

# $f$ -inclusion indexes between fuzzy sets

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## Abstract

We introduce the notion of  $f$ -inclusion, which is used to describe different kinds of subsethood relations between fuzzy sets by means of monotonic functions  $f: [0, 1] \rightarrow [0, 1]$ . We show that these monotonic functions can be considered indexes of inclusion, since the greater the function considered, the more restrictive is the relationship. Finally, we propose a general index of inclusion by proving the existence of a representative  $f$ -inclusion for any two ordered pairs of fuzzy sets. In such a way, our approach is different from others in the literature in not taking a priori assumptions like residuated implications or  $t$ -norms.

**Keywords:**  $f$ -inclusion, index of subsethood, fuzzy sets.

## 1. Introduction

Subsethood is one of the most basic binary relations between sets. However, so far there is not a consensus in the generalization of such a relation in fuzzy set theory. The first proposal (from Zadeh in [14]) focused on identifying the inclusion between fuzzy sets with the point-wise ordering between membership functions. That definition has been criticized in some approaches (for instance in [4, 1]) for being rigid and for the lack of softness according to the spirit of fuzzy logic.

Many proposals have been addressed to incorporate degrees to Zadeh's inclusion, unified under the name of *measures of inclusion*. Basically, there are three different kinds of approaches in the literature: those based on cardinality [10, 8]; those based on logic implications [1, 6]; and those based on axiomatic definitions [13, 2]. Certainly, our approach is related to these three kind of approaches, but it really diverges from all of them. For instance, the notion of  $f$ -inclusion presented in this paper generalizes, somehow, those inclusions based on logic residuated implications; however, whereas in such approaches degrees of inclusion depend strongly on an implication operator chosen a priori, our inclusion index determines, somehow, the best operator to represent the inclusion between two fuzzy sets without taking into account any a priori assumptions.

It is worth to note that defining measures of inclusion goes beyond mere theoretical interest. For instance fuzzy inclusion can be linked with Social Science by mainstream statistical techniques [11],

and with image processing by fuzzy mathematical morphology [7].

The structure of the paper is the following: In Section 4 we provide the notion of  $f$ -inclusion and some of its properties. In Section 5 we present the definition of an index of inclusion by means of  $f$ -inclusion relations. Finally, in Section 6 we present the conclusions and future works.

## 2. Preliminaries

In this section we recall some well-known notions in order to make this paper as self-contained as possible.

**Definition 1** A fuzzy set  $A$  is a pair  $(\mathcal{U}, \mu_A)$  where  $\mathcal{U}$  is a set (called the universe of  $A$ ) and  $\mu_A$  is a mapping from  $\mathcal{U}$  to  $[0, 1]$  (called the membership function of  $A$ ).

Note that a fuzzy set is fully determined by its membership function. Hence, for the sake of clarity, we use the same notation for fuzzy sets and membership functions (i.e.  $A(x) = \mu_A(x)$ ) and the universe is omitted whenever it does not generate any misunderstanding.

As usual, we will consider the standard framework of  $t$ -norms and their residuated implications as the underlying logical computational framework.

**Definition 2** A  $t$ -norm is any mapping  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  such that:

- $x_1 * y \leq x_2 * y$

for all  $x_1, x_2, y \in [0, 1]$  such that  $x_1 \leq x_2$  (monotonicity).

- $1 * x = x$
- $x * y = y * x$  (commutativity)
- $x * (y * z) = (x * y) * z$  (Associativity)

for all  $x, y, z \in [0, 1]$ :

Another notion used in this paper is that of residuated implication. Let  $*$  be a  $t$ -norm, an operator  $\rightarrow$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is a residuated implication<sup>1</sup> with respect to  $*$  if:

$$x * y \leq z \iff x \leq y \rightarrow z \quad (1)$$

for all  $x, y, z \in [0, 1]$ . It is important to take into account that given a residuated implication there is

<sup>1</sup>It is usual to call use the term residuated pair to refer to the pair  $(*, \rightarrow)$ .

just one  $t$ -norm satisfying equation (1), called the *adjoint property*. Moreover, from equation (1) we can infer that  $\rightarrow$  satisfies the following properties:

- $x_2 \rightarrow y \leq x_1 \rightarrow y$
- $y \rightarrow x_1 \leq y \rightarrow x_2$

for all  $x_1, x_2, y \in [0, 1]$  such that  $x_1 \leq x_2$  (antitonicity in the first argument and monotonicity in the second).

