On A Multiple Hilbert's Type Integral Inequality with Non-homogeneous Kernel

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Abstract— **By introducing the norm $\|x\|$ and two parameters α, β, a multiple Hilbert's type integral inequality with a non-homogeneous kernel and a best possible constant factor is given.**

Keywords-Inequalities; Hilbert's inequality; multiple Hilbert's integral inequality; best possible constant

I. INTRODUCTION

The well known Hardy-Hilbert's integral inequality is given by (see [2, 8])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left[\int_0^\infty f^p(t)dt\right]^{1/p}\left[\int_0^\infty g^q(t)dt\right]^{1/q}, \quad (1.1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible.

Hardy-Hilbert's integral inequality is inequality with homogeneous kernel, it is important in analysis and applications. During the past few years, many researchers obtained various generalizations, variants and extensions of the inequality of (1.1) (see [1, 6, 7, 9-12] and the references cited therein).

Recently, Yang gave a inequality with non-homogeneous kernel (see [15]).

If $p>1, 1/p+1/q=1, 0<\alpha<\gamma, \beta>-1, f,g\geq 0,$ satisfy

$$0<\int_0^\infty t^{p(1+\alpha)-1}f^p(t)dt<\infty, 0<\int_0^\infty t^{q(1+\alpha)-1}g^q(t)dt<\infty,$$

then one has

$$\int_0^\infty\int_0^\infty \left(\min\{1.xy\}\right)^\gamma |\ln(xy)|^\beta f(x)g(y)dxdy < \left[\frac{1}{(\gamma-\alpha)^{\beta+1}}+\frac{1}{\alpha^{\beta+1}}\right]\Gamma(\beta+1) \times\left[\int_0^\infty t^{p(1+\alpha)-1}f^p(t)dt\right]^{1/p}\left[\int_0^\infty t^{q(1+\alpha)-1}g^q(t)dt\right]^{1/q}, \quad (1.2)$$

where the constant factor $\left[\frac{1}{(\gamma-\alpha)^{\beta+1}}+\frac{1}{\alpha^{\beta+1}}\right]\Gamma(\beta+1)$ is the best possible.

At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hilbert's type integral inequalities have been studied. Y. Hong, B. Yang and J. Kuang etc. obtained some multiple Hilbert's type integral inequalities (see [3, 4, 7, 13, 14]).

The main objective of this paper is to build multiple Hilbert's type integral inequalities with a non-homogeneous kernel and a best constant factor of (1.2). For this reason, we introduce signs as

$\mathbb{R}_+^n = \{x=(x_1,x_2,\cdots,x_n): x_1,x_2,\cdots,x_n>0\}$,

$\|x\| = (x_1^2+x_2^2+\cdots+x_n^2)^{\frac{1}{2}}$, , and we agree on $\|x\|<c$ representing $\{x\in\mathbb{R}_+^n : \|x\|<c\}$.

II. LEMMAS

Lemma 2.1.([14]) If $p>0, \alpha>0$, $f(\tau)$ is a measurable function, then

$$\int_{x_1,x_2,\cdots,x_n>0; x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\leq 1} f\left(x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\right)dx_1dx_2\cdots dx_n = \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^n\Gamma\left(\frac{n}{\alpha}\right)}\int_0^1 f(\tau)\tau^{\frac{n}{\alpha}-1}d\tau \quad (2.1)$$

$$\int_{x_1,x_2,\cdots,x_n>0; x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\geq 1} f\left(x_1^\alpha+x_2^\alpha+\cdots+x_n^\alpha\right)dx_1dx_2\cdots dx_n = \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^n\Gamma\left(\frac{n}{\alpha}\right)}\int_1^\infty f(\tau)\tau^{\frac{n}{\alpha}-1}d\tau \quad (2.2)$$

Lemma 2.2. If $n-1<\alpha<n-1+\gamma, \beta>-1,$ Define the weight function $w(y)$ as:

$$w(y)=\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^{\gamma}\left|\ln(\|x\|\|y\|)\right|^{\beta}\|x\|^{-1-\alpha}\,d\;.x$$

Then

$$w(y)=\|y\|^{\alpha-n+1}\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)} \times\left[\frac{1}{(\gamma+n-1-\alpha)^{\beta+1}}+\frac{1}{(\alpha-n+1)^{\beta+1}}\right]\Gamma(\beta+1). \tag{2.3}$$

