The Connectivity of Faulty Folded Hypercube Networks

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***Abstract*—The connectivity of a topology of a given network is one of the most important issues in determining whether a network can simulate routings of various lengths. In this paper, it shows that any two distinct nodes x, y are connected by paths of every length from $d(x,y)$ to $2^n - 1$ in folded hypercube networks $FQ_n - F_e$ with $|F_e| \leq n-1$ if each node is incident with at least $m\,(2 \leq m \leq n)$ fault-free links, where $d(x,y)$ is the distance between x, y, and F_e is the set of fault links.**

Keywords-network connectivity; folded hypercube networks

I. Introduction

The n-dimensional hypercube Q_n (or n-cube) is one of the most important topology of networks duo to its excellent properties such as regularity, recursive structure, small diameter, vertex and edge transitive and relatively short mean distance (Xu, J.M. 2003). In order to improve the performance of hypercube, the folded hypercube FQ_n has been proposed (Saad, Y. & Schultz, M.H. 1988).

In a large-scale hypercube network, should any component fail, it's desirable that the rest of the network continue to operate in spite of the failure. This leads to the graph-embedding problem with faulty edges and/or vertices. This problem has received much attention (Fu, J.S. 2006)-(Xu, J.M. & Ma, M.J. 2006).

A graph G with at least three vertices is pancyclic if there exist cycles of all lengths from 3 to $|V(G)|$. A graph is edge-pancyclicity (vertex-pancyclicity) if each edge (vertex) is contained in cycles of every length from 3 to $|V(G)|$. Further, a graph G is bipancyclicity if G is a bipartite graph whose cycles are necessarily of even length. A bipartite graph G is bipanconnected if G contains a path of length l connecting any distinct vertices x, y with $d(x,y) \leq l \leq |V(G)|-1$ such that $2\,|\,(l - d(x,y))$, where $d(x,y)$ denotes the shortest path between x and y. In this paper, we primarily explore the bipanconnection of a conditional faulty folded hypercube.

The problem of embedding paths in an n-dimensional hypercube and folded hypercube has been well studied. (Fu, J.S. 2006) showed that $Q_n (n \geq 3)$ with faulty vertices $|F_v| \leq n-2$ contains a path joining any two different vertices x and y in $Q_n - F_v$ with length of at least $2^n - 2|F_v| - 1$ (or $2^n - 2|F_v| - 2$) when $d(x,y)$ is odd (or even). (Hsieh, S.Y. et al. 2009) proved that if the folded hypercube FQ_n has just only one fault node, then FQ_n contains cycles of every even length from 4 to $2^n - 2$ when n is even, and cycles of every odd length from $n+1$ to $2^n - 1$ when n is odd. (Liu, M. & Liu, H.M. 2012) addressed that there exists a cycle passing through all nodes in $FQ_n - F_e$ with the number of faulty edges $|F_e| \leq n-1$ when n is odd. (Ma, M.J. et al. 2006) further demonstrated that $FQ_n - F_e (n \geq 3)$ with $|F_e| \leq 2n-3$ contains a fault-free cycle passing through all nodes if each vertex is incident with at least two fault-free edges. (Kuo, C.N. & Hsieh, S.Y. 2010) improved the conclusion of (Ma, M.J. et al. 2006) and proved that $FQ_n - F_e$ with $|F_e| = 2n-3$ contains a fault-free cycle of every even length from 4 to 2^n. (Kuo, C.N. et al. 2013) discussed the fault tolerance properties of FQ_n with both fault vertices and fault edges occurring and obtained that (i) FQ_n contains a fault-free path of length at least $2^n - 2|F_v| - 1$ (resp. $2^n - 2|F_v| - 2$) between any two fault-free nodes of odd (resp. even) distance if $|F_v| + |F_e| \leq n-1$ when n is odd, (ii) FQ_n contains a fault-free path of length at least $2^n - 2|F_v| - 1$ between any two fault-free nodes if $|F_v| + |F_e| \leq n-2$ when n is even.

In this paper, under the conditional fault model, we show that every pair of vertices x and y with distance $d(x,y)$ in $FQ_n - F_e (n \geq 3)$ with $|F_e| \leq n-1$ are connected by paths of every length from $d(x,y)$ to $2^n - 1$ if each vertex is linked with at least m $(2 \leq m \leq n)$ fault-free edges.

