Logic Value Fuzzy Subgroup Of a Group

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***Abstract*—The aims of this article are to establish the relationship between the fuzzy algebra and the classical logic and to study fuzzy algebra by the use of classical logic methods. We firstly introduces the concept of the logic value fuzzy subgroup, and studies the relationship between the fuzzy subgroup and its dual. It is pointed out that H is a logic value（normal） fuzzy subgroup of a group G if and only if for all value assignment v, the core and the v- dual of H is the (normal) subgroup of G. Secondly, we study the properties of the logical value fuzzy subgroup, the logical value normal fuzzy subgroup and its quotient groups. Finally, We study the properties of homomorphic image of the logic value fuzzy subgroup. The research of this paper can help to establish the relationship between the fuzzy algebra and the classical logic.**

***Keywords*—group; subgroup; normal subgroup; sentence; value assignment; tautology; dual.**

I. INTRODUCTION

Since A.Rosenfeld introduced the concept of fuzzy subgroup[1], fuzzy algebra has been developed greatly[2]. At present, there are three main methods to research fuzzy algebra such as the mthod of membership function of fuzzy set [2,3], the method of the "point - set" neighborhood relation [4,5,6], and the mthod of probability theory [7].

It is well known that the classical algebra theory has the intimate connection with the classical logic [8]. The theory of fuzzy subgroups has been studied by using fuzzy logic in [5,7]. However, this intimate connection is still not completely understood. This paper attempts to establish a connection between classical propositional logic and fuzzy algebra.

In this paper, we introduce the necessary knowledge in section 2. In section 3 and section 4, the concept of logical value fuzzy subgroup and the logic value normal fuzzy subgroup are proposed and their properties are studied. In section 5, the properties of homomorphic image of the logic value fuzzy subgroups. In section 6, we introduce the concept of the strong dual H^* and the v- dual H_v^* of logic value fuzzy subgroup H and describe the logic value fuzzy subgroup by strong dual and v- dual .

II. PRELIMINARY

Let S be the set of sentential variable in propositional logic system L^P. $F(S)$ is the formula set in propositional logic L^P and $\neg,\wedge,\vee,\rightarrow,\leftrightarrow$ is the connective in L^P. A value assignment is a mapping $v:S\rightarrow\{0,1\}$. Let $A\in F(S)$ be a formula, if for all value assignment v we get $v(A)=1$, then we call A is a tautology, which is remarked as $\vDash A$ or $\vdash A$.
When $A,B\in F(S)$ and for all value assignment v, $v(A)=v(B)$,then we denote $A=B$.

Obviously, $v(A\rightarrow B)=1$ if and only if $v(A)\le v(B)$.

III. LOGIC VALUE FUZZY SUBGROUP

Let G be a group and $F(S)$ be the formula set in propositional logic L^P.

Definiton 1. Let $H:G\rightarrow F(S)$ be a mapping, if H satisfies:

(1) $\vdash H(e)$,

(2) $\forall x,y\in G,\ \ \vdash H(x)\wedge H(y)\rightarrow H(xy)$,

(3) $\forall x\in G,\ \ \vdash H(x)\rightarrow H(x^{-1})$.

Then we call H a logic value fuzzy subgroup of G.

The following conclusions are obvious.

Theorem 1. The mapping $H:G\rightarrow F(S)$ is a logic value fuzzy subgroup of G if and only if for all value assignment $v:S\rightarrow\{0,1\}$, we have

(i) $v(H(e))=1$

(ii) $v(H(x,y))\ge\min\{v(H(x)),v(H(y))\},\forall x,y\in G$

(iii) $v(H(x^{-1}))\ge v(H(x)),\forall x\in G.$

Theorem 2. Let $H:G\rightarrow F(S)$ be a mapping ,then H is a logic value fuzzy subgroup of G if and only if for all value assignment $v:S\rightarrow\{0,1\}$,

$$H_v=\{x\in G\,|\,v(H(x))=1\}$$

is the subgroup of G, H_v is called the v- core of H.

