"A COMPARITIVE STUDY OF MATRIX INVERSION BY RECURSIVE ALGORITHMS THROUGH SINGLE AND DOUBLE BORDERING"

Prof. I. Ramabhadra Sarma [1] **Dr. B. Rami Reddy** [2]

[1&2] Depts. (P.G.&U.G.) Of Mathematics, Hindu College,Guntur – 522 002, Andhra Pradesh, INDIA.
Ph: 0863 – 350818 (Res) E-Mail: rbhavanamr@rediffmail.com

"ABSTRACT"

We present recursive algorithms for finding the inverse of any nonsingular square matrix and in particular a positive definite symmetric (PDS) matrix through single bordering and double bordering and obtain the operational counts for these algorithms comparing them with the existing methods, we ultimately establish that inversion by our double bordering is much superior to Cholesky and single bordering methods in the case of PDS and CSP (centrosymmetric, symmetric and positive definite) matrices.

Keywords: *centrosymmetric matrix, bordering, operational count.*

1 Introduction

Single bordering of a matrix is a partition along a row and a column on the border. In the case of a (PDS) matrix of order n the resulting submatrix of order (n-1) by single bordering remains PDS. Using this property we derive recursion methods for obtaining the inverse of a given PDS matrix by finding the inverses lower order matrices. Single bordering does not inherit the property of centrosymmetriy for submatrices. However, if we remove the first and last row as well as the first and last columns of a centrosymmetric (symmetric) matrix, the resultant submatrix of order (n-2) remains centrosymmetric (symmetric). This type of partition called **double bordering** also provides recursive methods for finding the inversion of submatrices starting from the center.

Motivated by these ideas we make detailed investigations of inversion methods involving single and double bordering for inversion of PDS and CSP matrices. When a square matrix A is PDS, all the leading minors are PDS and hence all these are nonsingular. Thus the inversion of A is guaranteed. For computational purposes, all the entries of a PDS matrix need not be stored. Further, the operational count reduces substantially. If A is a nonsingular matrix of order n, then there exists an elementary matrix B such that all the the leading principal minors of AB are nonsingular.

2 Single bordering

2.1 Inversion through single bordering of a general nonsingular matrix

Theorem 2.1.1 If A_1 is a nonsingular square matrix of order $(n-1)$ and $A = \begin{bmatrix} A_1 & a \\ b^T & a_{nn} \end{bmatrix}$, then A is nonsingular if and only if $(a_{nn} - b^T A_1^{-1} a)$ is nonzero. In this case

$$A^{-1} = \begin{bmatrix} X & x \\ y^T & x_{nn} \end{bmatrix} \quad \text{Where } X = A_1^{-1}(I - a\,y^T) \;,$$

$x_{nn} = (a_{nn} - b^T A_1^{-1} a)^{-1}$, $x = -A_1^{-1} a\, x_{nn}$, $y^T = -x_{nn}\, b^T A_1^{-1}$.

Corollary 2.1.2 If A is a symmetric matrix of order n and $A = \begin{bmatrix} A_1 & a \\ a^T & a_{nn} \end{bmatrix}$ where A_1 is nonsingular and is of order $(n-1)$, then A is nonsingular iff $(a_{nn} - a^T A_1^{-1} a)$ is nonzero. In this case $A^{-1} = \begin{bmatrix} X & x \\ x^T & x_{nn} \end{bmatrix}$ where, $x = -A_1^{-1} a\, x_{nn}$ and

$$X = A_1^{-1}(I - a\, x^T).$$

2.1.3 An algorithm for computation of A^{-1} when all the leading principal minors of A are invertible.

Let A be a nonsingular matrix of order n whose leading principal minors are all non-singular. Let $A = (a_{ij})$. Define the column vectors a_i and b_i as