### 3. $f$ -inclusion indexes

Our notion of  $f$ -inclusion between two fuzzy sets is given as follows.

**Definition 3** *Let  $A$  and  $B$  be two fuzzy sets and  $f: [0, 1] \rightarrow [0, 1]$  a monotonic mapping such that  $f(x) \leq x$ . We say that  $A$  is  $f$ -included in  $B$  (denoted by  $A \subseteq_f B$ ) if and only if the inequality  $f(A(u)) \leq B(u)$  holds for all  $u \in \mathcal{U}$ .*

Obviously, although the definition above is given in terms of “ $A$  is  $f$ -included in  $B$ ”, we can say “ $B$   $f$ -includes  $A$ ” as well. In addition, for the sake of the presentation, we denote by  $\Omega$  the set of monotonic mappings such that  $f(x) \leq x$ .

Second, note that  $f$ -inclusion is actually a crisp relation. However, although the word “inclusion” has been used, such a crisp relation is not transitive, so it is not an order relation. The use of this terminology is motivated by the goal of the approach; namely to assign degrees of inclusion between fuzzy sets via mappings. Somehow, these mappings determine bounds on the possible values of two fuzzy sets. Specifically, fixed a mapping  $f \in \Omega$  and a value of  $A(u)$  (with  $u \in \mathcal{U}$ ), the  $f$ -inclusion determines a lower bound of the value of  $B(u)$ . Moreover, as  $f$  is monotonic, the greater the value of  $A(u)$ , the greater the lower bound, and thus, the greater can be the value of  $B(u)$ . Note that in the case that  $A(u) = 0$ , the  $f$ -inclusion does not impose any restriction on the value of  $B(u)$ , since  $f(0) = 0$ . This represents the fact that the empty set is contained in every set.

Note that, as the restriction depends uniquely of the mapping chosen, each mapping  $f$  determines a different “kind of  $f$ -inclusion”. Note also that the greater the mapping  $f$ , the more restrictive is the  $f$ -inclusion relation (see Proposition 5 below). Therefore, every  $f$ -inclusion can be identified with an index (or degree) of inclusion.

In this paper, it is worth to keep the following idea in mind:

*inclusion is not represented by just one  $f$ -inclusion but with the whole set of them, by identifying each  $f \in \Omega$  with an index.*

and, for instance, the pseudo transitivity property described in Proposition 2 should be read this way.

### Related approaches

The notion of  $f$ -inclusion can be linked with the degrees of inclusion proposed by approaches based on implications [1, 6] as follows. Given two fuzzy sets  $A$  and  $B$  and a residuated implication  $\rightarrow$ , the degree of inclusion of  $A$  in  $B$  is defined as:

$$Inc_d(A, B) = \inf_{u \in \mathcal{U}} (A(u) \rightarrow B(u))$$

Let us study the case where  $A$  is included in  $B$  with at least a degree of inclusion  $\alpha \in [0, 1]$ ; i.e. that  $\inf_{u \in \mathcal{U}} (A(u) \rightarrow B(u)) \geq \alpha$  holds for all  $u \in \mathcal{U}$ . By the adjoint property (1), we have the following chain of equivalent statements:

$$\inf_{u \in \mathcal{U}} (A(u) \rightarrow B(u)) \geq \alpha$$

if and only

$$A(u) \rightarrow B(u) \geq \alpha \text{ for all } u \in \mathcal{U}$$

if and only if

$$B(u) \geq A(u) * \alpha \text{ for all } u \in \mathcal{U}.$$

Note that the last inequality is in accordance with Definition 3, since the function  $f_\alpha: [0, 1] \rightarrow [0, 1]$  defined by  $f_\alpha(x) = x * \alpha$  is monotonic and  $f_\alpha(x) \leq x$  for all  $x \in [0, 1]$ . Hence, we are able to represent the restriction imposed by the equality  $Inc_d(A, B) = \alpha$  by means of the notion of  $f$ -inclusion.