II. Preliminaries

We follow (Xu, J.M. 2003) for graph-theoretical terminology and notation not defined here. A network is usually modelled by a simple connected graph $G = (V, E)$, where

$V = V(G)$ (or $E = E(G)$) is the set of vertices(or edges) of G. We define the vertex x to be a neighbor of y if $xy \in E(G)$. A graph G is bipartite if X, Y are two disjoint subsets of $V(G)$ such that $E(G) = \{xy \mid x \in X, y \in Y\}$. A graph $P = (u_1, u_2, \cdots, u_k)$ is called a path if the vertices $u_1, u_2, \cdots, u_k$ are distinct and any two consecutive vertices u_i and u_{i+1} are adjacent. u_1 and u_k are called the end-vertices of P. If $u_1 = u_k$, the path $P(u_1, u_k)$ is called a cycle (denoted by C). The length of a path P (a cycle C), denoted by $l(P)$ (or $l(C)$), is the number of edges in P (or C). In general, the distance of two vertices x, y is the length of the shortest (x, y)-path.

The n-dimensional hypercube Q_n (or, n-cube) can be represented as an undirected graph with 2^n vertices. Every vertex $x \in Q_n$ is labeled as a binary string $x_1 x_2 \cdots x_n$ of length n from $00\cdots0$ to $11\cdots1$. Two vertices u and v are adjacent if their binary strings differ in exactly one bit. For convenience, we call $e \in E(Q_n)$ an edge of dimension i if its end-vertices' strings differ in i th-bit. In the rest of this paper, denote $x^i = x_1 x_2 \cdots \overline{x_i} \cdots x_n$, $\overline{x_i} = 1 - x_i$, $x_i \in \{0,1\}$. The Hamming distance of two vertices $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$ is as

$$H(x, y) = \sum_{i=1}^{n} | x_i - y_i |$$

The number of different bits between them. Let $d(x, y)$ be the shortest distance of x and y. Note that Q_n is a bipartite graph, and for any two distinct vertices x, y of Q_n, $d(x, y) = H(x, y)$.

As a variant of hypercube, the n-dimensional folded hypercube FQ_n is obtained by adding more edges between its vertices.

Definition 1. The n-dimensional folded hypercube FQ_n is a graph with $V(FQ_n) = V(Q_n)$. Two vertices $x = x_1 x_2 \cdots x_n$ and y is connected by an edge if and only if

(i) $y = x_1 x_2 \cdots \overline{x_i} \cdots x_n$ (denoted by x^i), or

(ii) $y = \overline{x_1}\,\overline{x_2} \cdots \overline{x_i} \cdots \overline{x_n}$ (denoted by $\overline{x}$).

Therefore, the hypercube Q_n is a spanning subgraph of the folded hypercube FQ_n obtained by removing the second type of edges $x\overline{x} (x \in V(FQ_n))$, called complementary edges of FQ_n and denoted by $E_c = \{x\overline{x} \mid x \in V(FQ_n)\}$.

The first type of edges are defined to be the hypercube edges, and denoted by $E_i = \{xx^i\}, i = 1, 2, \cdots, n$.

The following results are useful in the proof of our method.

Lemma 1. $FQ_n - E_i$ is isomorphic to Q_n for any $i \in \{1, 2, \cdots, n, c\}$

Lemma 2 (Sun, C.M. 2012). Each edge of $Q_n - F_e$ with $|F_e| \le n-1$ is contained in cycles of every even length from 4 to 2^n.

Lemma 3 (Fu, J.S. 2006). Each edge of FQ_n is contained in every even cycle of length from 4 to 2^n if n is odd.

Lemma 4 (Fu, J.S. 2002). For any two vertices $u, v \in Q_n$, if $d(u,v) = d$, then there exists a path connecting u, v of length as $d, d+2, \cdots, c$, where $c = 2^n - 1$ (or $2^n - 2$) for d is odd (or even).