Proof: Let H be a logic value fuzzy subgroup of G and $v:S\rightarrow\{0,1\}$ be a value assignment. Then

(i) $e\in H_v$,

(ii) Let $x,y\in H_v$, then

$v(H(xy))\ge\min\{v(H(x)),v(H(y))\}=1,$

and consequently $xy\in H_v$.

(iii) Let $x\in H_v$,then $v(H(x^{-1}))\ge v(H(x))=1$, and consequently $x^{-1}\in H_v$.

Therefore H_v is the subgroup of G.

Conversely, Let $v: S \to \{0,1\}$ be a value assignment, $x, y \in G$. Let $\min\{v(H(x)), v(H(y))\} = 1$, then $v(H(x)) = v(H(y)) = 1$. Thus we get $x \in H_v, y \in H_v$ and $xy \in H_v$, so $v(H(xy)) = 1$, that is

$$v(H(xy)) \geq \min\{v(H(x)), v(H(y))\}.$$

Similarly, $v(H(x^{-1})) \geq v(H(x))$.

According to the theorem 1, we know that H is a logic value fuzzy subgroup of G.

Theorem 3. Let H_1, H_2 be the logic value fuzzy subgroups of G, we define

$$(H_1 \cap H_2)(x) \triangleq (H_1)(x) \wedge (H_2)(x),$$

Then $H_1 \cap H_2$ is the logic value fuzzy subgroup of G.

Proof : Let v be a value assignment, then

$$(H_1 \cap H_2) = \{x \in G \mid v((H_1 \cap H_2)(x)) = 1\} = \{x \in G \mid v(H_1(x) \wedge H_2(x)) = 1\}$$
$$= \{x \mid v(H_1(x)) = 1\} \cap \{ \mid v(H_2(x)) = 1\} = (H_1)_v \cap (H_2)_v$$

Since $(H_1)_v, (H_2)_v$ are the subgroups of G, so $(H_1 \cap H_2)_v$ is a subgroup of G. According to theorem 2, $H_1 \cap H_2$ is a logic value fuzzy subgroup of G.

IV. LOGIC VALUE NORMAL FUZZY SUBGROUP

Definition 2. Let $N: G \to F(S)$ be a logic value fuzzy subgroup of G, if

$$\vdash N(x) \to N(y^{-1}xy), \forall x, y \in G,$$

Then N is called a logic value normal fuzzy subgroup of G.

Theorem 4. Let $N: G \to F(S)$ be a logic value fuzzy subgroup of G, if

$$\vdash N(x) \to N(y^{-1}xy), \forall x, y \in G,$$

Then the following conclusions are equivalent:

(1) N is a logic value normal fuzzy subgroup of G.

(2) For all value assignment, $v: S \to \{0,1\}$, we have $v(N(y^{-1}xy)) \geq v(N(x))$.

(3) For all value assignment, $v: S \to \{0,1\}$, we have $v(N(xy)) = v(N(yx))$.

(4) For all value assignment, $v: S \to \{0,1\}, v((xN)(y)) = v((Nx)(y))$, where $xN, Nx: G \to F(S)$ satisfy

$$(xN)(y) = N(x^{-1}y), (xN)(y) = N(yx^{-1}).$$

Proof: $(1) \Rightarrow (2)$ Let $v: S \to \{0,1\}$ be a value assignment, by definition 1, we have

$$v(N(x) \to N(y^{-1}xy)) = 1 \Rightarrow v(N(y^{-1}xy)) \geq v(N(x)).$$

$(2) \Rightarrow (3)$ $\quad v(N(xy)) = v(N(y^{-1}yxy)) \geq v(N(yx))$, similarly, $v(N(yx)) \geq v(N(xy))$, so $v(N(xy)) = v(N(yx))$.

$(3) \Rightarrow (4)$

$$v((xN)(y)) = v(N(x^{-1}y)) = v(N(yx^{-1})) = v((Nx)(y)).$$

The following conclusions are obvious.

Theorem 5. The mapping $N: G \to F(S)$ is a logic value normal fuzzy subgroup of G if and only if for all value assignment $v: S \to \{0,1\}$, $H_v = \{x \mid v(N(x)) = 1\}$ is a normal subgroup of G.