Proof. By (2.1), (2.2),

$$w(y)=\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n}\Gamma\left(\frac{n}{2}\right)}$$

$$\times\int_0^{\infty}\left(\min\{1,\|y\|t^{\frac{1}{2}}\}\right)^{\gamma}\left|\ln(\|y\|t^{\frac{1}{2}})\right|^{\beta}t^{-\frac{1+\alpha}{2}}t^{\frac{n}{2}-1}d\;t$$

$$=\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n}\Gamma\left(\frac{n}{2}\right)}2\|y\|^{\alpha-n+1}\int_0^{\infty}\left(\min\{1,u\}\right)^{\gamma}\left|\ln u\right|^{\beta}u^{n-\alpha-2}du$$

$$=\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\|y\|^{\alpha-n+1}\int_0^{1}\left(-\ln u\right)^{\beta}u^{n-\alpha-2+\gamma}du$$

$$+\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\|y\|^{\alpha-n+1}\int_1^{\infty}\left(\ln u\right)^{\beta}u^{n-\alpha-2}du.$$

Notice that

$$\int_0^1\left(-\ln u\right)^{\beta}u^{n-\alpha-2+\gamma}du=\int_0^{\infty}t^{\beta}e^{-(n-\alpha-1+\gamma)t}dt$$

$$=\frac{1}{(\gamma+n-1-\alpha)^{\beta+1}}\Gamma(\beta+1),$$

$$\int_1^{\infty}\left(\ln u\right)^{\beta}u^{n-\alpha-2}du=\int_0^{\infty}t^{\beta}e^{(n-\alpha-1)t}dt$$

$$=\frac{1}{(\alpha-n+1)^{\beta+1}}\Gamma(\beta+1).$$

The lemma 2.2 is proved.

Lemma 2.3. If $p>1,1/p+1/q=1,$ $n-1<\alpha<n-1+\gamma, \beta>-1,\ n\in\mathbb{Z},\varepsilon>0,$ then

$$\tilde{I}=\int_{\|x\|\le 1}\int_{\|y\|>1}\left(\min\{1,\|x\|\|y\|\}\right)^{\gamma}\left|\ln(\|x\|\|y\|)\right|^{\beta}\|x\|^{-\alpha+\frac{\varepsilon}{p}-1}\|y\|^{-\alpha-\frac{\varepsilon}{q}-1}dxdy$$

$$=\left[\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\right]^2\Gamma(\beta+1)$$

$$\times\left[\frac{1}{(\gamma+n-1-\alpha)^{\beta+1}}+\frac{1}{(\alpha-n+1)^{\beta+1}}\right]\frac{1}{\varepsilon}(1+o(1)).$$

$$\varepsilon\to 0^+. \tag{2.5}$$

Proof. By (2.1), (2.2),

$$\tilde{I}=\int_{\|y\|>1}\|y\|^{-\alpha-\frac{\varepsilon}{q}-1}$$

$$\times\left(\int_{\|x\|\le 1}\left(\min\{1,\|x\|\|y\|\}\right)^{\gamma}\left|\ln(\|x\|\|y\|)\right|^{\beta}\|x\|^{-\alpha+\frac{\varepsilon}{p}-1}dx\right)dy$$

$$=\int_{\|y\|>1}\|y\|^{-\alpha-\frac{\varepsilon}{q}-1}$$

$$\left(\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n}\Gamma\left(\frac{n}{2}\right)}\int_0^1\left(\min\{1,\|y\|t^{\frac{1}{2}}\}\right)^{\gamma}\left|\ln(\|y\|t^{\frac{1}{2}})\right|^{\beta}t^{\frac{1}{2}\left(n-\alpha+\frac{\varepsilon}{p}-1\right)-1}dt\right)dy$$

$$=\int_{\|y\|>1}\|y\|^{-n-\varepsilon}\left(\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\int_0^{\|y\|}\left(\min\{1,u\}\right)^{\gamma}\left|\ln u\right|^{\beta}t^{n-2-\alpha+\frac{\varepsilon}{p}}du\right)dy$$

$$=\int_{\|y\|>1}\|y\|^{-n-\varepsilon}\left(\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\int_0^{1}\left(-\ln u\right)^{\beta}u^{n-\alpha-2+\gamma+\frac{\varepsilon}{p}}du\right)dy$$

$$=\int_{\|y\|>1}\|y\|^{-n-\varepsilon}\left(\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\int_1^{\|y\|}\left(\ln u\right)^{\beta}u^{n-\alpha-2+\frac{\varepsilon}{p}}du\right)dy$$

$$=\left(\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\right)^2\frac{1}{\varepsilon}\int_0^1(-\ln u)^\beta u^{n-\alpha-2+\gamma+\frac{\varepsilon}{p}}du$$

$$+\frac{1}{2}\left(\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\right)^2\int_1^\infty t^{\frac{-\varepsilon}{2}-1}\left(\int_1^{t^{\frac{1}{2}}}(\ln u)^\beta u^{n-2-\alpha+\frac{\varepsilon}{p}}du\right)dt$$