$a_i = (a_{1,i+1} \quad a_{2,i+1} \quad \ldots \quad a_{i,i+1})^T$ and

$b_i = (a_{i+1,1} \quad a_{i+1,2} \quad \ldots \quad a_{i+1,i})^T$, $1 \le i \le n-1$

Step 1: $A_1 = (a_{11}) \;\Rightarrow\; A_1^{-1} = (a_{11}^{-1})$

Step i+1: For $i = 1,2,\ldots n-1$, define

$$A_{i+1} = \begin{bmatrix} A_i & a_i \\ b_i^T & a_{i+1,i+1} \end{bmatrix}.$$

Compute $x_{i+1,i+1} = (a_{i+1,i+1} - b_i^T A_i^{-1} a_i)^{-1}$, $y_i^T = -x_{i+1,i+1}\, b_i^T A_i^{-1}$, $x_i = A_i^{-1} a_i\, x_{i+1,i+1}$, $X_i = A_i^{-1}(I - a_i\, y_i^T)$.

Then, $A_{i+1}^{-1} = \begin{bmatrix} X_i & x_i \\ y_i^T & x_{i+1,i+1} \end{bmatrix}$.

2.1.4 Operational Count for A^{-1} when A is a general matrix

The total number S of arithmetic operations required for computation of A_n^{-1} recursively is given by

$$S = n^3 + n(n+1)^2$$

2.2 A recursive algorithm for finding A^{-1} when A is PDS matrix of order n.

Step 1: $A_1 = (a_{11}) \Rightarrow A_1^{-1} = (a_{11}^{-1})$.

Step i: For i = 2,3, . . . n, define $A_i = \begin{bmatrix} A_{i-1} & a_{i-1} \\ a_{i-1}^T & a_{ii} \end{bmatrix}$,

where

$a_{i-1} = (a_{i,1}, a_{i,2}, \ldots, a_{i,i-1})^T$.

Compute $p_{i-1} = A_{i-1}^{-1} a_{i-1}$, $b_{i-1} = a_{i-1}^T A_{i-1}^{-1} a_{i-1}$,

$x_{ii} = (a_{ii} - b_{i-1})^{-1}$ $x_{i-1} = (-x_{ii}) A_{i-1}^{-1} a_{i-1}$,

$B_{i-1} = (A_{i-1}^{-1} a_{i-1}) x_{i-1}^T$, $X_{i-1} = (A_{i-1}^{-1} - B_{i-1})$.

Since $(A_{i-1}^{-1})^T = A_{i-1}^{-1}$.

Then, $A_i^{-1} = \begin{bmatrix} X_{i-1} & x_{i-1} \\ x_{i-1}^T & x_{ii} \end{bmatrix}$ $2 \le i \le n$.

Finally, $A^{-1} = A_n^{-1}$.

2.2.1 Operational Count for A^{-1}

The total number S of arithmetic operations Required for computation of A_n^{-1} recursively is given by

$$S = \frac{n(n+1)(4n-7)}{3} + 3n.$$

3 Double bordering

The double bordered form of a matrix of order n is given by [1]

$$R = \begin{bmatrix} r_{11} & a^T & r_{1n} \\ b & M & c \\ r_{n1} & d^T & r_{nn} \end{bmatrix} \quad (2.1)$$

Where r_{11}, r_{1n}, r_{n1} and r_{nn} are scalars.

In particular when R is a CSP matrix,

$$R = \begin{bmatrix} r_{11} & a^T & r_{1n} \\ a & M & Ja \\ r_{1n} & a^T J & r_{11} \end{bmatrix} \quad (2.2)$$

Theorem 3.1.1 When R has the form (2.1) and M is nonsingular, then R is nonsingular iff

$H = \begin{bmatrix} r_{11} - a^T M^{-1} b & r_{1n} - a^T M^{-1} c \\ r_{n1} - d^T M^{-1} b & r_{nn} - d^T M^{-1} c \end{bmatrix}$ is nonsingular. In this case, the inverse of R is given by

$$R^{-1} = S = \begin{bmatrix} s_{11} & x^T & s_{1n} \\ y & B & u \\ s_{n1} & v^T & s_{nn} \end{bmatrix}$$

