

# Fuzzy Community detection based on grouping and overlapping functions

Daniel Gómez<sup>1</sup> J. Tinguaro Rodríguez<sup>2</sup> Javier Montero<sup>3</sup> Javier Yáñez<sup>2</sup>

<sup>1</sup>Facultad de Estudios Estadísticos, Complutense University, Madrid, Spain.

<sup>2</sup>Facultad de Matemáticas, Complutense University, Madrid, Spain.

<sup>3</sup>Instituto IGEO-UCM (CISC-UCM), Facultad de Matemáticas, Complutense University, Madrid, Spain.

## Abstract

One of the main challenges of fuzzy community detection problems is to be able to measure the quality of a fuzzy partition. In this paper, we present an alternative way of measure the quality of a fuzzy community detection output based on n-dimensional grouping and overlapping functions that generalize the classical modularity for crisp community detection problems and also for crisp overlapping community detection problems.

**Keywords:** Overlaps functions, Grouping functions, Community detection.

## 1. Introduction.

Aggregation is a basic and necessary tool for most knowledge based systems. An *aggregation operator* [1, 6, 7, 8, 9, 10, 16] is usually defined as a real function  $A_n$  that, from  $n$  data items  $x_1, \dots, x_n$  in  $[0, 1]$ , produces an aggregated value  $A_n(x_1, \dots, x_n)$  in  $[0, 1]$  [5, 10]. Usually, some desirable properties any aggregation operator should satisfy are assumed (see [5, 10] for more details): these are some boundary conditions (for all  $n$ ,  $A_n(0, \dots, 0) = 0$  and  $A_n(1, \dots, 1) = 1$ ), as well as monotonicity and continuity in each variable. Also, other properties could be imposed as those that are studied in [4, 14, 26, 27, 29].

The concept of overlap as a bivariate aggregation operator was introduced in [2] to measure the degree of overlap of an object in a fuzzy classification system with two classes. This concept has been applied to some interesting situations, in which it is necessary to know the degree of overlap of objects in two-class classification systems, as the image segmentation problem described in [19] (in which it is necessary to discriminate between object and background) or in the framework of preference relations [3].

Obviously, there are situations in which we need to measure the degree of overlapping of an object in a fuzzy classification system with more than two classes. So with the aim to generalize this concept, in [13], the overlap concept was generalized into a  $n$ -dimensional framework. With this generalization of the overlap function it is possible to analyze most relevant properties and applications.

In this work, it is proposed an application of the  $n$ -dimensional overlaps and grouping functions to community detection problems into a fuzzy framework.

Large and complex networks representing relationships among set of entities have been one of the focuses of interest of scientists in many fields in the recent years. Various complex network examples include social network, worldwide web network, telecommunication network and biological network. One of the most important problems in social network analysis is to *describe/explain its community structure*. Generally, a community in a network is a subgraph whose nodes are densely connected within itself but sparsely connected with the rest of the network.

Community detection problems has been widely studied during the last decade (see [11, 15]) with many applications to several disciplines. Discovering inherent communities and structures in a social network must be a main objective when we pursue a better understanding of a given network. Nevertheless, real communities in complex network, often present overlap, such that each vertex may occur in more than one community. Community detection problems with overlapping communities has been also studied in literature (see [31]) with different proposes. One one hand, one of the aims of this problems is to know the communities allowing some key nodes to belong to more than one community. One of the other hand, the other (and related) aim is to detect and identify those nodes (usually addressed as *overlapping nodes*) that belong to more than one community. Overlapping nodes may play a special role in a complex network system and it is a very interesting issue how to detect them. Most known algorithms such as divisive algorithm [12] or agglomerative [11] cannot detect them.

As it is pointed in [18], two distinct types of overlapping are possible: crisp (where each node belongs fully to each community of which it is a member) and fuzzy (where each node belongs to each community to a different extent). So taking into account this classification of overlapping community detection problems, we have tree possibilities: classical community detection problems (in which non overlapping communities are allowed), overlapping community detection problems (in which a node could

belong to more than one community) and fuzzy community detection problems (in which each node has a degree of membership of each community). There are two main challenges in fuzzy community detection, one of them is the develop of algorithms that produce a fuzzy clustering of the nodes in the network. And the other, is to quantify the quality of this performance.