Lemma 5 (Saad, Y. & Schultz, M.H. 1988). For any two vertices $u, v \in Q_n$, if $d(u,v) = k$, then there are n internal disjoint paths from u and v such that there are k paths of length k and $n-k$ paths of length $k+2$.

Lemma 6 (Sun, C.M. 2012). For any two distinct vertices x and y of Q_n with at most $n-2$ faulty edges, there exists a non-faulty (x, y)-path in Q_n of length l such that

if $|F_e| < d(x, y)$, then

$d(x, y) \le l \le 2^n - 1$ and $2 \mid (l - d(x, y))$;

if $|F_e| \ge d(x, y)$, then

$d(x, y) + 2 \le l \le 2^n - 1$ and $2 \mid (l - d(x, y))$.

Lemma 7 (Sun, C.M. 2012). There exist exactly $n-1$ disjoint cycles in Q_n of length 4 that contain an edge xy in common.

By Lemma 7, we can immediately induct the following conclusion.

Lemma 8. There exist n disjoint cycles in an n-dimensional folded hypercube FQ_n of length 4 that contain an edge in common.

Lemma 9(Xu, J.M. & Ma, M.J. 2006). Let x, y be any two vertices of n-dimensional folded hypercube. If $H(x, y) = k$, then there exist $n+1$ internal vertex-disjoint paths between x and y, among which there are k paths of length k and $n-k+1$ paths of length $k+2$ when

$1 \le k \le \lceil n/2 \rceil$, or k paths of length $n-k+3$ and $n-k+1$ paths of length $n-k+1$ when $\lceil n/2 \rceil \le k \le n$.

Lemma 10. n-dimensional folded hypercube FQ_n can be portioned into two subgraphs, denoted by FQ_n^0 and FQ_n^1, which are $(n-1)$-cubes along any dimension i such that for any $x = x_1x_2 \cdots x_i \cdots x_n \in FQ_n^0$ satisfying $x_i = 0$. $x = x_1x_2 \cdots x_i \cdots x_n \in FQ_n^1$ satisfying $x_i = 1$.

III. THE BIPANCONNECTIVITY OF FOLDED HYPERCUBE

Theorem. For any two distinct vertices x, y in FQ_n ($n \ge 3$) with $|F_e| \le n-1$, when n is an odd integer, there exists a fault-free (x, y)-path of length l such that

if $|F_e| < d(x, y)$, then

$d(x, y) \le l \le 2^n - 1$ and $2 \mid (l - d(x, y))$;

if $|F_e| \ge d(x, y)$, then

$d(x, y) + 2 \le l \le 2^n - 1$ and $2 \mid (l - d(x, y))$.

Proof. Notice that for any two distinct vertices x, y in FQ_n , if $d(x, y) \le \lceil n/2 \rceil$, then $d(x, y) = H(x, y)$; if $\lceil n/2 \rceil < H(x, y) \le n$, $d(x, y) = n - H(x, y) + 1$. Then we consider the cases when $d(x, y) = H(x, y)$ and $d(x, y) = n - H(x, y) + 1$.

Case 1. $d(x, y) = H(x, y)$

By Lemma 10, FQ_n can be partitioned into two $(n-1)$-cubes along some dimension i such that the vertices x, y are in the same $(n-1)$-cube and $|E_i \cap F_e| + |E_c \cap F_e| \ge 1$. Without loss of generality, assume that $x, y \in FQ_n^0$.

Case1.1.

$$f_0 = |F_e \cap E(FQ_n^0)| = n-2, \ f_1 = |F_e \cap E(FQ_n^1)| = 0,$$

$$|(E_i \cup E_c) \cap F_e| = 1$$

Select a faulty edge e and regard e as fault-free, then $f_0 - 1 = n - 3$. With $d(x, y) = H(x, y)$, in FQ_n^0, there exists a (x, y)-path P of length l from $d(x, y) + 2$ to $2^{n-1} - 1$ and $2 \mid (l - d(x, y))$. Next, we need to consider the case whether the faulty edge e is in any fault-free (x, y)-path or not.

(1) If this faulty edge $e = uv$ is in some (x, y)-path $P \subseteq FQ_n^0$ of length l.