Where $\begin{bmatrix} s_{11} & s_{1n} \\ s_{n1} & s_{nn} \end{bmatrix} = H^{-1}$

$$(y \quad u) = -M^{-1}(s_{11} b + s_{n1} c \quad s_{1n} b + s_{nn} c)$$

$$\begin{bmatrix} x^T \\ v^T \end{bmatrix} = \begin{bmatrix} (s_{11} a^T + s_{1n} d^T) M^{-1} \\ (s_{n1} a^T + s_{nn} d^T) M^{-1} \end{bmatrix} = -H^{-1} \begin{bmatrix} a^T \\ d^T \end{bmatrix} M^{-1}$$

and $B = M^{-1}\left(I - b\,x^T - c\,v^T\right)$

3.1.2 Recursive algorithm for finding R_{2n+1}^{-1} by double bordering

Let $R = (r_{i,j})$ be the given matrix of order 2n+1.

We assume that the central element $r_{n+1,n+1} \neq 0$.

Step 1: Let $R_1 = (r_{n+1,n+1}) \Rightarrow R_1^{-1} = (r_{n+1,n+1})^{-1}$.

Step i+1: For i = 1, 2. . . n ,define

$R_{2i+1} = \begin{bmatrix} r_{n+1-i,n+1-i} & a_{2i-1}^T & r_{n+1-i,n+1+i} \\ b_{2i-1} & R_{2i-1} & c_{2i-1} \\ r_{n+1+i,n+1-i} & d_{2i-1}^T & r_{n+1+i,n+1+i} \end{bmatrix}$ where,

$M = R_{2i-1}$ and

$$a_{2i-1} = \begin{bmatrix} r_{n+1-i,n+2-i} \\ r_{n+1-i,n+3-i} \\ \cdots \\ r_{n+1-i,n+i} \end{bmatrix}, b_{2i-1} = \begin{bmatrix} r_{n+2-i,n+1-i} \\ r_{n+3-i,n+1-i} \\ \cdots \\ r_{n+i,n+1-i} \end{bmatrix},$$

$$c_{2i-1} = \begin{bmatrix} r_{n+2-i,n+1+i} \\ r_{n+3-i,n+1+i} \\ \cdots \\ r_{n+i,n+1+i} \end{bmatrix}, \quad d_{2i-1} = \begin{bmatrix} r_{n+1+i,n+2-i} \\ r_{n+1+i,n+3-i} \\ \cdots \\ r_{n+1+i,n+i} \end{bmatrix}.$$

It may be observed that R_3 is the innermost 3×3 matrix (at the center), R_5 is the innermost 5×5 matrix,etc.

Compute $p_{2i-1} = R_{2i-1}^{-1} b_{2i-1}$, $q_{2i-1} = R_{2i-1}^{-1} c_{2i-1}$, Compute $t_1 = a_{2i-1}^T p_{2i-1}$, $t_2 = a_{2i-1}^T q_{2i-1}$, $t_3 = d_{2i-1}^T p_{2i-1}$, $t_4 = d_{2i-1}^T q_{2i-1}$.

Compute $h_1 = r_{n+1-i,n+1-i} - t_1$,

$h_2 = r_{n+1-i,n+1+i} - t_2$, $h_3 = r_{n+1+i,n+1-i} - t_3$, and

$h_4 = r_{n+1+i,n+1+i} - t_4$.

Then $H = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$.