In this paper, we present an alternative way of measure the quality of a fuzzy community detection output based on n-dimensional grouping and overlapping functions that generalize the classical modularity for crisp community detection problems and also for crisp overlapping community detection problems.

## 2. Preliminaries

In this section, we recall some concepts and properties of bivariate and n-dimensional overlap and grouping functions, which were initially proposed in [2, 19], and that were extended to the n-dimensional case in [13].

### 2.1. Bivariate overlap and grouping functions

The definition of an overlap function and some basic results about it were presented in [2, 19]. Particularly, an overlap function is defined as a particular type of bivariate aggregation function characterized by a set of symmetry, natural boundary and monotonicity properties.

#### Definition 2.1.

$$G_O : [0, 1]^2 \longrightarrow [0, 1]$$

is an overlap function if and only if the following holds:

1.  $G_O$  is symmetric.
2.  $G_O(x, y) = 0$  if and only if  $xy = 0$ .
3.  $G_O(x, y) = 1$  if and only if  $x = 1$  and  $y = 1$ .
4.  $G_O$  is nondecreasing.
5.  $G_O$  is continuous.

Let us observe (as it is showed in [2, 19, 3]), that overlaps functions are closely related with t-norms but present some differences since the associative property is not required for the former. In the following example, we can see an instance of an aggregation function that is an overlapping function, but not a t-norm if  $p > 1$ .

**Example 2.1.** It is easy to see that the bivariate aggregation function  $G_p(x, y) = (\min\{x, y\})^p$  is an overlapping function, since the properties (1)-(5) are satisfied. But let us also note that, when  $p > 1$ , the bivariate function  $G_p$  is not associative, and thus it is not a t-norm.

Let us now recall in the notion of grouping function, proposed in [2, 19] as a natural complement of an overlap function. Given two degrees of membership  $x = \mu_A(c)$  and  $y = \mu_B(c)$  of an object  $c$  into classes  $A$  and  $B$ , a grouping function is supposed to yield the degree  $z$  up to which the combination (grouping) of the two classes  $A$  and  $B$  is supported, that is, the degree up to which either  $A$  or  $B$  (or both) hold.

**Definition 2.2.** A grouping function is a function

$$G_G : [0, 1]^2 \longrightarrow [0, 1]$$

that satisfies the following conditions:

1.  $G_G$  is symmetric.
2.  $G_G(x, y) = 0$  if and only if  $x = y = 0$ .
3.  $G_G(x, y) = 1$  if and only if  $x = 1$  and  $y = 1$ .
4.  $G_G$  is non-decreasing.
5.  $G_G$  is continuous.

### 2.2. n-dimensional overlap functions

In [13], previous ideas presented for two sets or classes were extended into a more general case. Sometimes, an object may belong to more than two classes, and thus it could be interesting to measure the degree of overlap of this object with respect to the classification system given by the available classes.

**Definition 2.3.** An n-dimensional aggregation function  $G_O : [0, 1]^n \longrightarrow [0, 1]$  is an n-dimensional overlap function if and only if:

1.  $G_O$  is symmetric.
2.  $G_O(x_1, \dots, x_n) = 0$  if and only if  $\prod_{i=1}^n x_i = 0$ .
3.  $G_O(x_1, \dots, x_n) = 1$  if and only if  $x_i = 1$  for all  $i \in \{1, \dots, n\}$ .
4.  $G_O$  is increasing.
5.  $G_O$  is continuous.

In a similar way, the grouping concept can be also extended into a more general framework, Given  $n$  degrees of membership  $x_i = \mu_{C_i}(c)$  for  $i = 1, \dots, n$  of an object  $c$  into classes  $C_1, \dots, C_n$ , a grouping function is supposed to yield the degree  $z$  up to which the combination (grouping) of the  $n$  classes  $C_1, \dots, C_n$  is supported.