Definition 3. Let A, B be the mapping from G to $F(S)$ and $\{A(x) \mid x \in G\}$ and $\{B(x) \mid x \in G\}$ are finite sets, we define

$$(AB)(y) = \bigvee_{x \in G} (A(yx^{-1}) \wedge B(x)),$$

AB is called the product of A and B.

Theorem 6. (1) If N_1, N_2 are logic value normal fuzzy subgroups of G, then $N_1 \cap N_2$ is a logic value normal fuzzy subgroup of G.

(2) If N is a logic value normal fuzzy subgroup of G, H is a logic value normal fuzzy subgroup of G and $\{N(x) \mid x \in G\}$ and $\{H(x) \mid x \in G\}$ are finite sets, then NG is a logic value normal fuzzy subgroup of G.

Proof : As

$$(NH)(y) = \bigvee_{x \in G} (N(yx^{-1}) \wedge H(x)),$$

we get

$$y \in (NH)_v \Leftrightarrow \exists x \in G, v(N(yx^{-1})) = v((Hx)) = 1$$
$$\Leftrightarrow \exists x \in G, x \in H_v, yx^{-1} \in H_v.$$
$$\Leftrightarrow y \in N_v H_v.$$

As N_v is a normal subgroup of G and H_v is a subgroup of G, so $N_v H_v$ is a subgroup of G. Thus $(NH)_v = N_v H_v$ is the subgroup of G.

Therefore NH is a logic value fuzzy subgroup of G.

Theorem 7. Let N be a logic value normal fuzzy subgroup of G and $\{N(x) \mid x \in G\}$ is a finite set, let $G/N = \{xN \mid x \in G\}$, then (1) G/N can form a group; (2) For all value assignment v, G/N_v is isomorphic to G/N.

Proof : (1) Let $xN, yN \in G/N$, we prove

$$(xN)(yN) = xyN,$$

that is for all value assignment v, we have

$$v((xN)(yN)(z)) = v(xy(N)(z)).$$

In fact, $v((xN)(yN)(z))=1 \Leftrightarrow \exists a \in G$ such that
$v(((xN)(za^{-1})) \wedge ((yN)(a^{-1})))=1$
$\Leftrightarrow \exists a \in G, v(N(x^{-1}za^{-1}) \wedge N(y^{-1}a))=1$
$\Leftrightarrow \exists a \in G, x^{-1}za^{-1} \in N_v, y^{-1}a \in N_v$
$\Leftrightarrow y^{-1}a \in N_v, a^{-1}(x^{-1}z) = a^{-1}(x^{-1}za^{-1})a \in N_v$
$\Leftrightarrow (xy)^{-1}z = y^{-1}a \cdot a^{-1}x^{-1}z \in N_v$
$\Leftrightarrow v(N((xy)^{-1}(z)))=1 \Leftrightarrow v((xyN)(z))=1$
Therefore, $v((xN)(yN)(z)) = v((xyN)(z))$, that is $(xN)(yN)=(xy)N$. Hence G/N forms a group.

(2) Let $\phi : G/N_v \to G/N, xN_v \mapsto xN$, then

$$xN_v = yN_v \Leftrightarrow y^{-1}x \in N_v$$
$$\Leftrightarrow v(N(y^{-1}x))=1 \Leftrightarrow v(N(x^{-1}y))=1,$$

So we get
$v((xN)(z))=1 \Leftrightarrow v(N(x^{-1}z))=1 \Leftrightarrow v(N(x^{-1}y \cdot y^{-1}z))=1$

Then

$$v(xN)(z)) \geq \min\{v(N(x^{-1}y)), v(N(y^{-1}z))\}$$
$$= v(N(y^{-1}z)) = v(yN)(z))$$

Similarly, $v((yN)(z)) \geq v((xN)(z))$.

Thus $v((yN)(z)) = v((xN)(z))$ and consequently $yN = xN$.

Therefore , ϕ is a mapping and ϕ is a surjection.