The notion of *fuzzy inclusion* given in [12] can be considered a specific case of  $f$ -inclusion as well. Specifically, given  $\alpha \in [0, 1]$ , the relation determined by the “*fuzzy inclusion of degree  $\alpha$* ” coincides with the  $(x + \alpha - 1)$ -inclusion.

### 4. Properties of $f$ -inclusion indexes

As we stated above, the  $f$ -inclusion relation is not necessarily transitive and, hence, is not an order relation. Notice, however, that it does not mean that the notion of  $f$ -inclusion is unable to deal with subethood in fuzzy set theory. Actually, most approaches to deal with subethood in fuzzy environments are not based on defining order relations between fuzzy sets.

We introduce below some results relating the notion  $f$ -inclusion to the properties of *reflexivity*, *antisymmetry* and *transitivity*. The first one shows that the  $f$ -inclusion is always reflexive.

**Proposition 1** *Let  $A$  be a fuzzy set, then  $A \subseteq_f A$  for all  $f \in \Omega$ .*

*Proof:* The result comes from the fact that  $f(x) \leq x$  for all  $f \in \Omega$  and  $x \in [0, 1]$ . Hence,  $f(A(u)) \leq A(u)$  for all  $u \in \mathcal{U}$ .  $\square$

The following result relates the  $f$ -inclusion and transitivity in term of composition of mappings in  $\Omega$ .

**Proposition 2** Let  $A, B$  and  $C$  be three fuzzy sets and let  $f, g \in \Omega$ . Then,  $A \subseteq_f B$  and  $B \subseteq_g C$  implies  $A \subseteq_{g \circ f} C$ .

*Proof:* From  $A \subseteq_f B$  and  $B \subseteq_g C$  we have the inequalities  $f(A(u)) \leq B(u)$  and  $g(B(u)) \leq C(u)$  for all  $u \in \mathcal{U}$  respectively. From the former inequality and by using that  $g$  is monotonic, we have  $g(f(A(u))) \leq g(B(u))$  for all  $u \in \mathcal{U}$ . So by adding the latter inequality we have  $g(f(A(u))) \leq g(B(u)) \leq C(u)$  for all  $u \in \mathcal{U}$ ; or equivalently,  $A \subseteq_{g \circ f} C$ .  $\square$

To relate the  $f$ -inclusion to the antisymmetry note that, obviously, from  $A \subseteq_f B$  and  $B \subseteq_f A$  we cannot ensure that  $A = B$  (except the case  $f = id$ ). However, the difference between both fuzzy sets can be bounded as follows.

**Proposition 3** Let  $A$  and  $B$  be two fuzzy sets such that  $A \subseteq_f B$  and  $B \subseteq_g A$ . Then:

$$|A(u) - B(u)| \leq \sup_{x \in [0,1]} \{x - f(x), x - g(x)\}$$

for all  $u \in \mathcal{U}$ .

*Proof:* Let  $u \in \mathcal{U}$  and let us assume firstly that  $B(u) \leq A(u)$ . Then, as  $g(B(u)) \leq A(u)$  (because of  $B \subseteq_g A$ ), we have that:

$$B(u) - A(u) \leq B(u) - g(B(u))$$

Similarly, by assuming  $A(u) \leq B(u)$  and using that  $A \subseteq_f B$  we have that

$$A(u) - B(u) \leq A(u) - f(A(u))$$

So we can conclude that  $|A(u) - B(u)| \leq \sup_{x \in [0,1]} \{x - f(x), x - g(x)\}$   $\square$

Note that, as a consequence of the previous result, assuming both  $A \subseteq_f B$  and  $B \subseteq_g A$ , the closer to the identity the mappings  $f$  and  $g$  are, the more similar the fuzzy sets  $A$  and  $B$  are.

Let us see now other properties of the notion of  $f$ -inclusion. The next result shows that the relation  $\subseteq_f$  still holds when considering a "smaller" set to be contained or a "larger" set to be the container. This is,  $f$ -inclusion behaves naturally with respect to the ordering between fuzzy sets.

