Global Exponential Stability of Discrete-Time Complex-Valued Neural Networks with Time-Varying Delay

Zhenjiang Zhao[1, a], Qiankun Song[2,b]

[1]Department of Mathematics, Huzhou University, Huzhou, China

[2]Department of Mathematics, Chongqing Jiaotong University, Chongqing, China

[a]zhaozjcn@163.com, [b]qianlunson@163.com

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Abstract. In this paper, a class of discrete-time complex-valued neural networks with time-varying delays and impulses are considered. Based on M-matrix theory and analytic methods, several simple sufficient conditions checking the global exponential stability are obtained for the considered neural networks. The obtained results show that the stability still remains under certain impulsive perturbations for the neural network with stable equilibrium point, and the neural network with unstable equilibrium point can be stabilization by impose appropriate impulsive perturbations.

Introduction

The complex-valued neural networks (CVNN) have found important applications in various areas such as static image processing and solving nonlinear algebraic equations [1]. Some of these applications require that the designed CVNN has a unique stable equilibrium point. In hardware implementation, time delays occur due to finite switching speed of the amplifiers and communication time [2]. Therefore, study of CVNN with consideration of the delayed problem becomes extremely important to manufacture high quality CVNN. In recent years, some results concerning the stability of CVNN without or with delays have been reported, for example, see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein. In [2, 3, 4], authors investigated the stability of continuous-time CVNN without delays, and provided several stability criteria for the considered CVNN. In [5, 6, 7, 8], authors considered a class of continuous-time CVNN with constant delays, and obtained some sufficient condition of stability for the studied CVNN. In [9], a class of continuous-time CVNN with both discrete time-varying delays and unbounded distribute delays were considered, a main criterion for assuring the existence, uniqueness and exponential stability of the equilibrium point of the system are derived by using the vector Lyapunov function method, homeomorphism mapping lemma and the matrix theory. As pointed out in [10], in numerical simulation and practical implementation of the continuous-time neural networks, it is essential to formulate a discrete-time system that is an analogue of the continuous-time system. Therefore, it is of both theoretical and practical importance to study the dynamics of discrete-time neural networks. Some results on stability of discrete-time CVNN without or with delays have been reported [10, 11, 12].

However, besides delay effect, impulsive effect likewise exists in neural networks [13]. For instance, in implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants, which may be caused by switching phenomenon, frequency change or other sudden noise, that is, does exhibit impulsive effects. Therefore, it is necessary to consider both impulsive effect and delay effect on dynamical behaviors of neural networks. In [13], authors considered a class of continuous-time CVNN with impulses and three kinds of delays including leakage delay, discrete delay and distributed delay, and gave several delay-dependent stability criteria. To the best of our knowledge, few authors have considered the problem on stability of discrete-time CVNN with variable delays and impulses.

Motivated by the above discussions, the objective of this paper is to study the global exponential stability of discrete-time CVNN with variable delays and impulses.

Notations: For $u=(u_1,u_2,\cdots,u_n)^T$ and $A=(a_{ij})_{n\times n}\in R^{n\times n}$, let $|u|=(|u_1|,|u_2|,\cdots,|u_n|)^T$, $\|u\|=\left(\sum_{i=1}^{n}|u_i|^2\right)^{\frac{1}{2}}$, $|A|=(|a_{ij}|)_{n\times n}$. For integers a and b with $a<b$, $N[a,b]$ denotes the discrete interval given $N[a,b]=\{a,a+1,\cdots,b-1,b\}$, $C(N[-\tau,0],R^n)$ denotes the set of all functions $\varphi:N[-\tau,0]\to R^n$.

Model Description and Preliminaries

In this paper, we consider the global exponential stability of the following model

$$\begin{cases} u_i(m+1)=d_iu_i(m)+\sum_{j=1}^{n}a_{ij}f_j(u_j(m))+\sum_{j=1}^{n}b_{ij}f_j(u_j(m-\tau_{ij}(m)))+I_i & m\neq m_k, \\ u_i(m)=p_{ik}(u_1(m^-),u_2(m^-),\cdots,u_n(m^-))+J_{ik} & m=m_k, \\ u_i(s)=\phi(s), & s\in N[m_0-\tau,m_0], \end{cases} \tag{1}$$