Let $\phi(xN_v) = \phi(yN_v)$, thus $xN = yN$. Then for all value assignment v and $z \in G$, we have $v((yN)(z)) = v((xN)(z))$, that is $v(N(x^{-1}z)) = v(N(y^{-1}z)), \forall z \in G$, thus $v(N(x^{-1}y)) = v(N(y^{-1}y)) = v(N(e)) = 1$, then $x^{-1}y \in N_v$, we get $xN_v = yN_v$.

Therefore, ϕ is a isomorphism.

So G/N_v is isomorphic to G/N .

V. THE PROPERTY OF HOMOMORPHISM IMAGE ON THE LOGIC VALUE FUZZY SUBGROUP

Definition 4. Let $f : G_1 \to G_2$ be a surjective homomorphism of group ,

$$H : G_1 \to F(S), K : G_2 \to F(S)$$

be a mapping and $\{H(x) | x \in G\}$ be a finite set, then

$$f(H)(y) \triangleq \bigvee_{f(x)=y} H(x), \; f^{-1}(K)(x) \triangleq K(f(x)) .$$

Theorem 8. Let $f : G_1 \to G_2$ be surjective homomorphism of group . Then we have

(1) If H is a logic value (normal) fuzzy subgroup of G_1 and $\{H(x) | x \in G\}$ is a finite set, then $f(H)$ is a logic value (normal) fuzzy subgroup of G_2 .

(2) If K is a logic value (normal) fuzzy subgroup of G_1 , then $f^{-1}(K)$ is a logic value (normal) fuzzy subgroup of G_1 .

Proof : (1) Let H be a logic value (normal) fuzzy subgroup of G_1 , v be a value assignment and $y \in (f(H))_v$, then $v(f(H)(y)) = 1 \Leftrightarrow \exists x \in G, f(x) = y$ and $v(H(x)) = 1$

$\Leftrightarrow \exists x \in H_v$ such that $f(x) = y \Leftrightarrow y \in f(H_v)$,

Therefore, $(f(H))_v = f(H_v)$.

Because H_v is the subgroup of G_1 , $f(H_v)$ is the subgroup of G_2 .

Therefore, $f(H)$ is a logic value fuzzy subgroup of G_2 , when H is a logic value normal fuzzy subgroup of G_1 ,we can similarly prove that $f(H)$ is a logic value (normal) fuzzy subgroup of G_2 .

(2) $x \in (f^{-1}(K))_v \Leftrightarrow v(f^{-1}(K)(x)) = 1$
$\Leftrightarrow v(K(f(x))) = 1 \Leftrightarrow f(x) \in K_v \Leftrightarrow x \in f^{-1}(K_v)$,

Thus $(f^{-1}(K))_v = f^{-1}(K_v)$. Therefore when K is the logic value (normal) fuzzy subgroup of G_2 , $f^{-1}(K)$ is the logic value (normal) fuzzy subgroup of G_1 .

VI. THE DUAL OF LOGIC VALUE FUZZY SUBGROUP

Definition 5. Let $H : G \to F(S)$ be a mapping, $v : S \to \{0,1\}$ is a value assignment, $A \in F(S)$, let

$$H^*(A) = \{x | \vdash A \to H(x)\},$$
$$H_v^*(A) = \{x | \; v(A) \leq v(H(x))\}$$

H^* is called the strong dual of H and H_v^* is called the v- dual of H .

Theorem 9. (1) H is a logic value (normal) fuzzy subgroup of G if and only if for all $A \in F(S)$, $H^*(A)$ is a (normal) subgroup of G . (2) H is a logic value (normal) fuzzy subgroup of G if and only if for all $A \in F(S)$ and value assignment v , $H_v^*(A)$ is a (normal) subgroup of G .

Proof :Let H be a logic value (normal) fuzzy subgroup of G , $A \in F(S)$, as $H(e)$ is a tautology ,so $H^*(A) \neq \varnothing$.
Let $x, y \in H^*(A)$,then for all value assignment v , we have

$$v(A \to H(x)) = v(A \to H(y)) = 1 ,$$

Then

$$v(A) \leq \min\{v(H(x)), v(H(y))\} \leq v(H(xy)) ,$$

Thus $v(A \to H(xy)) = 1$, It follows that $\vdash A \to H(xy)$ and consequently $xy \in H^*(A)$.