Notice that

$$\int_0^1(-\ln u)^\beta u^{n-\alpha-2+\gamma+\frac{\varepsilon}{p}}du$$

$$=\frac{1}{(\gamma+n-1-\alpha+\frac{\varepsilon}{p})^{\beta+1}}\Gamma(\beta+1),$$

$$\int_1^\infty t^{\frac{-\varepsilon}{2}-1}\left(\int_1^{t^{\frac{1}{2}}}(\ln u)^\beta u^{n-2-\alpha+\frac{\varepsilon}{p}}du\right)dt$$

$$=2\int_1^\infty s^{-\varepsilon-1}\left(\int_1^s(\ln u)^\beta u^{n-2-\alpha+\frac{\varepsilon}{p}}du\right)ds$$

$$=2\int_1^\infty\left(\int_u^\infty s^{-\varepsilon-1}ds\right)(\ln u)^\beta u^{n-2+\alpha+\frac{\varepsilon}{p}}du$$

$$=\frac{2}{\varepsilon}\int_1^\infty(\ln u)^\beta u^{n-2-\alpha-\frac{\varepsilon}{q}}du$$

$$=\frac{2}{\varepsilon}\frac{1}{(\alpha-n+1+\frac{\varepsilon}{q})^{\beta+1}}\Gamma(\beta+1).$$

We have (2.5). The lemma 2.3 is proved.

III. MAIN RESULTS

Theorem 3.1. If $p>1, 1/p+1/q=1, n\in\mathbb{Z}$, $n-1<\alpha<n-1+\gamma, \beta>-1, f, g\geq 0$, satisfy

$$0<\int_{\mathbb{R}_+^n}\|x\|^{(1+\alpha)p-n}f^p(x)dx<\infty,$$

$$0<\int_{\mathbb{R}_+^n}\|y\|^{(1+\alpha)q-n}g^q(y)dy<\infty. \tag{3.1}$$

Then

$$I:=\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta$$
$$\times f(x)g(y)dxdy$$

$$<C_n(\alpha,\beta,\gamma)\left[\int_{\mathbb{R}_+^n}\|x\|^{(1+\alpha)p-n}f^p(x)dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_{\mathbb{R}_+^n}\|y\|^{(1+\alpha)q-n}g^q(y)dy\right]^{\frac{1}{q}}; \tag{3.2}$$

Where the constant factor

$$C_n(\alpha,\beta,\gamma)=\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}$$

$$\times\left[\frac{1}{(\gamma+n-1-\alpha)^{\beta+1}}+\frac{1}{(\alpha-n+1)^{\beta+1}}\right]\Gamma(\beta+1).$$

is the best possible.

Proof. By Hölder's inequality, one has

$$I=\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta\frac{\|x\|^{\frac{1+\alpha}{q}}}{\|y\|^{\frac{1+\alpha}{p}}}f(x)$$
$$\times\frac{\|y\|^{\frac{1+\alpha}{p}}}{\|x\|^{\frac{1+\alpha}{q}}}g(y)dxdy$$

$$\leq\left\{\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta\frac{\|x\|^{(p-1)(1+\alpha)}f^p(x)}{\|y\|^{1+\alpha}}dxdy\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta\frac{\|y\|^{(q-1)(1+\alpha)}g^q(y)}{\|x\|^{1+\alpha}}dxdy\right\}^{\frac{1}{q}} \tag{3.3}$$

$$=\left[\int_{\mathbb{R}_+^n}\|x\|^{(p-1)(1+\alpha)}w(x))f^p(x)dx\right]^{\frac{1}{p}}$$
$$\times\left[\int_{\mathbb{R}_+^n}\|y\|^{(q-1)(1+\alpha)}w(y)g^q(y)dy\right]^{\frac{1}{q}}. \tag{3.4}$$