for $m\geq m_0$, $i=1,2,\cdots,n$, $k=1,2,\cdots$, where n corresponds to the number of units in the neural network; $u_i(m)$ corresponds to the state of the ith unit at time m; f_j is the activation function; $\tau_{ij}(m)$ corresponds to the transmission delay along the axon of the jth unit from the ith unit and satisfies $0\leq\tau_{ij}(m)\leq\tau$ (τ is a nonnegative integer); $D=diag\{d_1,d_2,\cdots,d_n\}$ ($0\leq d_i\leq 1$), $A=(a_{ij})_{n\times n}$ and $B=(b_{ij})_{n\times n}$ are constant matrix. m_k are called impulsive moments and satisfy $0\leq m_1\leq m_2\leq\cdots$, $\lim_{k\to+\infty}m_k=+\infty$; $p_{ik}(u_1(m^-),\cdots,u_n(m^-))$ represents impulsive perturbations of the ith unit at time m_k; I_i and J_{ik} are constants.

If $p_{ik}(u_1,\cdots,u_n)=u_i$ and $J_{ik}=0$ $(i=1,2,\cdots,n;\quad k=1,2,\cdots)$ then model (1) turns to non-impulsive discrete-time CVNN with variable delays

$$u_i(m+1)=d_iu_i(m)+\sum_{j=1}^{n}a_{ij}f_j(u_j(m))+\sum_{j=1}^{n}b_{ij}f_j(u_j(m-\tau_{ij}(m)))+I_i \quad i=1,2,\cdots,n. \tag{2}$$

In stability analysis of model (1), we make the following assumptions:

(*H1*) If $(u_1^*,u_2^*,\cdots,u_n^*)^T$ is an equilibrium point of model (2), then the impulsive jumps of model (1) satisfy the following conditions $u_i^*=p_{ik}(u_1^*,u_2^*,\cdots,u_n^*)+J_{ik}$, $k=1,2,\cdots,i=1,2,\cdots,n.$

(*H2*) There exist a positive diagonal matrix $F=diag(F_1,F_2,\cdots,F_n)$ such that $|f_i(u_1)-f_i(u_2)|\leq F_i|u_1-u_2|$, for all $u_1,u_2\in C$, $i=1,2,\cdots,n$.

(*H3*) There exist nonnegative matrices $P_k=diag\{P_{1k},P_{2k},\cdots,P_{nk}\}$ such that $|p_{ik}(u_1,\cdots,u_n)-p_{ik}(v_1,\cdots,v_n)|\leq P_{ik}|u_i-v_i|$, for all $(u_1,\cdots,u_n)^T\in C^n$, $(v_1,\cdots,v_n)^T\in C^n$, $i=1,2,\cdots,n$, $k=1,2,\cdots$.

Main Result

Theorem 1: Under assumptions (*H1*)-(*H3*), model (1) has a unique equilibrium point, which is globally exponentially stable, if the following conditions are satisfied

(i) $W=E-D-(|A|+|B|)F$ is a non-singular M-matrix.

(ii) There exists a constant λ such that

$$\frac{\ln \gamma_k}{m_k - m_{k-1}} \le \lambda < \varepsilon, \quad k = 1,2,\cdots, \tag{3}$$

where

$$\gamma_k \ge \max\{1, P_{1k}, P_{2k}, \cdots, P_{nk}\}, \tag{4}$$

for $k = 1,2,\cdots$, and

$$-\xi_i(1-d_i) + \sum_{j=1}^{n} \xi_j F_j (|a_{ij}| + e^{\varepsilon\tau} |b_{ij}|) < 0, \tag{5}$$

for $i = 1,2,\cdots,n$, $\xi = (\xi_1, \xi_2, \cdots, \xi_n) > 0$ satisfies $W\xi > 0$.

Proof. Let $\varphi(u) = (\varphi_1(u), \varphi_2(u), \cdots, \varphi_n(u))^T$, where

$$\varphi_i(u) = -(1-d_i)u_i + \sum_{j=1}^{n} a_{ij} f_j(u_j) + \sum_{j=1}^{n} b_{ij} f_j(u_j) + I_i, \quad i = 1,2,\cdots,n.$$

In the following, we shall prove that $\varphi(u)$ is a homeomorphism of C^n onto itself.

First, we prove that $\varphi(u)$ is an injective map on C^n.

