The Application of Cauchy Inequality in Middle School Mathematics

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Abstract. Cauchy inequality is widely applied in the middle school mathematics. This paper first studies the Cauchy inequality, then discusses the application skills of Cauchy inequality in the middle school mathematics, illustrates applications of the Cauchy inequality in the proof of the complex inequalities and the extreme value problems, and finally summarizes the application scope of Cauchy inequality and solution procedure.

Introduction

Cauchy inequality is widely applied in the middle school mathematics. Many authors[1-9] studied the problems in the middle school mathematics by using Cauchy inequality. Cauchy inequality can be stated as follows:

Suppose that $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots, b_n$ are two groups of real numbers, then inequality

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \ge \left(\sum_{k=1}^{n} a_k b_k \right)^2 , \tag{1}$$

holds. Equality holds when and only when $a_1 : a_2 : \cdots : a_n = b_1 : b_2 : \cdots : b_n$.

Cai [3] discussed the various equivalent forms of the Cauchy inequality.

Let $a_1, a_2, \cdots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \right) \ge n^2 . \tag{2}$$

Equality holds when and only when $a_1 = a_2 = \cdots = a_n$.

Let $a_1, a_2, \cdots, a_n$ are real numbers, then

$$n \sum_{k=1}^{n} a_k^2 \ge \left(\sum_{k=1}^{n} a_k \right)^2 . \tag{3}$$

Equality holds when and only when $a_1 = a_2 = \cdots = a_n$.

In (3), if $a_1, a_2, \cdots, a_n$ are positive real numbers, then Cauchy inequality has the following equivalent forms:

$$\sum_{k=1}^{n} a_k b_k \sum_{k=1}^{n} \frac{a_k}{b_k} \ge \left(\sum_{k=1}^{n} a_k \right)^2 , \tag{4}$$

$$\sum_{k=1}^{n} b_k \sum_{k=1}^{n} \frac{a_k^2}{b_k} \ge \left(\sum_{k=1}^{n} a_k \right)^2 , \tag{5}$$

$$\sqrt{\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k} \geq \sum_{k=1}^{n} \sqrt{a_k b_k}\ . \tag{6}$$

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Using the condition of equality holding in Cauchy inequality, we prove identical equations.

Example 1 If $a\sqrt{1-b^2}+b\sqrt{1-a^2}=1$, then $a^2+b^2=1$.

Proof. By Cauchy inequality, we have

$$a\sqrt{1-b^2}+b\sqrt{1-a^2} \leq \left[a^2+\left(1-a^2\right)\right]\left[\left(1-b^2\right)+b^2\right]=1$$

Since $a\sqrt{1-b^2}+b\sqrt{1-a^2}=1$, we have

$$a\sqrt{1-b^2}+b\sqrt{1-a^2}=\left[a^2+\left(1+a^2\right)\right]\left[\left(1-b^2\right)+b^2\right].$$

So $\dfrac{\sqrt{1-b^2}}{a}=\dfrac{b}{\sqrt{1-a^2}}$, $a^2+b^2=1$.

Proving complex inequalities using Cauchy inequality

Example 2[3] If $a,b,c,d,$ are positive real numbers,and $a+b+c+d=4$,then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \geq 4+(a-b)^2. \tag{7}$$

Proof. Since

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}-(a+b+c+d)$$

$$=\left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{d}+d-2c\right)+\left(\frac{d^2}{a}+a-2d\right)$$

$$=\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-d)^2}{d}+\frac{(d-a)^2}{a}. \tag{8}$$

From (5) we get

$$\left[\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-d)^2}{d}+\frac{(d-a)^2}{a}\right](a+b+c+d)$$

$$\geq \left(|a-b|+|b-c|+|c-d|+|d-a|\right)^2. \tag{9}$$

Since $|b-c|+|c-d|+|d-a| \geq |a-b|$, we have

$$\left(|a-b|+|b-c|+|c-d|+|d-a|\right)^2 \geq 4(a-b)^2 \tag{10}$$

From (8), (9) and (10),we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \geq a+b+c+d+\frac{4(a-b)^2}{a+b+c+d}. \tag{11}$$

Since $a+b+c+d=4$, From (11),we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\geq 4+(a-b)^2.$$

Finding extreme value problems by using Cauchy inequality

Example 3[1] Analyse the maximum of $y=3\sqrt{x-5}+4\sqrt{9-x}$.

using Cauchy inequality $|a_1b_1+a_2b_2|\leq\sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)}$,we get

$$3\sqrt{x-5}+4\sqrt{9-x}\leq\sqrt{(3^2+4^2)\left[\left(\sqrt{x-5}\right)^2+\left(\sqrt{9-x}\right)^2\right]}=10.$$

Solving equation using Cauchy inequality

Example 4[3] Solve the equation

$$\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x,\ x>1. \tag{12}$$

In order to using Cauchy inequality, we construct two groups of real numbers $\sqrt{x-\frac{1}{x}},\sqrt{\frac{1}{x}},1,\sqrt{x-1}$.

Using Cauchy inequality (1), we obtain

$$\left(\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}\right)^2=\left(\sqrt{x-\frac{1}{x}}+\sqrt{\frac{1}{x}}\cdot\sqrt{x-1}\right)^2$$

$$\leq\left(x-\frac{1}{x}+\frac{1}{x}\right)(1+x-1)=x^2. \tag{13}$$

Since $\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}=x$, we have

$$\left(\sqrt{x-\frac{1}{x}}+\sqrt{\frac{1}{x}}\cdot\sqrt{x-1}\right)^2=\left(x-\frac{1}{x}+\frac{1}{x}\right)(1+x-1). \tag{14}$$

Using Cauchy inequality (1), we have

$$\sqrt{x-\frac{1}{x}}\cdot\sqrt{x-1}=\sqrt{\frac{1}{x}}\Leftrightarrow x\left(x^2-x-1\right)=0\Leftrightarrow x=\frac{1+\sqrt{5}}{2}.$$

After checking, we see that $x=\frac{1+\sqrt{5}}{2}$ is the root of equation (12).

Summary

This paper studies the Cauchy inequality, discusses the application skills of Cauchy inequality in the middle school mathematics, illustrates the applications of the Cauchy inequality in the proof of the complex inequalities and the extreme value problems, and finally summarizes the application scope of Cauchy inequality and solution procedure.

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