A New Smoothing Nonmonotone Trust Region Method for Solving Nonlinear Complementarity Problems

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Abstract—In this paper, we propose one new smoothing nonmonotone trust region method for solving nonlinear complementarity problems with P_0- functions. First, the nonlinear complementarity problem (NCP) is reformulated as a nonsmooth equation. Then on the basis of the reformulation, a smoothing nonmonotone trust region algorithm via a line search for solving the NCP with P_0- functions is proposed. When a trial step is not accepted, the method does not resolve the trust region subproblem but generates a iterative point whose steplength is generated by a formula. We prove that every accumulation point of the sequence generated by the algorithm is a solution of the NCP. Under a nonsingularity condition, the superlinear convergence of the algorithm is established without the strict complementarity condition.

Keywords-NCP, Trust region method, Fixed steplength, Nonmonotone technique

I. INTRODUCTION

The nonlinear complementarity problem, denoted by $NCP(F)$, is to find a vector $x \in R^n$ such that

$$x \geq 0, \ F(x) \geq 0, \ x^T F(x) = 0 \tag{1}$$

where $F : R^n \to R^n$ is assumed to be continuously differentiable.

In this paper, we propose and analyze a new smoothing nonmonotone trust region algorithm that combines elements of trust region methods with elements of line search methods. The new algorithm retains the quick convergence and stability of trust region methods, while significantly decreasing the average cost per iteration of the method. We first reformulate the NCP as a system of semismooth by using the Fischer-Burmeister function, then using Kanzow's [1] smooth approximation function to construct the smooth operator, we propose a smoothing nonmonotone trust region algorithm via a line search for solving the NCP with P_0- functions. When a trial step is not accepted, the method does not resolve the trust region subproblem but generates a iterative point whose steplength is generated by a formula. We prove that every accumulation point of the sequence generated by the algorithm is a solution of the NCP.

Compared with the method in [2], the superlinear convergence of the proposed algorithm is established without the strict complementarity condition.

We will use Kanzow's [1] smooth approximation

$$\varphi_\varepsilon(a,b) := \sqrt{a^2 + b^2 + 2\varepsilon} - a - b, \varepsilon > 0$$

for the Fischer-Burmeister function. The corresponding smooth operator $\Phi_\varepsilon : R^n \to R^n$ is defined by

$$\Phi_\varepsilon(x) = \begin{pmatrix} \varphi_\varepsilon(x_1, F_1(x)) \\ \dots \\ \varphi_\varepsilon(x_n, F_n(x)) \end{pmatrix}. \tag{2}$$

Denote $\Psi_\varepsilon(x) := \frac{1}{2} \Phi_\varepsilon(x)^T \Phi_\varepsilon(x)$. Then $NCP(F)$ can be approximated by the following nonlinear least square problems,

$$\min_{x \in R^n} \Psi_\varepsilon(x) \tag{3}$$

By the above discussions, we have that when $\varepsilon \to 0$ the problem (3) is equivalent to $NCP(F)$.

II. THE ALGORITHM

Define

$$Q_k(s) := \frac{1}{2} \left\| \Phi_{\varepsilon_k}(x_k) + \nabla \Phi_{\varepsilon_k}(x_k)^T s \right\|^2 \tag{4}$$

$$= \Psi_{\varepsilon_k}(x_k) + \nabla \Psi_{\varepsilon_k}(x_k)^T s + \frac{1}{2} s^T \nabla \Phi_{\varepsilon_k}(x_k) \nabla \Phi_{\varepsilon_k}(x_k)^T s \tag{5}$$

Then the trial step s_k will be obtained by solving the following trust region subproblem,

$$\min\{Q_k(s) \mid \|s\| \leq \Delta_k\} \tag{6}$$

where Δ_k is called trust region radius. Let s_k be the solution of the subproblem (6).