In fact, if there exist $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in R^n$ and $x \ne y$ such that $\varphi(x) = \varphi(y)$, then $(1-d_i)(x_i - y_i) = \sum_{j=1}^{n} (a_{ij} + b_{ij})(f_j(x_j) - f_j(y_j)) \quad i = 1,2,\cdots,n.$

It follows from (*H2*) that $(1-d_i)|x_i - y_i| \le \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) F_j |x_j - y_j|$ for $i = 1,2,\cdots,n.$ That is $W(|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|)^T \le 0.$

From W is an M-matrix, we can get that $x_i = y_i$, $i = 1,2,\cdots,n$, which is a contradiction. So $\varphi(u)$ is an injective on C^n.

Second, we prove that $\|\varphi(u)\| \to +\infty$ as $\|u\| \to +\infty$.

Let $\tilde{\varphi}(u) = (\tilde{\varphi}_1(u), \tilde{\varphi}_2(u), \cdots, \tilde{\varphi}_n(u))^T$, where $\tilde{\varphi}_i(u) = -(1-d_i)u_i + \sum_{j=1}^{n} (a_{ij} + b_{ij})(f_j(u_j) - f_j(0))$ for $i = 1,2,\cdots,n$. We have from

$$\begin{aligned} u^* \tilde{\varphi}(u) + \tilde{\varphi}^*(u) u &\le 2\sum_{i=1}^{n} \left(-(1-d_i)|u_i|^2 + \sum_{j=1}^{n} (|a_{ij}| + |b_{ij}|) F_j |u_i| \cdot |u_j| \right) \\ &= -2|u|^T W |u| \\ &\le -2\lambda_{\min}(W)\|u\|^2. \end{aligned}$$

When $\|u\| \ne 0$, we have $\|\tilde{\varphi}(u)\| \ge \lambda_{\min}(W)\|u\|$. Therefore $\|\tilde{\varphi}(u)\| \to +\infty$ as $u \to +\infty$, which implies $\|\varphi(u)\| \to +\infty$ as $u \to +\infty$. Thus $\varphi(u)$ is a homeomorphism of R^n to itself, which implies that model (2) has a unique equilibrium point $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^T$. From assumption (*H1*), we know that u^* is also a unique equilibrium point of model (1).

In the following, we will prove that this unique equilibrium point u^* of model (1) is globally exponentially stable. Let

$$y_i(m) = u_i(m) - u_i^*, \qquad \tilde{f}_j(y_j(m)) = f_j(y_j(m) + u_j^*) - f_j(u_j^*),$$

$$\tilde{p}_{ik}(y_1(m), \cdots, y_n(m)) = p_{ik}(y_1(m) + u_1^*, \cdots, y_n(m) + u_n^*) - p_{ik}(u_1^*, \cdots, u_n^*),$$

then model (1) can be rewritten as

$$\begin{cases} y_i(m+1) = d_i y_i(m) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(y_j(m)) + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(y_j(m-\tau_{ij}(m))), \quad m \neq m_k, \\ y_i(m) = \tilde{p}_{ij}(y_1(m^-), y_2(m^-), \cdots, y_n(m^-)), \quad m = m_k. \\ y_i(s) = \phi(s) - u_i^*, \quad s \in N[m_0 - \tau, m_0]. \end{cases} \tag{6}$$

It follows from (*H2*) that

$$|y_i(m+1)| \le d_i |y_i(m)| + \sum_{j=1}^{n} |a_{ij}| F_j |y_j(m)| + \sum_{j=1}^{n} |b_{ij}| F_j |y_j(m-\tau_{ij}(m))|, \quad m \neq m_k \tag{7}$$

for $i = 1,2,\cdots,n$, $k = 1,2,\cdots$.