Since $f_1 = 0$ and $|(E_i \cup E_c) \cap F_e| = 1$, then either the edges $u\bar{u}, v\bar{v}, \bar{u}\bar{v}$ or uu^i, vv^i, u^iv^i are fault-free. Without loss of generality, say, $u\bar{u}, v\bar{v}, \bar{u}\bar{v}$ are fault-free. Lemma 4 and $d(\bar{u}, \bar{v}) = 1$ guarantees fault-free $(\bar{u}, \bar{v})$-path $P_1 \subseteq FQ_n^1$ of every odd length $l_1 \in \{1, 3, \cdots, 2^{n-1} - 1\}$. $P^* = (P - uv) \cup u\bar{u} \cup P_1 \cup \bar{v}v$ is a fault-free path connecting x, y with length of $l^* = (l-1) + 1 + l_1 + 1 = l + l_1 + 1$, $d(x, y) + 4 \le l^* \le 2^n - 1$, $2 \mid (l^* - d(x, y))$.

Now, we consider a (x, y)-path of length $d(x, y) + 2$. With Lemma 5, there are $n-1$ internal disjoint paths between x and y in FQ_n^0, if every one of these $n-1$ paths contains at least one faulty edge, then there are at least $n-1$ fault edges in FQ_n^0, but $f_0 = n-2$. This means that there exists a fault-free path of length $d(x, y) + 2$.

(2) If this faulty edge $e = uv$ is not in any (x, y)-path.

That is, there exists a fault-free (x, y)-path $P \subseteq FQ_n^0$ of length l , where $d(x, y) \le l \le 2^{n-1} - 1$ and $2 \mid (l - d(x, y))$. Then we only need to find those fault-free (x, y)-paths of length l^* which has the same parity as $d(x, y)$ and ranges from 2^{n-1} to $2^n - 1$. Choose an edge ab in one of the shortest fault-free (x, y)-path $P \subseteq FQ_n^0$ of length $2^{n-1} - 1$ when $d(x, y)$ is odd (or $2^{n-1} - 2$ when $d(x, y)$ is even) such that the edges aa^i, bb^i are non-faulty. Because $f_1 = 0$, the edge a^ib^i is fault-free in FQ_n^1, and by Lemma 4, we can also construct a fault-free (a^i, b^i) -path $P_1 \subseteq FQ_n^1$ of every odd length l_1 , $l_1 \in \{1, 3, \cdots, 2^{n-1} - 1\}$. Thus the desired (x, y)-path P^* of length l^* in FQ_n is constructed as $P^* = (P_1 - ab) \cup aa^i \cup P_2 \cup b^ib$ with length of l^*.

If $d(x, y)$ is odd, $l^* = 2^{n-1} + l_1$, and $2^{n-1} + 1 \le l^* \le 2^n - 1$, where $2 \mid (l^* - d(x, y))$.

If $d(x, y)$ is even, $l^* = 2^{n-1} + l_1 - 1$, and $2^{n-1} \le l^* \le 2^n - 2$, where $2 \mid (l^* - d(x, y))$.

Case1.2.

$$f_0 = |F_e \cap E(FQ_n^0)| \le n-3, \ f_1 = |F_e \cap E(FQ_n^1)| \le n-3.$$

Without loss of generality, suppose that $f_0 \ge f_1$. If $f_0 < d(x,y) = H(x,y)$, by Lemma 6, there exists a fault-free path (x,y)-path P_1 of length l_1, where $d(x,y) \le l_1 \le 2^{n-1}-1$ and $2 \mid (l_1 - d(x,y))$. Choose an edge ab in some fault-free (x,y)-path of length $2^n - 1$. By Lemma 8, the edges aa^i, bb^i are fault-free. $H(a^i,b^i)=1$ and Lemma 6 guarantee that there being a fault-free (a^i,b^i)-path P_2 of length l_2, where $H(a^i,b^i) \le l_2 \le 2^{n-1}-1, 2 \mid (l_2 - H(a^i,b^i))$.

Thus $(P_1 - ab) \cup aa^i \cup P_2 \cup b^i b$ is a fault-free (x,y)-path of length $l_1 - 1 + 1 + l_2 + 1 = l_1 + l_2 + 1$.