Then either $x_k + s_k$ is accepted as a new iteration point or the trust-region radius is reduced according to a

comparison between the actual reduction of the objective function

$$ared_k(s_k) := \Psi_{\varepsilon_{l(k)}}(x_{l(k)}) - \Psi_{\varepsilon_k}(x_k + s_k) \quad (7)$$

and the reduction predicted by the model

$$pred_k(s_k) := Q_k(0) - Q_k(s_k) \quad (8)$$

i.e.,

$$r_k := \frac{ared_k(s_k)}{pred_k(s_k)}$$

where

$$\Psi_{\varepsilon_{l(k)}}(x_{l(k)}) = \max_{0 \le j \le m(k)} \{\Psi_{\varepsilon_{k-j}}(x_{k-j})\}, k = 0,1,2,..., \quad (9)$$

and

$$m(k) := \begin{cases} 0, & if\ r_k \ge c_2 \\ \min\{m(k-1)+1, M\}, & otherwise, \end{cases} \quad (10)$$

$m(0) := 0, M \ge 1$ is an integer constant. That is, if the reduction in the objective function is satisfactory, then we finish the current iteration by taking

$x_{k+1} := x_k + s_k$ and adjusting the trust-region radius; otherwise a new iterative point by $x_{k+1} = x_k + \alpha_k s_k$ is generated, where α_k is a steplength generated by

$$\alpha_k = -\frac{\delta \Psi_{\varepsilon_k}(x_k)^T s_k}{s_k^T \Phi_{\varepsilon_k}(x_k)\Phi_{\varepsilon_k}(x_k)^T s_k} \quad (11)$$

here $\delta > 0$ is a constants.

Algorithm 1(The smoothing nonmonotone trust region algorithm via a line search for $NCP(F)$)

Step 0. Choose parameters $0 < c_2 < c_1 < 1$, $0 < c_3 < 1 < c_4$, $\Delta_{min} > 0$, $v > 0$, $0 < \eta < 1$, $0 < \mu < 1$. Give a starting point $x_0 \in R^n$, an integer constant $M \ge 0$ and an initial trust-region radius $\Delta_{min} \le \Delta_0$; set $\Gamma_0 := (1+\mu)\|\Phi(x_0)\|$, $\beta_0 := \|\Phi(x_0)\|$, $\kappa := \sqrt{2n}$, $\varepsilon_0 := ((\mu/(2\Gamma_0\kappa))\beta_0^2)^2$, $m(0) := 0, M \ge 1$ and $k := 0$.

Step 1. Compute the solution $s_k \in R^n$ of the subproblem (6).

Step 2. Compute $ared_k(s_k)$, $pred_k(s_k)$ and $r_k := \frac{ared_k(s_k)}{pred_k(s_k)}$

If $k \ge 1$, compute $m(k)$ by (10). If $r_k \ge c_2$, then $x_{k+1} := x_k + s_k$ (12)

$$\Delta_{k+1} := \begin{cases} \max[c_4\Delta_k, \Delta_{min}], & if\ r_k > c_1 \\ \max[\Delta_k, \Delta_{min}], & otherwise \end{cases}$$

(2.10)

Otherwise generate α_k by (11) and set

$$x_{k+1} = x_k + \alpha_k s_k, \quad \Delta_k := c_3\|s_k\| \quad (13)$$

Step 3. If $\|\nabla\Psi(x_{k+1})\| = 0$, stop.

Step 4. If

$$\|\Phi(x_{k+1})\| \le \max\{\eta\beta_k, \mu^{-1}\|\Phi(x_{k+1}) - \Phi_{\varepsilon_k}(x_{k+1})\|\},$$

then set $\beta_{k+1} := \|\Phi(x_{k+1})\|$ and choose ε_{k+1} such that

$$0 < \varepsilon_{k+1} \le \min\{(\frac{\mu}{2\Gamma_0\kappa}\beta_{k+1}^2), \frac{\varepsilon_k}{4}, \overline{\varepsilon}(x_{k+1}, v\beta_{k+1})\} \quad (14)$$

where $\overline{\varepsilon}(\cdot, \cdot)$ is defined in Lemma 2.3 [3]; otherwise, let $\beta_{k+1} := \beta_k$ and $\varepsilon_{k+1} := \varepsilon_k$.

Step 5. Set $k := k + 1$, and go to Step 1.

Lemma 1 Suppose that F is a P_0 function, and Algorithm 2.1 does not stop finitely, then $s_k \ne 0$ and furthermore $pred_k(s_k) > 0, \forall k$.