**Proposition 4** Let  $A, B, C$  and  $D$  be fuzzy sets such that  $A(u) \leq B(u)$  and  $C(u) \leq D(u)$  for all  $u \in \mathcal{U}$ . Then  $B \subseteq_f C$  implies  $A \subseteq_f D$

In other words, the greater the fuzzy set, the more fuzzy sets  $f$ -includes and, respectively, the lesser the fuzzy set the bigger the number of fuzzy sets where it is  $f$ -included in.

The following result shows that  $f$ -inclusion can be related also to the point-wise ordering defined in  $\Omega$ .

**Proposition 5** Let  $A$  and  $B$  be two fuzzy sets and let  $f, g \in \Omega$  such that  $f \geq g$ . Then  $A \subseteq_f B$  implies  $A \subseteq_g B$ .

This proposition supports the interpretation of each mapping used to define  $f$ -inclusions as indexes of inclusion, since the lesser the mapping  $f \in \Omega$ , the weaker the restriction imposed by the  $f$ -inclusion. Furthermore, as  $\Omega$  is a bounded lattice, we have two extreme cases; namely, the weakest one given by the mapping  $f \equiv 0$ , and the strongest one given by the mapping  $f \equiv id$ . Let us study restrictions imposed by these two extreme  $f$ -inclusions. The 0-inclusion represents, simply, *zero index of inclusion*, because of, according to the result below, it does not impose any restriction.

**Proposition 6** Every pair of fuzzy sets  $A$  and  $B$  satisfies the relation  $A \subseteq_0 B$ .

The previous result says, somehow, that every fuzzy set is at least 0-included in any fuzzy set. The 0-inclusion becomes an interesting case when it is the only  $f$ -inclusion between two fuzzy sets.

**Proposition 7** Let  $A$  and  $B$  be two fuzzy sets. The only  $f$ -inclusion of  $A$  in  $B$  is  $A \subseteq_0 B$  if and only if there exists a sequence  $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{U}$  such that  $A(u_n) = 1$  for all  $n \in \mathbb{N}$  and  $\lim B(u_n) = 0$ .

*Proof:* We will prove that  $A \subseteq_f B$  with  $f \neq 0$  if and only if for every sequence  $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{U}$  such that  $\lim B(u_n) = 0$  it is not the case that  $A(u_n) = 1$  for every  $n \in \mathbb{N}$ .

Let us assume firstly that  $A \subseteq_f B$  with  $f \neq 0$  and let us consider  $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{U}$  a sequence such that  $\lim B(u_n) = 0$ . Then as  $0 \leq f(A(u_n)) \leq B(u_n)$  for all  $n \in \mathbb{N}$ , we have :

$$0 \leq \lim f(A(u_n)) \leq \lim B(u_n) = 0$$

i.e,  $\lim f(A(u_n)) = 0$ . Now, it cannot exist a subsequence  $\{u_{n_j}\}$  satisfying  $A(u_{n_j}) = 1$ ; otherwise, we would have  $f(1) = 0$  and, by properties of  $f \in \Omega$ , we would have  $f = 0$ .

Conversely, assuming that there is no sequence  $\{u_n\}$  such that  $A(u_n) = 1$  for all  $n \in \mathbb{N}$  and  $\lim B(u_n) = 0$ , we will build a function  $f \neq 0$  such that  $A \subseteq_f B$ . Consider the value

$$\delta = \inf\{B(u) : u \in \mathcal{U} \text{ and } A(u) = 1\}$$

Straightforwardly  $\delta \neq 0$  by the hypothesis assumed in this implication and, the function

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ \delta & \text{if } x = 1 \end{cases}$$

satisfies  $f \neq 0$  and  $f(A(u)) \leq B(u)$ . Note that case where  $\{B(u) : u \in \mathcal{U} \text{ and } A(u) = 1\} = \emptyset$  is also included in the proof.  $\square$

Note that, given two fuzzy sets  $A$  and  $B$ , if there exist an element  $u \in \mathcal{U}$  such that  $A(u) = 1$  and

$B(u) = 0$ , by the previous proposition, the only  $f$ -inclusion of  $A$  in  $B$  is  $\subseteq_0$  (i.e. no inclusion). The converse is not true in general, since Proposition 7 only ensures that we can find an element in the universe  $\mathcal{U}$  with a value 1 in  $A$  and, as close as we want to 0 in  $B$  (see Example 1 below). Anyway, the converse of this simplified formulation in terms of existence holds when the universe is finite.