Compute

$$H^{-1}\begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}^{-1} = \begin{bmatrix} s_{n+1-i,n+1-i} & s_{n+1-i,n+1+i} \\ s_{n+1+i,n+1-i} & s_{n+1+i,n+1+i} \end{bmatrix}.$$

Compute

$$\left(y_{2i-1} \quad u_{2i-1}\right) = -R_{2i-1}^{-1}\left(s_{11}b + s_{n1}c \quad s_{1n}b + s_{nn}c\right)$$

Compute $\begin{bmatrix} x_{2i-1}^T \\ v_{2i-1}^T \end{bmatrix} = -\begin{bmatrix} (s_{11}a^T + s_{1n}d^T)M^{-1} \\ (s_{n1}a^T + s_{nn}d^T)M^{-1} \end{bmatrix}$.

Compute $B_{2i-1} = R_{2i-1}^{-1}\left(I - b_{2i-1}x_{2i-1}^T - c_{2i-1}v_{2i-1}^T\right)$

Then, $R_{2i+1}^{-1} = \begin{bmatrix} s_{n+1-i,n+1-i} & x_{2i-1}^T & s_{n+1-i,n+1+i} \\ y_{2i-1} & B_{2i-1} & u_{2i-1} \\ s_{n+1+i,n+1-i} & v_{2i-1}^T & s_{n+1+i,n+1+i} \end{bmatrix}$.

For $i = n$, we obtain

$$R^{-1} = R_{2n+1}^{-1} = \begin{bmatrix} s_{11} & x_{2n-1}^T & s_{1,2n+1} \\ y_{2n-1} & B_{2n-1} & u_{2n-1} \\ s_{n1} & v_{2n-1}^T & s_{nn} \end{bmatrix}.$$

3.13 Operational count for finding R^{-1}

The total number S of arithmetic operations required for finding R^{-1} is given by

Case (i): When R is a matrix of order 2n+1,

$S = 16n^3 + 12n^2 + 7n + 1$.

Case (ii): When R is a matrix of order 2n,

$S = 16n^3 - 12n^2 + 7n$.

3.2 Inversion of a PDS matrix

When R is a PDS matrix of order n, then we deduce from Theorem 3.1.1

$$R^{-1} = S = \begin{bmatrix} s_{11} & x^T & s_{1n} \\ x & B & u \\ s_{1n} & u^T & s_{nn} \end{bmatrix} \text{ where}$$

$$H = \begin{bmatrix} r_{11} - a^T M^{-1} a & r_{1n} - a^T M^{-1} c \\ r_{1n} - c^T M^{-1} a & r_{nn} - c^T M^{-1} c \end{bmatrix},$$

$$H^{-1} = \begin{bmatrix} s_{11} & s_{1n} \\ s_{1n} & s_{nn} \end{bmatrix}, \ (x \quad u) = -M^{-1}(a \quad c)H^{-1},$$

$$B = M^{-1}\left(I - ax^T - cu^T\right).$$

3.2.1 Recursive algorithm for finding R^{-1}

Step 1: Let $R_1 = (r_{n+1,n+1}) \Rightarrow R_1^{-1} = (r_{n+1,n+1})^{-1}$.

Step i+1: For i = 1, 2. . . n ,define

$$R_{2i+1} = \begin{bmatrix} r_{n+1-i,n+1-i} & a_{2i-1}^T & r_{n+1-i,n+1+i} \\ a_{2i-1} & R_{2i-1} & c_{2i-1} \\ r_{n+1+i,n+1-i} & c_{2i-1}^T & r_{n+1+i,n+1+i} \end{bmatrix}$$

$$a_{2i-1} = \begin{bmatrix} r_{n+1-i,n+2-i} \\ r_{n+1-i,n+3-i} \\ \cdots \\ r_{n+1-i,n+i} \end{bmatrix}, \quad c_{2i-1} = \begin{bmatrix} r_{n+2-i,n+1+i} \\ r_{n+3-i,n+1+i} \\ \cdots \\ r_{n+i,n+1+i} \end{bmatrix}$$

Compute $p_{2i-1} = R_{2i-1}^{-1} b_{2i-1}$, $q_{2i-1} = R_{2i-1}^{-1} c_{2i-1}$,

Compute $t_1 = a_{2i-1}^T p_{2i-1}$, $t_2 = t_3 = a_{2i-1}^T q_{2i-1}$, $t_4 = c_{2i-1}^T q_{2i-1}$.