**Definition 2.4.** An n-dimensional function

$$G_G : [0, 1]^n \longrightarrow [0, 1]$$

is an n-dimensional grouping function if and only if it satisfies the following conditions:

1.  $G_G$  is symmetric.
2.  $G_G(x) = 0$  if and only if  $x_i = 0$ , for all  $i = 1, \dots, n$ .
3.  $G_G(x) = 1$  if and only if there exist  $i \in \{1, \dots, n\}$  with  $x_i = 1$ .

4.  $G_G$  is non-decreasing.
5.  $G_G$  is continuous.

Again, continuous t-conorms (their  $n$ -ary forms) and their convex combinations are prototypical examples of  $n$ -ary grouping functions.

**Example 2.2.** The following aggregation functions are examples of  $n$ -dimensional grouping functions:

- The maximum powered by  $p$ .  $G_G(x_1, \dots, x_n) = \max_{1 \leq i \leq n} \{x_i^p\}$  with  $p > 0$ .
- The Einstein sum aggregation operator.  

$$ES(x_1, \dots, x_n) = \frac{\sum_{i=1}^n x_i}{1 + \prod_{i=1}^n (x_i)}$$

### 3. Modularity measure in fuzzy community detection problems

Modularity is one of the most used measures to represent the quality of a partition in graph and is the key stone in many community detection algorithms. For unsupervised clustering algorithms, can be used to determine the optimal number of communities or can be used to compare the performance among several algorithms. This measure was initially defined [12] for crisp partitions and crisp graphs as follow: Given a partition  $\mathcal{C}$  of a network  $G = (V, E)$ , the modularity is defined as:

$$Q_{GN} = \frac{1}{2m} \sum_{i,j \in V} \left[ A(i, j) - \frac{k_i k_j}{2m} \right] \delta(c_i c_j), \quad (1)$$

where  $m$  is the number of edges in the graph,  $k_i$  is the degree of node  $i$ ,  $A_{ij}$  is the adjacency matrix of the graph and  $\delta(c_i c_j)$  is equal to 1 if nodes  $i$  and  $j$  belong to the same cluster and 0 otherwise. The modularity of a partition represents the fraction of edges that fall within the given groups minus the expected such fraction if edges were distributed at random.

**Remark 1.** Let us observe that this definition allows to measure the performance of a crisp clustering of a graph with overlapping communities (i.e. a node can belong to more than one class).

In a fuzzy framework, it is important to note that there are few methods that produce a fuzzy partitions of a network (see for example [31]). Taking into account this, few efforts has been dedicated to extend the crisp modularity measure into a more general scenario. Now we will give a very short review of the different extensions of modularity measure in a fuzzy scenario (see [11] for more details).

In [32] it is presented one of the first definition that permits to measure one the modularity of a fuzzy network partition. In that paper, the authors propose a fuzzy modularity measure based on an  $\alpha$  value in the following sense.

**Definition 3.1.** Given a fuzzy classification of a set of nodes  $V$  in  $C$  communities (i.e.  $\mu_c(i)$  for all  $c \in \{1, \dots, C\}$  and  $i \in V$ ). The crisp community  $V_c$  is defined as  $V_c = \{i \in V / \mu_c(i) \geq \alpha\}$ . Taking into account this the fuzzy modularity is defined as:

$$Q_{Zang}(\alpha) = \sum_{c=1}^C \left[ \sum_{i,j \in V_c} \frac{((\mu_c(i) + \mu_c(j))/2) A_{ij}}{2m} - \dots \right] \left[ \dots \left( \sum_{i \in V_c, j \notin V_c} \frac{((\mu_c(i) + 1 - \mu_c(j))/2) A_{ij}}{2m} \right)^2 \right]$$

Previous definition presents some problems since the modularity of a fuzzy partition depends on an  $\alpha$  value, so for each  $\alpha$  we have a different modularity measure. In [21], it is presented an alternative definition of fuzzy modularity which not depends on  $\alpha$  value.