**Example 1** *On the universe  $\mathcal{U} = [1, \infty) \subseteq \mathbb{R}$  consider the fuzzy sets  $A$  and  $B$  defined by  $A(u) = 1$  and  $B(u) = 1/u$  for all  $u \in \mathcal{U}$ . Note that  $B(u) \neq 0$  for all  $u \in \mathcal{U}$  and the only  $f$ -inclusion of  $A$  in  $B$  is  $\subseteq_0$ . On the other hand, if we consider  $f$ -inclusions of  $B$  in  $A$  we have that  $B \subseteq_f A$  for any function  $f \in \Omega$ .*

**Corollary 1** *Let  $A$  and  $B$  be two fuzzy sets defined on a finite universe  $\mathcal{U}$ . Then there exist  $u \in \mathcal{U}$  such that  $A(u) = 1$  and  $B(u) = 0$  if and only if the only  $f$ -inclusion of  $A$  in  $B$  is  $\subseteq_0$ .*

The importance of the 0-inclusion comes from the fact that it is the least element in  $\Omega$ . In the opposite case we have the  $f$ -inclusion given by the mapping identity, i.e. the greatest element in  $\Omega$ . The following result shows the properties of  $id$ -inclusion.

**Corollary 2** *Let  $A$  and  $B$  be two fuzzy sets. The following statements are equivalent:*

1.  $A \subseteq_{id} B$ .
2.  $A(u) \leq B(u)$  for all  $u \in \mathcal{U}$ .
3.  $A \subseteq_f B$  for all  $f \in \Omega$ .

*Proof:* (1)  $\iff$  (2) is straightforward.

(1)  $\iff$  (3) holds by taking into account that  $id$  is the greatest element in  $\Omega$  and by applying Proposition 5.  $\square$

This result is closely related to the axiomatic approaches of measures of inclusion between fuzzy sets [13, 2]. In those approaches the greatest degree of inclusion holds if and only if Zadeh's inclusion relation is satisfied (i.e. item (2) in the previous result). Furthermore, note also that (3) means that  $id$ -inclusion is the most restrictive condition imposed by  $f$ -inclusions; therefore, it makes sense considering it as the strongest index of  $f$ -inclusion.

## 5. Representing the inclusion of fuzzy sets

In this section we describe how to represent the inclusion between two fuzzy sets by using the notion of  $f$ -inclusion. A standard procedure would be to fix a specific  $f$ -inclusion and to determine whether two fuzzy sets satisfy or not the relation  $\subseteq_f$ . Note that this procedure would have the same shortcomings of Zadeh's definition of inclusion about lack of softness. In this paper, we proceed in a different way: given two fuzzy sets  $A$  and  $B$ , we search for the best  $f$ -inclusion "representing the inclusion of

$A$  in  $B$ ." As we emphasized in the previous section, mappings in  $\Omega$  can (and must) be considered as indexes of inclusion in the sense "the greater the mapping the stronger the inclusion." Therefore, the index of inclusion of  $A$  in  $B$  can be defined by the greatest element  $f$  in  $\Omega$  such that  $A \subseteq_f B$ . Note that this index of inclusion is well-defined if and only if such a greatest element exists. The proof of such existence is not hard, consider for each pair of fuzzy sets  $A$  and  $B$  the following subset of  $\Omega$ :

$$\mathcal{Inc}(A, B) = \{f \in \Omega \mid A \subseteq_f B\}$$

As  $\Omega$  has a structure of complete lattice, we can ensure that supremum of  $\mathcal{Inc}(A, B)$  exists. Finally, as a direct consequence of the following result, such a supremum is in fact a maximum.