Compute $h_1 = r_{n+1-i,n+1-i} - t_1$,

$h_2 = h_3 = r_{n+1-i,n+1+i} - t_2$, and

$h_4 = r_{n+1+i,n+1+i} - t_4$.

Write $H = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_4 \end{bmatrix}$.

Compute

$$H^{-1} = \begin{bmatrix} h_1 & h_2 \\ h_2 & h_4 \end{bmatrix}^{-1} = \begin{bmatrix} s_{n+1-i,n+1-i} & s_{n+1-i,n+1+i} \\ s_{n+1-i,n+1+i} & s_{n+1+i,n+1+i} \end{bmatrix}.$$

Compute $\left(x_i \quad u_i\right) = -R_{2i-1}^{-1}\left(a_{2i-1} \quad c_{2i-1}\right)H^{-1}$.

Compute

$$B_{2i-1} = R_{2i-1}^{-1}\left(I - a_{2i-1}\, x_{2i-1}^T - c_{2i-1}\, u_{2i-1}^T\right).$$

Then, $R_{2i+1}^{-1} = \begin{bmatrix} s_{n+1-i,n+1-i} & x_{2i-1}^T & s_{n+1-i,n+1+i} \\ x_{2i-1} & B_{2i-1} & u_{2i-1} \\ s_{n+1-i,n+1+i} & u_{2i-1}^T & s_{n+1+i,n+1+i} \end{bmatrix}$.

For $i = n$, we obtain

$$R^{-1} = R_{2i+1}^{-1} = \begin{bmatrix} s_{1,1} & x_{2n-1}^T & s_{1,2n+1} \\ x_{2i-1} & B_{2n-1} & u_{2n-1} \\ s_{1,2n+1} & u_{2n-1}^T & s_{n,n} \end{bmatrix}.$$

3.2.2 Operational Count for R^{-1} when R is PDS

The total number S of arithmetic operations required for finding R^{-1} is given by

Case (i) When R is of order 2n+1,

$$8n^3 + 9n^2 + 6n + 1$$

Case (ii) When R is of order 2n,

$$8n^3 - 3n^2 + 3n$$

3.3 Inversion of a CSP matrix

If R is a CSP matrix and has the form (2.2), then

$$R^{-1} = \begin{bmatrix} s_{11} & x^T & s_{1n} \\ x & B & jx \\ s_{1n} & x^T j & s_{11} \end{bmatrix} \text{ Where}$$

$$s_{11} = p/(p^2 - q^2),\ s_{1n} = -q/(p^2 - q^2),$$

$$x = -(s_{11} I + s_{1n} J) M^{-1} a,$$

$$B = M^{-1}\left(I - ax^T - Jax^T J\right), \quad p = r_{11} - a^T M^{-1} a$$

and $q = r_{1n} - a^T J (M^{-1} a)$.

3.3.1 Recursive algorithm for finding R^{-1} when R is a CSP matrix

Step 1: Let $R_1 = (r_{n+1,n+1}) \Rightarrow R_1^{-1} = (r_{n+1,n+1})^{-1}$.

Step i+1: For i = 1, 2. . . n ,define

$$R_{2i+1} = \begin{bmatrix} r_{n+1-i,n+1-i} & a_{2i-1}^T & r_{n+1-i,n+1+i} \\ a_{2i-1} & R_{2i-1} & Ja_{2i-1} \\ r_{n+1+i,n+1-i} & a_{2i-1}^T & r_{n+1+i,n+1+i} \end{bmatrix},$$

where $a_{2i-1} = \begin{bmatrix} r_{n+1-i,n+2-i} \\ r_{n+1-i,n+3-i} \\ \cdots \\ r_{n+1-i,n+i} \end{bmatrix}$.