Since W is an M-matrix, there exists a vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ such that $-\xi_i(1-d_i) + \sum_{j=1}^{n} \xi_j F_j (|a_{ij}| + |b_{ij}|) < 0$ for $i = 1,2,\cdots,n$. We can choose a small enough $\varepsilon > 0$ such that

$$-\xi_i(1-d_i) + \sum_{j=1}^{n} \xi_j F_j (|a_{ij}| + e^{\varepsilon\tau} |b_{ij}|) < 0 \tag{8}$$

for $i = 1,2,\cdots,n$. Let $x_i(m) = e^{\varepsilon(m-m_0)} |y_i(m)|, \quad i = 1,2,\cdots,n.$ Then, we have from inequality (7) that

$$\begin{aligned} x_i(m+1) &= e^{\varepsilon(m+1-m_0)} |y_i(m+1)| \\ &\le e^{\varepsilon} \left(d_i x_i(m) + \sum_{j=1}^{n} |a_{ij}| F_j x_j(m) + \sum_{j=1}^{n} e^{\varepsilon\tau} |b_{ij}| F_j x_j(m-\tau_{ij}(m)) \right) \end{aligned} \tag{9}$$

for $i = 1,2,\cdots,n$. Let $l_0 = \dfrac{\|\phi - u^*\|}{\min\limits_{1\le i\le n}\{\xi_i\}}$, then

$$x_i(s) \le |y_i(s)| = |u_i(s) - u_i^*| \le \|\phi - u^*\| \le \xi_i l_0 \tag{10}$$

for $s \in N[m_0 - \tau, m_0]$, $i = 1,2,\cdots,n$. In following, we prove that for any $i \in \{1,2,\cdots,n\}$, inequality

$$x_i(m) \le \xi_i l_0, \quad m \in N[m_0, m_1) \tag{11}$$

holds.

In fact, if inequality (11) is not true, then there exists some r and $m^* \in N[m_0, m_1)$ such that $x_r(m^*+1) > \xi_r l_0$, and $x_j(m) \ge \xi_j l_0$ for $m \in N[m_0 - \tau, m^*]$, $j = 1,2,\cdots,n.$

However, from inequality (8) and (10), we have

$$\begin{aligned} x_r(m^*+1) &\le e^{\varepsilon} \left(d_r x_r(m^*) + \sum_{j=1}^{n} |a_{rj}| F_j x_j(m^*) + \sum_{j=1}^{n} e^{\varepsilon\tau} |b_{rj}| F_j x_j(m^* - \tau_{rj}(m^*)) \right) \\ &\le e^{\varepsilon} \left(d_r \xi_r + \sum_{j=1}^{n} |a_{rj}| F_j \xi_j + \sum_{j=1}^{n} e^{\varepsilon\tau} |b_{rj}| F_j \xi_j \right) l_0 \\ &\le \xi_r l_0, \end{aligned}$$

this is a contradiction. So inequality (11) is true. Thus,

$$|y_i(m)| \le \xi_i l_0 e^{-\varepsilon(m-m_0)}, \quad m \in N[m_0, m_1) \tag{12}$$

for $i = 1,2,\cdots,n.$

In the following, we will use the mathematical induction to prove that

$$|y_i(m)| \le \gamma_0 \gamma_1 \cdots \gamma_{k-1} \xi_i l_0 e^{-\varepsilon(m-m_0)}, \quad m \in [m_{k-1}, m_k), \quad i = 1,2,\cdots,n \tag{13}$$

for $k = 1,2,\cdots,$ where $\gamma_0 = 1$.

When $k=1$, from inequality (12) we know that inequality (13) holds.
Suppose that the following inequalities

$$\left|y_i(m)\right| \le \gamma_0\gamma_1\cdots\gamma_{k-1}\xi_i l_0 e^{-\varepsilon(m-m_0)}, \qquad m\in N[m_{k-1},m_k), \quad i=1,2,\cdots,n \tag{14}$$

hold for $k=1,2,\cdots,h.$

From assumption ($H3$) and inequality (14), we know that the second equation of model (6) satisfies

$$\left|y_i(m_h)\right| \le P_{ih}\left|y_j(m_h^-)\right| \le P_{ih}\gamma_0\gamma_1\cdots\gamma_{h-1}\xi_i l_0 e^{-\varepsilon(m_h-m_0)} \qquad i=1,2,\cdots,n. \tag{15}$$

It follows from inequality (4) and (15) that

$$\left|y_i(m)\right| \le \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h\xi_i l_0 e^{-\varepsilon(m-m_0)}, \qquad m\in N[m_0-\tau,m_h], \quad i=1,2,\cdots,n$$

Thus

$$x_i(m) \le \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h\xi_i l_0, \qquad m\in N[m_0-\tau,m_h], \quad i=1,2,\cdots,n \tag{16}$$

In the following, we will prove that

$$x_i(m) \le \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h\xi_i l_0, \qquad m\in N[m_h,m_{h+1}), \quad i=1,2,\cdots,n \tag{17}$$

holds.