Similarly we can prove $x^{-1} \in H^*(A)$ from $x \in H^*(A)$.

Therefore $H^*(A)$ is the subgroup of G.

When H is a logic value normal fuzzy subgroup of G, similarly we can prove $H^*(A)$ is the normal subgroup of G.

(1) "$\Leftarrow$" Let $A = H(x) \wedge H(y)$, as $H^*(A)$ is the normal subgroup of G and $x \in H^*(A), y \in H^*(A)$, so $xy \in H^*(A)$, we have that $\vdash A \to H(xy)$, that is

$\vdash H(x) \wedge H(y) \to H(xy)$.

Let $A = H(x)$, thus $H^*(A)$ is the subgroup of G and $x \in H^*(A)$, then $x^{-1} \in H^*(A)$. We have that

$$\vdash H(x) \to H(x^{-1}).$$

Let A be a tautology, according to $H^*(A)$ is the subgroup of G, we know $e \in H^*(A)$,then $\vdash A \to H(e)$, thus $\vdash H(e)$. H is a logic value fuzzy subgroup of G.

Similarly, for all $A \in F(S)$, $H^*(A)$ is a normal subgroup of G. therefore H is the logic value normal fuzzy subgroup of G.

(2) "$\Leftarrow$"Because H is the logic value fuzzy subgroup of G,we get $\vdash H(e)$. Therefore for all formula $A \in F(S)$, there is $e \in H_v^*(A)$.

Let $x, y \in H_v^*(A)$, which results in

$v(A) \le v(H(x)), v(A) \le v(H(y))$.

Thus $v(A) \le \min\{v(H(x)), v(H(y))\} \le v(H(xy))$, then $xy \in H_v^*(A)$.

Let $x \in H_v^*(A)$, thus $v(A) \le v(H(x)) \le v(H(x^{-1}))$, then $x^{-1} \in H_v^*(A)$, therefore $H_v^*(A)$ is a subgroup of G.

Similarly, we can prove that $H^*(A)$ is the subgroup of G when H is a logic value normal fuzzy subgroup of G for all formula A.

"$\Leftarrow$"let $v(A) = 1, x, y \in H_v$, hence $v(H(x)) = v(H(y)) = 1$. So we have $v(A) \le v(H(x)), v(A) \le v(H(y))$, that is $x, y \in H_v^*(A)$, thus $xy \in H_v^*(A)$, then $v(A) \le v(H(xy))$. We have $v(H(xy)) = 1$. Hence H_v is the subgroup of G.

Therefore H is a logic value normal fuzzy subgroup of G.

The others are similar.

We are able to get the following conclusions obviously.

Theorem 10.

(1) $(H_1 \cap H_2)^*(A) = H_1^*(A) \cap H_2^*(A)$

(2) $(H_1 \cap H_2)_v^*(A) = (H_1)_v^*(A) \cap (H_2)_v^*(A)$

(3) $H^*(A \vee B) = H^*(A) \cap H^*(B)$

(4) $H_v^*(A \vee B) = H_v^*(A) \cap H_v^*(B)$

(5) $H^*(A \wedge B) \supseteq H^*(A) \cup H^*(B)$

(6) $H_v^*(A \wedge B) = H_v^*(A) \cup H_v^*(B)$

(7) $H_v^*(A) = \begin{cases} G, & v(A) = 0 \\ H_v, & v(A) = 1 \end{cases}$

(8) $H^*(A) = \begin{cases} \{x \mid \vdash H(x)\}, & A \text{ is a tautology} \\ G, & A \text{is a contradictory} \end{cases}$

(9) $H^*(A) \cap H^*(\neg A) = \{x \mid \vdash H(x)\}$

VII. CONCLUSION

This article presents the concepts of logic value fuzzy subgroup, the core H_v, the strong dual H^* and the v- dual H_v^* of logic value fuzzy subgroup. It is pointed out that we are able to describe the logic value fuzzy subgroup H of G by use of H_v, H^*, H_v^*. The research of this paper can be seen that the existing fuzzy algebra theory is easy to generalize to the theory of logic value fuzzy algebra.

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