Compute $a_{2i-1}^T R_{2i-1}^{-1} a_{2i-1}$.

Compute $p = r_{n+1-i,n+1-i} - a_{2i-1}^T R_{2i-1}^{-1} a_{2i-1}$,

$$q = r_{n+1-i,n+1+i} - a_{2i-1}^T J R_{2i-1}^{-1} a_{2i-1},$$

$$s_{n+1-i,n+1-i} = p/(p^2 - q^2),$$

$$s_{n+1-i,n+1+i} = -q/(p^2 - q^2),$$

$$x_{2i-1} = -\left(s_{n+1-i,n+1-i} I + s_{n+1-i,n+1+i} J\right) R_{2i-1}^{-1} a \text{ and}$$

$$B_{2i-1} = R_{2i-1}^{-1}\left(I - a_{2i-1} x_{2i-1}^T - Ja_{2i-1} x_{2i-1}^T J\right).$$

Then, $R_{2i+1}^{-1} = \begin{bmatrix} s_{n+1-i,n+1-i} & x_{2i-1}^T & s_{n+1-i,n+1+i} \\ x_{2i-1} & B_{2i-1} & Jx_{2i-1} \\ s_{n+1-i,n+1+i} & x_{2i-1}^T J & s_{n+1+i,n+1+i} \end{bmatrix}$

For $i = n \quad R^{-1} = R_{2n+1}^{-1}$.

3.3.2Operational Count for R^{-1}

The total number S of arithmetic operations required for finding R^{-1} is given by

Case (i) when R is of order 2n+1,

$$S = \frac{1}{3}\left(20n^3 + 18n^2 + 10n + 3\right)$$

Case (ii) when R is of order 2n,

$$S = \frac{1}{3}\left(20n^3 - 12n^2 + 7n\right).$$

4 Comparison with competitive methods

A	Operational count: Single bordering	Double bordering of a matrix of order s: S=2n+1	Double bordering of a matrix of order s: S=2n	Existing methods
Nonsingular	$2s^3 - 2s^2 + s$	$\frac{4s^3 - 6s^2 + 7s - 3}{2}$	$\frac{4s^3 - 6s^2 + 7s}{2}$	$\frac{4s^3 + s^2 + 3s}{2}$ Factorization
PDS	$\frac{4s^3 - 3s^2 + 2s}{3}$	$\frac{4s^3 - 3s^2 + 6s - 3}{4}$	$\frac{4s^3 - 3s^2 + 6s}{4}$	$\frac{10s^3 + 3s^2 + 5s}{6}$ Cholesky
CSP	$\frac{4s^3 - 3s^2 + 2s}{3}$	$\frac{5s^3 - 6s^2 + 7s}{6}$	$\frac{10s^3 - 6s^2 + 7s}{6}$	

5 Conclusion

1 At each of recursion, the matrices retain the original properties of the given matrix,(PDS,CSP). Hence, there is no need for any checks doing recursion

2 For a general nonsingular matrix, both single and double bordering methods are superior to the existing factorization method, while the double bordering method is superior compared to the single bordering method. For example, n=31 (odd), the factorization, single and double bordering methods respectively have the operational counts as 60109,57961,56806. For n=30 (even), , the factorization, single and double bordering methods respectively have the operational counts as 54495,52230,51405.

3 For a PDS matrix, double bordering method is much superior compared to the single bordering and

Cholesky methods. For a large n, the operational counts for double bordering, single bordering, and Cholesky methods are respectively of orders n^3, $4n^3/3$ and $5n^3/3$ (approximately).For example, n=31 (odd),the operational counts for double bordering, single bordering and Cholesky method respectively are 29116,38781,50158. For n=31 (odd), the operational counts for double bordering, single bordering and Cholesky method respectively are 26370,35120,45475.

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