**Lemma 1** *Let  $A$  and  $B$  be two fuzzy sets and let  $\{f_i\}_{i \in \mathbb{I}} \subseteq \Omega$ . If  $A$  is  $f_i$ -included in  $B$  for all  $i \in \mathbb{I}$ , then  $A$  is  $\sup_{i \in \mathbb{I}}\{f_i\}$ -included in  $B$ .*

*Proof:* It is well known that  $\sup_{i \in \mathbb{I}}\{f_i\}$  is given by  $f(x) = \sup_{i \in \mathbb{I}}\{f_i(x)\}$  for all  $x \in [0, 1]$ . Moreover, since  $\Omega$  is a complete lattice,  $\sup_{i \in \mathbb{I}}\{f_i\} \in \Omega$  as well. Now, since  $A$  is  $f_i$ -included in  $B$  for all  $i \in \mathbb{I}$ , we have that  $f_i(A(u)) \leq B(u)$  for all  $u \in \mathcal{U}$  and  $i \in \mathbb{I}$ . Therefore,  $\sup_{i \in \mathbb{I}}\{f_i\}(A(u)) \leq B(u)$  for all  $u \in \mathcal{U}$ , or equivalently,  $A$  is  $\sup_{i \in \mathbb{I}}\{f_i\}$ -included in  $B$ .  $\square$

So, as a straightforward consequence of the previous proposition, the greatest element of  $\mathcal{Inc}(A, B)$  exists, let us denote it by  $f_{AB}$ . Note also that such a greatest element is a *natural* and *representative* index of inclusion of  $A$  in  $B$ .

It is worth to note that  $f_{AB}$  is defined without taking into account any *a priori* assumption (such as the choice of a residuated lattice, or any operator, or any kind of parameter), hence it is indeed "natural." Moreover, it is also "representative" because, thanks to Proposition 5, the set of mappings  $f \in \Omega$  such  $A$  is  $f$ -included in  $B$  can be characterized in terms of this greatest element, since:

$$\mathcal{Inc}(A, B) = \{f \in \Omega \mid f \leq f_{AB}\}$$

In the rest of the section we focus on proving that such a representative element coincides with the following expression:

$$f_{AB}(x) = \min \left\{ x, \inf_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\} \right\} \quad (2)$$

The proof of this assertion has been divided into the three lemmas below. The first of them shows that  $f_{AB}$  is in  $\Omega$ , so it makes sense to talk about  $f_{AB}$ -inclusion.

**Lemma 2** *Let  $A$  and  $B$  be fuzzy sets, then the mapping  $f_{AB}$  (defined in equation (2)) is in  $\Omega$ .*

*Proof:* Obviously,  $f_{AB}(x) \leq x$  by definition. Let us show now that it is monotonic. Consider

$x, y \in [0, 1]$  such that  $x \leq y$ . Straightforwardly, we have the inclusion

$$\{u \in \mathcal{U} \mid y \leq A(u)\} \subseteq \{u \in \mathcal{U} \mid x \leq A(u)\}$$

Therefore, by definition of infimum we have:

$$\begin{aligned} f_{AB}(x) &= \min\{x, \inf_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}\} \\ &\leq \min\{y, \inf_{u \in \mathcal{U}} \{B(u) \mid y \leq A(u)\}\} = f_{AB}(y). \end{aligned}$$

In other words,  $f_{AB}$  is monotonic.  $\square$

Let us show now that for all pair of fuzzy sets  $A$  and  $B$ ,  $A$  is  $f_{AB}$ -included in  $B$ ; note that the definition of  $f_{AB}$  depends on the chosen ordering between  $A$  and  $B$ , i.e. generally  $f_{AB} \neq f_{BA}$ .

**Lemma 3** *Let  $A$  and  $B$  be two fuzzy sets, then  $A$  is  $f_{AB}$ -included in  $B$ .*

*Proof:* We need to show that  $f_{AB}(A(u)) \leq B(u)$  for all  $u \in \mathcal{U}$ . That inequality can be achieved by definition of  $f_{AB}$ , since:

$$\begin{aligned} f_{AB}(A(u)) &= \min\{A(u), \inf_{v \in \mathcal{U}} \{B(v) \mid A(u) \leq A(v)\}\} \\ &\leq \min\{A(u), B(u)\} \leq B(u) \end{aligned}$$

for all  $u \in \mathcal{U}$ .  $\square$

Finally, the third lemma proves the maximality, i.e.  $f_{AB}$  is the greatest mapping in  $\Omega$  satisfying that  $A$  is  $f$ -included in  $B$ .