**Definition 3.2.** Given a fuzzy classification of a set of nodes  $V$  in  $C$  communities (i.e.  $\mu_c(i)$  for all  $c \in \{1, \dots, C\}$  and  $i \in V$ ). The crisp community  $V_c$  is defined as  $V_c = \{i \in V / \mu_c(i) = \max\{\mu_k(i) \mid 1 \leq k \leq C\}\}$ . Taking into account this the fuzzy modularity is defined as:

$$Q_{Liu} = \sum_{c=1}^C \left[ \sum_{i,j \in V_c} \frac{((\mu_c(i) + \mu_c(j))/2) A_{ij}}{2m} - \dots \right] \left[ \dots - \left( \sum_{i \in V_c, j \notin V_c} \frac{((\mu_c(i) + 1 - \mu_c(j))/2) A_{ij}}{2m} \right)^2 \right]$$

Obviously, the main difference between  $Q_{Liu}$  and  $Q_{Zang}$  is the definition of  $V_c$ . Let us observe that if the partition is a Ruspini partition (i.e. for all  $i \in V$ ,  $\sum_{c \in C} \mu_c(i) = 1$ ) is possible to find  $\alpha$  values in which both measures coincides.

The third well-known generalization of the crisp modularity in a fuzzy framework is given in [23]. In that paper, the Kronecker delta  $\delta(c_i c_j)$  that appears in the classical formula (defined in [12]), is replaced by  $s_{i,j}$ ,  $s_{ij}$  being the sum of the products of the belonging coefficients of  $i$  and  $j$  in communities to which they both belong. Formally, the definition given in [23] can be expressed as follows:

$$Q_{NE} = \frac{1}{2m} \sum_{i,j \in V} \left[ A_{ij} - \frac{k_i k_j}{2m} \right] s_{ij}, \quad (2)$$

where  $s_{ij} = \sum_{c \in C} \mu_c(i) \mu_c(j)$ . In previous formula it is imposed that the fuzzy partition of the graph be a Ruspini partition in the sense that  $\sum_{c \in C} \mu_c(i) = 1$ . Imposing the Ruspini condition it is guarantee that  $s_{ij}$  belong to the unit interval  $[0, 1]$ .

Although the fuzzy modularity presented in [23] are close to the correct generalization of the modularity measure into a fuzzy scenario, they present some deficiencies. The most important one, is that

it is necessary to impose that the fuzzy partition  $\mu$  of the set of nodes be a Ruspini partition, and thus is not a generalization of the classical modularity measure where there exist overlapping communities. So if we have a crisp or fuzzy clustering in which for one node  $\sum_{c=1}^C \mu_c(i) > 1$ ,  $Q_{NE}$  does not performed well.

In the original definition of Girvan-Newman modularity,  $\delta(c_i c_j)$  represents the true value associated with the assertion "node  $i$  and node  $j$  belong to the same community". In  $Q_{NE}$  this "degree of true of node  $i$  and node  $j$  belong to the same community" is replaced by  $s_{ij} = \sum_{c \in \mathcal{C}} \mu_c(i) \mu_c(j)$  which as a different meaning. As it is pointed in [18], previous modularity measures don't permits overlapping in the sense that  $\sum_{c \in \mathcal{C}} \mu_c(i) > 1$  (in the crisp or fuzzy case) and thus is not a generalization of the crisp  $GN$  modularity measure with overlapping nodes.

Now let us try to quantify the assertion "node  $i$  and node  $j$  belong to the same community" by means of overlapping and global functions. Let us observe that the  $\delta(c_i c_j)$  value of  $GN$  crisp modularity measure takes the value 1 if  $i, j$  both belong to some of the communities of the network  $\{C_1, \dots, C_k\}$ . If we denote by  $C^i = \cup_{l \in C_i} C_l$  and  $C^j$  the set of communities to which nodes  $i$  and  $j$  belong respectively, then  $C^i \cap C^j = \cup_{i,j \in C_l} C_l$  represents the set of communities in which  $i$  and  $j$  belong simultaneously. In a crisp scenario,  $\delta(c_i c_j) = 1$  if and only if  $\cup_{l \in C_i} C_l \neq \emptyset$ . In a fuzzy framework, this union could be represented by a grouping function  $G_G$  and the intersection or the condition of  $i, j \in C_l$  as a overlapping function.