If inequality (17) is not true, then there exists some l and $m^{**}\in N[m_h,m_{h+1})$ such that $x_l(m^{**}+1) > \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h\xi_l l_0$ and $x_j(m) \le \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h\xi_j l_0$ for $m\in N[m_0-\tau,m^{**}]$, $j=1,2,\cdots,n$.

However, from inequality (8) and (9), we have

$$\begin{aligned} x_l(m^{**}+1) &\le e^{\varepsilon}\left(d_l x_l(m^{**}) + \sum_{j=1}^{n}\left|a_{lj}\right|F_j x_j(m^{**}) + \sum_{j=1}^{n} e^{\varepsilon\tau}\left|b_{lj}\right|F_j x_j\left(m^{**}-\tau_{lj}(m^{**})\right)\right) \\ &\le \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h e^{\varepsilon}\left(d_l\xi_l + \sum_{j=1}^{n}\left|a_{lj}\right|F_j\xi_j + \sum_{j=1}^{n} e^{\varepsilon\tau}\left(\left|\alpha_{lj}\right|+\left|\beta_{lj}\right|\right)F_j\xi_j\right)l_0 \\ &\le \gamma_0\gamma_1\cdots\gamma_{h-1}\gamma_h\xi_l l_0, \end{aligned}$$

this is a contradiction. So inequality (17) is hold.

By the mathematical induction, we can conclude that inequality (13) holds.

From inequality (3), (13) and the definition of l_0, we have

$$\left|y_i(m)\right| \le e^{\lambda(m_1-m_0)}e^{\lambda(m_2-m_1)}\cdots e^{\lambda(m_{k-1}-m_{k-2})}\xi_i l_0 e^{-\varepsilon(m-m_0)} \le \frac{\xi_i}{\min\limits_{1\le i\le n}\{\xi_i\}}\left\|\phi-u^*\right\|e^{-(\varepsilon-\lambda)(m-m_0)}$$

for $m\in N[m_{k-1},m_k)$, $i=1,2,\cdots,n$, $k=1,2,\cdots$. So $\left\|u(m)-u^*\right\| \le M\left\|\phi-u^*\right\|e^{-(\varepsilon-\lambda)(m-m_0)}$ for $m\in N[m_0,+\infty)$, where $M=\left(\sum_{j=1}^{n}\xi_j^2\right)^{\frac{1}{2}} \Big/ \min\limits_{1\le i\le n}\{\xi_i\}$. The proof is completed.

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References

[1] A. Hirose: *Complex-Valued Neural Networks: Theories and Applications* (World Scientific, Singapore 2003).

[2] A. Hirose: Electronics Letter Vol. 28 (1992), p. 1492.

[3] S. Jankowski, A. Lozowski and J. Zurada: IEEE Trans. Neural Netw. Vol. 7 (1996), p. 1491.

[4] D. Lee: IEEE Trans. Neural Netw. Vol. 12 (2001), p. 1260.

[5] J. Hu, J. Wang: IEEE Trans. Neural Netw. Learning Syst. Vol. 23 (2012), p. 853.

[6] B. Zhou, Q. Song: IEEE Trans. Neural Netw. Learning Syst. Vol. 24 (2013), p. 1227.

[7] Z. Zhang, C. Lin and B. Chen: IEEE Trans. Neural Netw. Learning Syst. Vol. 25 (2014), p. 1704.

[8] T. Fang, J. Sun: IEEE Trans. Neural Netw. Learning Syst. Vol. 25 (2014), p. 1709.

[9] X. Xu, J. Zhang and J. Shi: Neurocomputing Vol. 128 (2014), p. 483.

[10] V. Rao, G. Murthy: International Journal of Neural Systems Vol. 18 (2008), p. 165.

[11] W. Zhou, J. Zurada: IEEE Trans. Circuits Syst. II Vol. 56 (2009), p. 669.

[12] C. Duan, Q. Song: Discrete Dynamics in Nature and Society Vol. 2010 (2010), Article ID 368379.

[13] X. Chen, Q. Song, Y. Liu and Z. Zhao: Abstract and Applied Analysis Vol. 2014 (2014), Article ID 397532.