**Lemma 4** *Let  $A$  and  $B$  be two fuzzy sets and let  $f \in \Omega$ . Then  $f > f_{AB}$  implies that  $A$  is not  $f$ -included in  $B$ .*

*Proof:* Consider  $f \in \Omega$  such that  $f > f_{AB}$  and let us show that  $A$  is not  $f$ -included in  $B$ . As  $f > f_{AB}$ , there exists  $\alpha \in [0, 1]$  such that  $f(\alpha) > f_{AB}(\alpha)$ . Note that, as  $f \in \Omega$ , this strict inequality implies that  $f_{AB}(\alpha) = \inf_{u \in \mathcal{U}} \{B(u) \mid \alpha \leq A(u)\}$  since, otherwise  $f_{AB}(\alpha) = \alpha = f(\alpha)$ , contradiction. Furthermore, note that the previous expression for  $f_{AB}(\alpha)$  implies that there exist elements  $u \in \mathcal{U}$  such that  $\alpha \leq A(u)$ .

Now, let us prove that we can assume that such an element  $\alpha$  is in the support of the fuzzy set  $A$ . That is, let us show that there exists  $u \in \mathcal{U}$  such that  $f(A(u)) > f_{AB}(A(u))$ . Let us assume by *reductio ad absurdum* that for all  $v \in \mathcal{U}$  such that  $\alpha \leq A(v)$  we have  $f(A(v)) \leq f_{AB}(A(v))$ . By the monotonicity of  $f$  and the previous assumption, we obtain the inequality  $f(\alpha) \leq f(A(v)) \leq f_{AB}(A(v))$  for all  $v \in \mathcal{U}$  such that  $\alpha \leq A(v)$ . Thus:

$$\begin{aligned} f(\alpha) &\leq \inf_{v \in \mathcal{U}} \{f_{AB}(A(v)) \mid \alpha \leq A(v)\} \\ &\text{(by definition of } f_{AB}) \\ &= \inf_{\alpha \leq A(v)} \{\min\{A(v), \inf_{u \in \mathcal{U}} \{B(u) \mid A(v) \leq A(u)\}\}\} \\ &\text{(by properties of infima)} \\ &\leq \inf_{u \in \mathcal{U}} \{B(u) \mid \alpha \leq A(u)\} \\ &\text{(by the remark in the first paragraph of the proof)} \\ &= f_{AB}(\alpha) \end{aligned}$$

So we have obtained that  $f(\alpha) \leq f_{AB}(\alpha)$ , which contradicts our assumption  $f(\alpha) > f_{AB}(\alpha)$ . Therefore, there exists  $u \in \mathcal{U}$  such that:

$$f(A(u)) > f_{AB}(A(u)).$$

Straightforwardly we have the equality  $f_{AB}(A(u)) = \inf_{v \in \mathcal{U}} \{B(v) \mid A(u) \leq A(v)\}$ . So, as  $f(A(u)) > f_{AB}(A(u))$ ,  $f(A(u))$  cannot be a lower bound of the set  $\{B(v) \mid A(u) \leq A(v)\}_{v \in \mathcal{U}}$ . Hence, there exists  $w \in \mathcal{U}$  such that  $A(u) \leq A(w)$  and  $f(A(u)) > B(w)$ . By using the monotonicity of  $f$  in the former inequality we have that  $f(A(u)) \leq f(A(w))$ , and by adding to it the latter one we conclude that  $f(A(w)) \geq f(A(u)) > B(w)$ . In other words,  $A$  is not  $f$ -included in  $B$ .  $\square$

By joining the three previous lemmas we can obtain the following result:

**Theorem 1** *Let  $A$  and  $B$  two fuzzy sets, then the greatest element of  $\mathcal{Inc}(A; B)$  is:*

$$f_{AB}(x) = \min\{x, \inf_{u \in \mathcal{U}} \{B(u) \mid x \leq A(u)\}\}$$

*Proof:* By Lemmas 2 and 3 we have that  $f_{AB}(x) \in \mathcal{Inc}(A; B)$ . By the complete lattice structure of  $\Omega$ , we know that there exist the suprema of  $\mathcal{Inc}(A; B)$ . Now, by Lemma 4, we have that there are not upper bounds of  $\mathcal{Inc}(A; B)$  strictly greater than  $f_{AB}$ . But as  $f_{AB}(x) \in \mathcal{Inc}(A; B)$ , then  $f_{AB}(x)$  is necessarily the greatest element of  $\mathcal{Inc}(A; B)$ .  $\square$

The following result characterizes the two extreme cases of inclusion under this approach; namely, when  $f_{A,B} = 0$  and  $f_{A,B} = id$ . Note that the former case ( $f_{AB} = 0$ ) can be interpreted as  $A$  is not included in  $B$  at all, whereas the latter case ( $f_{AB} = id$ ) as definitely  $A$  is included in  $B$ .