Just to put and example, let  $i$  and  $j$  be two nodes with membership function to three communities  $\mu(i) = (\mu_{C_1}(i), \mu_{C_2}(i), \mu_{C_3}(i)) = (0.9, 1, 0)$  and  $\mu(j) = (\mu_{C_1}(j), \mu_{C_2}(j), \mu_{C_3}(j)) = (0.4, 0.5, 1)$ . The degree of true of the fact that nodes  $i$  and  $j$  belong simultaneously to the community  $C_1$  could be measure as the degree of overlap that this community has over the nodes  $i$  and  $j$ , i.e.  $G_O(\mu_{C_1}(i), \mu_{C_1}(j)) = G_O(0.9, 0.4)$ . So after comparing the degree to which  $i$  and  $j$  belong to the communities  $C_1, C_2$  and  $C_3$ , we have to aggregate this three values into one using a grouping function. Thus

$$s_{ij} = G_G(G_O(\mu_{C_1}(i), \mu_{C_1}(j)), \dots, G_O(\mu_{C_3}(i), \mu_{C_3}(j))) \quad (3)$$

If we take for example the overlapping function  $G_O(x, y) = \min\{x, y\}^{1/2}$  and the global grouping function  $G_G(x_1, \dots, x_n) = \max\{x_i, \dots\}$ , then we have that  $s_{ij} = \max\{0.4^{1/2}, 0.5^{1/2}, 0\} = 0.5^{1/2} = 0.70$ .

Taking into account previous considerations, in the next definition we present an extension of classical  $GN$  modularity to evaluate the performance of a fuzzy classification (not necessarily Ruspini partition) of set of nodes in graph based on grouping and overlapping functions.

**Definition 3.3.** Given a fuzzy partition  $\mathcal{C}$  of a

graph  $(V, E)$  with membership functions  $\mu_c : V \rightarrow [0, 1]$ , for all  $c \in \mathcal{C}$ , the modularity is defined as:

$$\tilde{Q}(\mathcal{C}) = \frac{1}{2m} \sum_{i,j \in V} \left[ A_{ij} - \frac{k_i k_j}{2m} \right] G_G \{ G_O(\mu_c(i), \mu_c(j)) \mid c \in \mathcal{C} \}, \quad (4)$$

where  $A_{ij}$  is the adjacent matrix of the crisp graph,  $m$  is the number of links of this graph,  $k_i$  is the degree of node  $i$  in the graph,  $G_G$  is a global grouping function  $G_G : [0, 1]^{|\mathcal{C}|} \rightarrow [0, 1]$ , and  $G_O$  is a bivariate overlapping function,  $G_O : [0, 1]^2 \rightarrow [0, 1]$ .

Obviously, if the partition  $\mathcal{C}$  is crisp, this definition coincides with the classical modularity definition. Let us observe that our definition permits to measure the performance of a crisp overlapping classification and also a fuzzy overlapping classification in the sense  $\sum_{c \in \mathcal{C}} \mu_c(i) > 1$ . It can be proved easily that our fuzzy modularity measure  $\tilde{Q}$  is a generalization of the classical  $Q_{GN}$  when there exist overlapping communities. And the  $Q_{Zang}$ ,  $Q_{NE}$  cannot provide a suitable measure when the partition is crisp with overlapping communities. Let us stress the relevance of our fuzzy modularity measure, since it allows to measure the performance of a fuzzy network clustering, being an extension of the classical modularity measure with crisp overlapping communities.

**Remark 2.** Although the main aim of this work is to build a modularity measure that permits to deal with classical, overlapping and fuzzy community detection problems, we would like to emphasize the importance of these measures. Once we have proposed a fuzzy modularity measure that can be used in the three scenarios (non overlapping, overlapping and fuzzy), we could extend the algorithms that produces fuzzy communities (as for example the Zang algorithm, or the Nepusz [23] algorithm) using this new measure to deal with three different but related problems:

- Fuzzy community detection problems. That produce a fuzzy clustering of the set of nodes. (This class of algorithms using required an accuracy measure to know the number of fuzzy communities of the network. )
- Identify, in a crisp way the overlapping nodes of the network.
- Rank the nodes based on its overlapping degree.