Theorem 2 If F is a P_0 function, then the sequence x_k generated by Algorithm 2.1 remains in the level set

$$L_0 := \{x \in R^n | \Psi(x) \le \Psi(x_0)\} \text{ with } \delta_0 = (\frac{1+\mu}{1-\mu})^2$$

Theorem 3 Suppose that F is a P_0 function. Then Algorithm 2.1 is well defined.

III. GLOBAL CONVERGENCE

Theorem 1 Suppose that F is a P_0 function. Then there exists a positive constant δ_2 such that

$$pred_k(s_k) \ge \delta_2\|\nabla\Psi_{\varepsilon_k}(x_k)\|\min[\Delta_k, \frac{\|\nabla\Psi_{\varepsilon_k}(x_k)\|}{\|\nabla\Phi_{\varepsilon_k}(x_k)\|^2}] \quad (15)$$

for all k, where s_k is the solution to (2.3).

Lemma 2 Suppose that F is a P_0 function. Then there exists a positive constant δ_3 such that

$$|\Psi_{\varepsilon_k}(x_k) - \Psi_{\varepsilon_k}(x_k + s_k) - pred_k(s_k)| \le \delta_3\|s_k\|^2, \forall k \quad (16)$$

Theorem 3 Suppose that F is a P_0 function, and there exists $\varepsilon > 0$ such that $\|\nabla\Psi_{\varepsilon_k}(x_k)\| \ge \varepsilon$ for all k. Then there exists a constant $\delta_4 > 0$ such that

$$\Psi_{\varepsilon_{l(k)}}(x_{l(k)}) - \Psi_{\varepsilon_{k+1}}(x_{k+1}) \geq \delta_4 \min\{\Delta_k, \frac{\varepsilon}{\|\nabla\Phi_{\varepsilon_k}(x_k)\|^2}\} \quad (17)$$

holds for all k.

Lemma 4 Suppose that F is a P_0 function, and there exists $\varepsilon > 0$ such that $\|\nabla\Psi_{\varepsilon_k}(x_k)\| \geq \varepsilon$ for all k. Then there exists $\nu \in (0,1)$ such that

$$\lim_{k\to\infty}\{\nu\Delta_k, \frac{\varepsilon}{Z_k}\} = 0$$

, where $Z_k := 1 + \max_{1\leq i\leq k}\|\nabla\Phi_{\varepsilon_i}(x_i)\|^2$.

Lemma 5 Suppose that F is a P_0 function, and there exists $\varepsilon > 0$ such that $\|\nabla\Psi_{\varepsilon_k}(x_k)\| \geq \varepsilon$ for all k. Then the following inequality

$$\|s_k\| \geq \min\{1, \delta_2\varepsilon(1-c_2)\}/Z_k,$$ holds for $k \in J$ sufficiently large.

Lemma 6 Suppose that F is a P_0 function, and there exists $\varepsilon > 0$ such that $\|\nabla\Psi_{\varepsilon_k}(x_k)\| \geq \varepsilon$ for all k. Then the following inequality

$$\Delta_k \geq c_3 \min\{1, \delta_2\varepsilon(1-c_2)\}/Z_k,$$ holds for sufficiently large k.

Theorem 7 Assume that F is a P_0 function. Let $\{x_k\}$ be a sequence generated by Algorithm 2.1. Then every accumulation point of the sequence $\{x_k\}$ is a solution of $NCP(F)$.

IV. LOCAL CONVERGENCE

Theorem 1 Suppose F is a P_0-function. If for an accumulation point x^* of the sequence $\{x_k\}_{k\in\hat{K}}$, all $V = \partial_C\Phi(x^*)$ are nonsingular, then the following statements hold:

1) The whole sequence $\{x_k\}$ converges to x^*.

2) Eventually all iterations are successful and the trust region radius in subproblem (3.3) is inactive; i.e., for all k sufficiently large, we have

$$x_{k+1} = x_k + s_k, s_k = -(\Phi'_{\varepsilon_k}(x_k))^{-1}\Phi_{\varepsilon_k}(x_k)$$

3) The sequence $\{x_k\}$ converges to x^* superlinearly.

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