**Corollary 3** *Let  $A$  and  $B$  two fuzzy sets, then:*

- $f_{AB} = 0$  if and only if there is a sequence  $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{U}$  such that  $A(u_n) = 1$  and  $\lim B(u_n) = 0$ .
- $f_{AB} = id$  if and only if  $A(u) \leq B(u)$  for all  $u \in \mathcal{U}$ .

*Proof:* The first item can be proved by noting that  $f_{AB} = 0$  implies that the only  $f$ -inclusion between  $A$  and  $B$  is the 0-inclusion. Then, by applying Propositions 7 the result is straightforwardly reached. To prove the second item, take into account that  $id$ -inclusion relation is equivalent to  $A(u) \leq B(u)$  for all  $u \in \mathcal{U}$ .  $\square$

The following result focus on two special kind of fuzzy sets: those defined on a finite universe and those defined by continuous membership functions from an interval of  $\mathbb{R}^n$ . In both cases, the function  $f_{AB}(x)$  can be defined by using the operator minimum instead of infimum.

**Corollary 4** Let  $A$  and  $B$  be two fuzzy sets defined on a finite universe. Then:

$$f_{AB}(x) = \min\{x, \min_{u \in \mathcal{U}}\{B(u) \mid x \leq A(u)\}\}$$

**Corollary 5** Let  $A$  and  $B$  be two fuzzy sets defined on the universe  $\mathcal{U} = [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$  ( $n \in \mathbb{N}$ ) by continuous membership functions. Then:

$$f_{AB}(x) = \min\{x, \min_{u \in \mathcal{U}}\{B(u) \mid x \leq A(u)\}\}$$

*Proof:* The proof arises simply by taking into account that for every  $x \in [0, 1]$ , the expression given for  $f_{AB}$  in Theorem 1 is the infimum of a continuous mapping (namely  $g(u) = \min\{x, B(u)\}$ ) on a closed subset (namely  $\{u \in \mathcal{U} \mid x \leq A(u)\}$ ). Therefore, such an infimum is in fact a minimum.  $\square$

## 6. Conclusion and Future work

We have shown how to represent the inclusion of two fuzzy sets via a mapping without taking into account any preliminary assumption (as could be fixing an operator, a function or a value). Specifically we started by defining the notion of  $f$ -inclusion, a binary crisp relation which determines, somehow, an index of inclusion between fuzzy sets. Subsequently, we have shown that, given two fuzzy sets  $A$  and  $B$ , the set of all  $f$ -inclusions of  $A$  in  $B$  has a representative element. This representative element (denoted above by  $f_{AB}$ ) can be considered a natural index of inclusion of  $A$  in  $B$  and some results concerning with its structure have been presented.

We propose three different lines of future research. For instance, a measure of inclusion (in the sense of obtaining a real value for each ordered pair of fuzzy sets) can be defined by using this approach. Specifically, as done in [3], to measure *contradiction*. In addition, a comparison with the axiomatic approaches of subsethood measures will be done as well.

An application of this theoretical approach could be done by identifying the inclusion with the logic connective of implication. Thus, given the information provided by two fuzzy sets  $A$  and  $B$ ,  $f_{AB}$  could be associated to an implication to represent the information we can infer to  $B$  from the information of  $A$ . This implication could be of interest for databases based on logic programming [9] under missing information.

Finally, an index of similarity can be defined in a natural way from an index of inclusion; just by identifying the equality = with both  $\subseteq$  and  $\supseteq$ . It is worth mentioning that similarity relations and measures has been used in several practical areas, as for instance in Decision Making [5].

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