On the other hand, if $f_0 \ge d(x,y)$, by Lemma 6, there are fault-free (x,y)-paths of length from $d(x,y)+2$ to $2^{n-1}-1$. The method of constructing such a fault-free (x,y)-path is similar to the case of $f_0 < d(x,y)$.

In particular, when $f_0 = f_1 = 0$, that is, all faulty edges are in $E_i \cap E_c$, by Lemma 1 and Lemma 6, the theorem is true .

Case 2. $d(x,y) \ne H(x,y)$.

By definition of FQ_n, $d(x,y) \ne H(x,y)$ implies that $\lceil n/2 \rceil < H(x,y) \le n$ and $d(x,y) = n - H(x,y) + 1$. The shortest path connecting x and y contains a complementary edge. Now, we can partition FQ_n into two subcubes such that the nodes x, y are not in the same subcube and $|E_i \cap F_e| + |E_c \cap F_e| \ge 1$. Suppose that $x \in FQ_n^0$ and $y \in FQ_n^1$. There are several cases need to be discussed.

Case 2.1. $f_0 = n-2, f_1 = 0$.

Choose a non-faulty edge $ax \in FQ_n^0$ such that the edge $a\bar{a}$ is fault-free (if $x\bar{x}$ is a non-fault link, $x = a$ is feasible), $d(\bar{a},y) = d(x,y) - 2$ if ax is on a shortest (x,y)-path, and $d(\bar{a},y) = d(x,y)$ if ax is not on a shortest (x,y)-path. Note that $f_1 = 0$ implies that FQ_n^1 is fault-free. By Lemma 4, we can find a fault-free $(\bar{a},y)$-path P_1 of length l_1, $d(x,y)+2 \le l_1 \le 2^{n-1}-1$ (or $d(x,y) \le l_1 \le 2^{n-1}-1$), l_1 and $d(x,y)$ have the same parity. Select a fault-free (x,y)-path $P = xa \cup a\bar{a} \cup P_1$ with length of $l = l_1 + 2$. Since $d(x,y) = n - H(x,y) + 1$, then $d(x,y) \le l_1 \le 2^{n-1}-1$ (or $d(x,y)+2 \le l_1 \le 2^{n-1}-1$) and $2 \mid (l - d(x,y))$. Similar to Case 1.1, those desired fault-free (x,y)-paths of length from $2^{n-1}+3$ to $2^n - 1$ can be constructed.

Case 2.2. $f_0 \le n-3,\ f_1 \le n-3$.

Because $f_0 \le n-3$, choose a neighbor a of x in FQ_n^0 such that $ax, a\bar{a}$ are fault-free. Let $xa \cup a\bar{a} \cup P[\bar{a},y]$ be a fault-free (x,y)-path, where $P[\bar{a},y] \in FQ_n^1$ is a path connecting $\bar{a}$ and y.

If $|F_e| < d(x,y)$, we have $f_1 < H(\bar{a},y)$ because $|E_i \cap F_e| + |E_c \cap F_e| \ge 1$ and $d(x,y) = 2 + H(\bar{a},y)$. By Lemma 6, there exists a fault-free $(\bar{a},y)$-path of length from $H(\bar{a},y)$ to $2^{n-1}-1$. Thus the desired paths, whose lengths are from $d(x,y)$ to $2^{n-1}-1$, is constructed as $xa \cup a\bar{a} \cup P[\bar{a},y]$.

If $|F_e| \ge d(x,y)$, by Lemma 8, there exists a shortest path P of length $d(x,y)+2$ between x and y in FQ_n. Therefore, we only need to find fault-free paths of length from $2^{n-1}+3$ to 2^n-1. By Lemma 6 and $d(x,y) = H(x,a) = 1$, there are fault-free (x,a)-paths $P[x,a]$ of length l' from $d(x,y)+2$ to $2^{n-1}-1$. Choose a fault-free $(\bar{a},y)$-path $P[\bar{a},y]$ of length $2^{n-1}-1$. Then set $P^* = P[x,a] \cup a\bar{a} \cup P[\bar{a},y]$ to be a new fault-free path of length l^* from $2^{n-1}+3$ to $2^n - 1$, where $2 \mid (l^* - d(x,y))$ .The proof is finished.

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