Now we present a simple example of how to deal with these problems based on the Zang algorithm with this new Fuzzy modularity measure.

**General algorithm :**

- For each possible number of communities  $c \in \{2, \dots, n\}$ .
- (1) Obtain a fuzzy clustering of the network  $\mu_1, \dots, \mu_c$  with  $c$  classes (Zang algorithm).
- (2) Compute the Fuzzy modularity function of the previous fuzzy partition  $\tilde{Q}(\mu)$ .

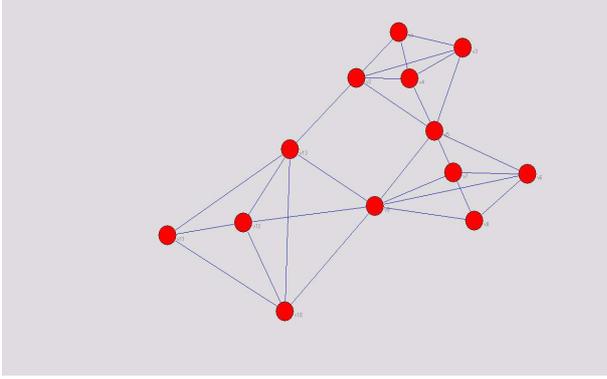


Figure 1: Zang network with overlapping communities.

- (3) Pick the number of classes  $c$  and the corresponding fuzzy partition  $\mu$  that maximizes the modularity function  $\tilde{Q}(\mu)$ .
- (4) For all  $i \in V$  compute the overlapping degree as  $G_{GO}(i) = G_{GO}(\mu_1(i), \dots, \mu_c(i))$ .
- (5) Determine the  $\alpha$  value, for which the crisp clustering obtained by the  $\alpha$ -cut maximizes  $\tilde{Q}(\mu)$ .

Let us observe that previous general algorithm produces as an output: a fuzzy clustering of the network, a degree of overlapping for each node, and also a crisp clustering network with overlapping communities in which the overlapping nodes are identify in a crisp way. Also let us note that we can have different algorithms changing the sub-algorithm (step 2) in which in which the fuzzy clustering of the network is obtained for a fixed number of classes  $c$ .

#### 4. The Zang example

To conclude this work, we just present an example of how this new measure could be used.

In [32], it is introduced a situation in which there exist clearly overlapping nodes (see Figure 1). This situation clearly presents three overlapping communities with some nodes as 5, 9, 2 or 13 (especially 5 and 9) that could belong to more than one community.

In Table 1, we show the fuzzy classification of the 13 nodes using previous scheme. Our algorithms reaches the maximum fuzzy modularity  $\tilde{Q}(\mu)$  with three classes. The function that have been used in this table to represent the bi-variate overlap is  $\max\{i \neq j \text{Min}^2\{x_i, x_j\}\}$ . Let us note that in general, n-dimensional overlapping function are not suitable for detecting overlapping nodes, since they are only able to detect nodes that belong to all communities, so we have used a composition of grouping and bi-variate overlapping functions. From the Table 1 we can detect that nodes 5, 9, 2 or 13 are the most overlapping nodes and could belong to more than one community. After the defuzzification, we have that there exist three communities  $C = \{C_1, C_2, C_3\}$  with  $C_1 = \{5, 6, 7, 8, 9\}$ ,

Table 1: Fuzzy classification using our algorithm in the Zang network. Also we show the degree of overlap for each node (and its corresponding rank) using different class of aggregation functions.