According to the condition of taking equality in Hölder's inequality, if this inequality takes the form of an equality, then there exist constants C_1 and C_2, such that they are not all zero, and

$$C_1\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta\frac{\|x\|^{(p-1)(1+\alpha)}}{\|y\|^{1+\alpha}}f^p(x)$$

$$=C_2\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta\frac{\|y\|^{(q-1)(1+\alpha)}}{\|x\|^{1+\alpha}}g^q(y),$$

a.e.. in $\mathbb{R}_+^n\times\mathbb{R}_+^n$. It following that

$$C_1\|x\|^n\|x\|^{(1+\alpha)p-n} f^p(x) = C_2\|y\|^n\|y\|^{(1+\alpha)q-n} g^q(y) = C$$

(constant), a.e.. in $\mathbb{R}_+^n\times\mathbb{R}_+^n$, which contradicts (3,1), hence we have

$$I < \left[\int_{\mathbb{R}_+^n}\|x\|^{(p-1)(1+\alpha)} w(x)) f^p(x)dx\right]^{\frac{1}{p}} \times\left[\int_{\mathbb{R}_+^n}\|y\|^{(q-1)(1+\alpha)} w(y)g^q(y)dy\right]^{\frac{1}{q}}. \quad (3.5)$$

By lemma 2.2, we have (3.2).

If the constant factor $C_n(\alpha,\beta,\gamma)$ in (3.2) is not the best possible, then exists a positive number k (with $k < C_n(\alpha,\beta,\gamma)$), such that(3.2) is still valid if one replaces $C_n(\alpha,\beta,\gamma)$ by k.

For $\varepsilon > 0,$ sitting

$$f_\varepsilon(x) = \begin{cases}\|x\|^{-\alpha+\frac{\varepsilon}{p}-1}, & \|x\|\le 1\\ 0, & \|x\|>1,\end{cases} \quad g_\varepsilon(y) = \begin{cases}\|y\|^{-\alpha-\frac{\varepsilon}{q}-1}, & \|y\|>1\\ 0, & \|y\|\le 1,\end{cases}$$

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta f_\varepsilon(x)g_\varepsilon(y)dxdy$$

$$< k\left[\int_{\mathbb{R}_+^n}\|x\|^{p(1+\alpha)-n} f_\varepsilon^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+^n}\|y\|^{q(1+\alpha)-n} g_\varepsilon^q(y)dy\right]^{\frac{1}{q}},$$

by lemma 2.3

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta f_\varepsilon(x)g_\varepsilon(y)dxdy$$

$$= \int_{\|x\|\le 1}\int_{\|y\|>1}\left(\min\{1,\|x\|\|y\|\}\right)^\gamma\left|\ln(\|x\|\|y\|)\right|^\beta$$

$$\|x\|^{-\alpha+\frac{\varepsilon}{p}-1}\|y\|^{-\alpha-\frac{\varepsilon}{q}-1} dxdy$$

$$= \left[\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\right]^2 \Gamma(\beta+1)$$

$$\times\left[\frac{1}{(\gamma+n-1-\alpha)^{\beta+1}} + \frac{1}{\left(\alpha-n+1\right)^{\beta+1}}\right]\frac{1}{\varepsilon}(1+o(1)).$$

$$\left[\int_{\mathbb{R}_+^n}\|x\|^{p(1+\alpha)-n} f_\varepsilon^p(x)dx\right]^{\frac{1}{p}}\left[\int_{\mathbb{R}_+^n}\|y\|^{q(1+\alpha)-n} g_\varepsilon^q(y)dy\right]^{\frac{1}{q}}$$

$= \dfrac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\cdot\dfrac{1}{\varepsilon}$. Hence, we have

$$\left[\frac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\right]^2 \Gamma(\beta+1)$$

$$\times\left[\frac{1}{(\gamma+n-1-\alpha)^{\beta+1}} + \frac{1}{\left(\alpha-n+1\right)^{\beta+1}}\right]\frac{1}{\varepsilon}(1+o(1)).$$

$\le k\cdot\dfrac{\Gamma^n\left(\frac{1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}\cdot\dfrac{1}{\varepsilon}$ .For $\varepsilon\to 0^+$, we have $C_n(\alpha,\beta,\gamma)\le k.$ This contradicts the fact that $k < C_n(\alpha,\beta,\gamma)$, hence the constant factor in（3.2） is the best possible.

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