers.

Thus, as it's possible to obtain a fuzzy complex number starting from of the gradual, it is also possible to describe in the gradual the representation of a fuzzy complex number. In other words, it is possible, given a fuzzy complex number, to obtain a set of gradual complexes that represents them, as the following definition shows.

Definition 3.11 *Given a fuzzy complex number of Zang* $\hat{Z} = (\tilde{A},\tilde{B})$*, with* $\tilde{A}$ *and* $\tilde{B}$ *as triangular or trapeizodal fuzzy numbers. Thus, the set of gradual complex numbers* $G_{\hat{Z}}$ *that represents the fuzzy complex* $\hat{Z}$ *in the gradual, is defined by the following assignment function:*

$$A_{G_{\hat{Z}}}(\alpha) = \{(x,y)|R_{\hat{Z}}(x) = \alpha\ e\ Im_{\hat{Z}}(y) = \alpha\}. \quad (9)$$

According to [5], given a fuzzy complex number $\hat{Z}$, we have that the set α-curt of $\hat{Z}$ as a closed rectangular region. Thus, the assignment function above associates for each α four points, that give rise to four straight lines in the plane $\mathbb{R}^3$, that not necessarily coincide when $\alpha = 1$. Graphically explaining, those four segments of lines represent the " structure ' of a fuzzy complex number of Zang. However, the fuzzy complex number of

[3] Re_z is the real part of a gradual complex number z

[4] Im_z is the imaginary part of a gradual complex number z

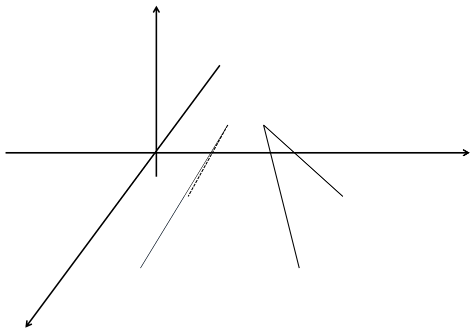

Figure 3: Gradual complex set representing a fuzzy complex of Zhang

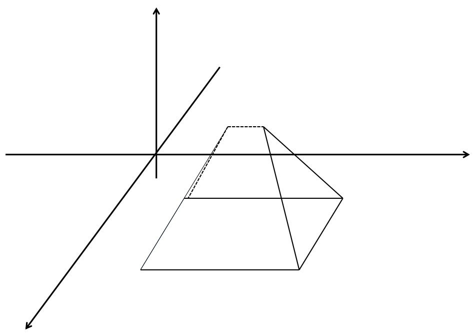

Figure 4: Fuzzy complex number obtained from a pair of gradual complexes

Zang is all the content limited by four straight lines that set the gradual complex numbers. In the figures 3 and 4 below, the difference can be seen.

4. Final Remarks

In this paper we developed the idea of Gradual Complex Numbers providing its algebraic and polar forms. Furthermore, we've shown the relation between gradual complex numbers and fuzzy numbers. Finaly, was presented how we can build a fuzzy complex number starting from the notion of gradual and how we can obtain the representation of a fuzzy complex number through a set of gradual complex numbers.

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