Nodes	$\mu_{C_1}(i)$	$\mu_{C_2}(i)$	$\mu_{C_3}(i)$	G Overlap	Crisp C
1	0	0	0.9898	0	(0,0,1)
2	0.0152	0.0189	0.9898	0.019	(0,0,1)
3	0.0017	0	0.9901	0.001	(0,0,1)
4	0.0017	0	0.9901	0.001	(0,0,1)
5	0.9774	0	0.9728	0.994	(1,0,1)
6	0.99	0	0	0	(1,0,0)
7	0.99	0	0	0	(1,0,0)
8	0.9898	0	0	0	(1,0,0)
9	0.9898	0.3463	0	0.3512	(1,1,0)
10	0.0017	0.9901	0	0.001	(0,1,0)
11	0	0.9898	0	0	(0,1,0)
12	0.0017	0.9901	0	0.001	(0,1,0)
13	0.0173	0.9897	0.021	0.02	(0,1,0)

$C_2 = \{9, 10, 11, 12, 13\}$  and  $C_3 = \{1, 2, 3, 4, 5\}$ , two overlapping nodes 5 and 9 and the crisp modularity is  $Q_{GN}(C) = \tilde{Q}(C) = 0.4847$ . Most of the crisp overlapping detecting nodes algorithms as CONGA [17], NMF [25] or CFINDER [24] with  $k = 3, 4$  coincide with our algorithm in the crisp clustering  $C$  and also in the overlapping nodes.

To conclude this paper, we would like to emphasize the importance of including overlaps and grouping functions to define a new modularity measure that deals with a fuzzy classification (not necessarily a Ruspini partition) of the set of nodes. Actually, there not exists any method in literature that deals with the following three problems: the fuzzy classification problem (Fuzzy Community detection), the problem of ranking the overlapping nodes and the crisp identification of the overlapping nodes.

#### Acknowledgment

This research was partially supported by the Government of Spain (grant TIN2012-32482), the Government of Madrid (grant S2013/ICCE-2845) and the UCM (Research Group 910149).

#### References

- [1] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions, a Guide to Practitioners, Springer-Verlag, Berlin, 2007.
- [2] H. Bustince, J. Fernandez, R. Mesiar, J. Montero and R. Orduna, Overlap functions, Non-linear Analysis-Theory Methods and Applications 72 (2010) 1488-1499.
- [3] H. Bustince, M. Pagola, R. Mesiar, E. Hullermeier, F. Herrera, Grouping, Overlap, and

- Generalized Bientropic Functions for Fuzzy Modeling of Pairwise Comparisons, *IEEE Transaction on Fuzzy Systems* 20 (2012) 405-415.
- [4] H. Bustince, J. Montero, E. Barrenechea, M. Pagola, Semiautoduality in a restricted family of aggregation operators, *Fuzzy Sets and Systems* 158 (2007) 1360-1377.
- [5] T. Calvo, A. Kolesárová, M. KomornÁková, R. Mesiar, Aggregation operators, properties, classes and construction methods. In T. Calvo et al. (Eds.): *Aggregation Operators New trends and Applications*, Physica-Verlag, Heidelberg, 2022, pp. 3-104.
- [6] T. Calvo, G. Mayor, J. Torrens, J. Suñer, M. Mas and M. Carbonell, Generation of weighting triangles associated with aggregation functions. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(4) (2000) 417-451.
- [7] V. Cutello, J. Montero, Hierarchical aggregation of OWA operators: basic measures and related computational problems. *Int. J. Uncertainty Fuzziness Knowl. based Syst.* 3 (1995), 17-26.
- [8] V. Cutello, J. Montero. Recursive families of OWA operators. *Proceedings FUZZ-IEEE Conference (IEEE Press, Piscataway, 1994)*, 1137-1141.
- [9] V. Cutello, J. Montero. Recursive connective rules. *International Journal of Intelligent Systems* 14, 3-20, 1999.
- [10] D. Gómez, J. Montero, A discussion of aggregation functions, *Kybernetika* 40 (2004) 107-120.
- [11] S. Fortunato, Community detection in graphs. *Physics Reports*, 486(3) (2010) 75-174.
- [12] M. Girvan and M.E.J. Newman, Community structure in social and biological networks *Proc. Natl. Acad. Sci. USA* 99 (2002) 7821.
- [13] D. Gómez, J. T. Rodríguez, J. Montero, H. Bustince, E. Barreneche, n-Dimensional overlap functions. *Fuzzy Sets and Systems*. doi:10.1016/j.fss.2014.11.023.
- [14] D. Gómez, K. Rojas, J.T. Rodríguez, J. Montero, G. Beliakov, Consistency and stability in aggregation operators: An application to missing data problems, *International Journal of Computational Intelligence Systems* 7 (2014) 595-604.
- [15] D. Gómez, E. Zarrazola, J. Yáñez, J. Montero, A Divide-and-Link Algorithm for Hierarchical Clustering in Networks, *Information Sciences*, DOI: 10.1016/j.ins.2015.04.011.
- [16] M. Grabisch, J. Marichal, R. Mesiar and E. Pap, *Aggregation Functions*. *Encyclopedia of Mathematics and its Applications*, Cambridge University, New York, 2009.
- [17] S. Gregory, A fast algorithm to find overlapping communities in networks, *Lect. Notes Comput. Sci.* 5211 408 (2008).
- [18] S. Gregory, Fuzzy overlapping communities in networks, *Journal of Statistical Mechanics: Theory and Experiment*, P02017. (2011).
- [19] A. Jurio, H. Bustince, M. Pagola, A. Pradera, R. R. Yager, Some properties of overlap and grouping functions and their application to image thresholding, *Fuzzy Sets and Systems* 229 (2013) 69-90.
- [20] A. Kolesárová. Sequential aggregation. In: M. González et al. (Eds.), *Proceedings of the Fifth International Summer School on Aggregation Operators*, AGOP, Universitat de les Illes Balears, Palma de Mallorca, 2009, pp. 183-187.
- [21] J. Liu, Fuzzy modularity and fuzzy community structure in networks. *The European Physical Journal B-Condensed Matter and Complex Systems*, 77(4), (2010) 547-557.
- [22] J. Montero, D. Gómez, H. Bustince, On the relevance of some families of fuzzy sets, *Fuzzy Sets and Syst.* 158 (2007) 2429-2442.
- [23] T. Nepusz, A. Petroczi, L. Negyessy and F. Bazso, Fuzzy communities and the concept of bridgeness in complex networks *Physical Review. E* 77 016107. (2008)
- [24] G. Palla, I. DerÁ©nyi, I. Farkas and T. Vicsek, Uncovering the overlapping community structure of complex networks in nature and society *Nature* 435 (2004) 814.
- [25] I. Psorakis, S. Roberts and B. Sheldon, Efficient Bayesian community detection using non-negative matrix factorisation, arXiv:1009.2646 (2010)
- [26] K. Rojas, D. Gómez, J.T. Rodríguez, J. Montero, Some properties of consistency in the families of aggregation functions. *Advances in Intelligent and Soft Computing* 107:169-176 (P. Melo-Pinto, P. Couto, C. Serodio, J. Fodor, B. De Baets, Eds.) Springer, Berlin (2011).
- [27] K. Rojas, D. Gómez, J.T. Rodríguez, J. Montero, Strict Stability in Aggregation Operators, *Fuzzy sets and Systems* 228 (2013) 44-63.
- [28] R.R. Yager, On ordered weighted averaging aggregation operators in multi-criteria decision making, *IEEE Transactions on Systems, Man and Cybernetics* 18 (1988) 183-190.
- [29] R.R. Yager and A. Rybalov, Noncommutative self-identity aggregation, *Fuzzy Sets and Systems* 85 (1997) 73-82.
- [30] R.R. Yager, Prioritized aggregation operators, *International Journal of Approximate Reasoning* 48 (2008) 263-274.
- [31] J. Xie, S. Kelley, and B. K. Szymanski, Overlapping community detection in networks: The state-of-the-art and comparative study. *ACM Computing Surveys (CSUR)*, 45(4) (2013) 43.
- [32] S. Zhang, R. Wang and X. Zhang, Identification of overlapping community structure in complex networks using fuzzy C-means clustering. *Physica A* 374 483 (2007).