

The Properties of Convex Function and its Application

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Abstract. Convex function is a kind of very important function. In the proving inequality, optimization theory, operations research, control theory and other fields of applied mathematics, convex function has many applications. This paper introduces the definition and properties of convex function, and Hadamard inequality; then discusses applications of convex function in proving inequalities and verifying astringency of series.

Introduction

Convex function is an important concept in mathematical analysis, which relates to the proof of many mathematical propositions. In higher mathematics, we frequently encountered this kind of special function when we discuss properties of functions by derivative. Convex function has very practical value in many practical applied problems. Many authors [1-9] studied properties and applications of convex function.

Definition (see [1]) Suppose that $f(x)$ be a function defined on interval I . If the following relation

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2), \forall x_1, x_2 \in I, t \in (0,1), \quad (1)$$

holds. Then, $f(x)$ is called a convex function on interval I . If the inequality (1) is the strict inequality, Then $f(x)$ is called a strict convex function on interval I .

Lemma 1 (see [1]) If $f(x)$ is a differential function defined on the closed interval $[a, b]$. Then $f(x)$ is a convex function on the closed interval $[a, b]$ when and only when

$$f''(x) \geq 0, x \in (a, b). \quad (2)$$

$f(x)$ is a strict convex function when and only when

$$f''(x) > 0, x \in (a, b). \quad (3)$$

Lemma 2 (see [1]) If $f(x)$ is differential on the closed interval $[a, b]$, then the following statement are equivalence

$f(x)$ is a convex function on interval $[a, b]$;

$f'(x)$ is a increasing function on interval $[a, b]$;

For $\forall x_1, x_2 \in [a, b]$, the inequality

$$f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1) \quad (4)$$

holds.

Lemma 3 (see [2]) If $f(x)$ is convex function on the closed interval $[a, b]$, then the following inequality

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \varphi(x) dx \leq \frac{\varphi(a) + \varphi(b)}{2} \quad (5)$$

holds.

This paper firstly introduces the definition and properties of convex function, and Hadamard inequality; then discusses applications of convex function in proving inequalities and verifying astringency of series.

Application of convex function

The convexity of convex function and its definition is established on the bases of inequalities. This fact makes the convex function to be an important tool in proving inequality.

Example 1 (see [2]) Suppose that $\varphi(x)$ is a continuous function on interval $[0, a]$, and $f(y)$ is twice differentiable with $f''(x) \geq 0$, then

$$\frac{1}{a} \int_0^a f(\varphi(t)) dt \geq f\left(\frac{1}{a} \int_0^a \varphi(t) dt\right). \quad (6)$$

Proof. Let $b = \frac{1}{a} \int_0^a \varphi(t) dt$, then (6) can be rewritten

$$\frac{1}{a} \int_0^a f(\varphi(t)) dt \geq af(b).$$

Since $f''(x) \geq 0, x \in (-\infty, +\infty)$, by Lemma 1 and Lemma 2, we have $f(x) \geq f(b) + f'(b)(x-b)$, thus

$$f(\varphi(t)) \geq f(b) + f'(b)(\varphi(t) - b).$$

Integrating the two side of the above inequality, we have

$$\begin{aligned} \int_0^a f(\varphi(t)) dt &\geq \int_0^a f(b) dt + \int_0^a f'(b)(\varphi(t) - b) dt \\ &= af(b) + f'(b) \int_0^a \varphi(t) dt - af'(b) \cdot b \\ &= af(b) + f'(b) \int_0^a \varphi(t) dt - f'(b) \cdot a \cdot \frac{1}{a} \int_0^a (\varphi(t)) dt. \\ &= af(b) \end{aligned}$$

So we have proved the inequality

$$\frac{1}{a} \int_0^a f(\varphi(t)) dt \geq f\left(\frac{1}{a} \int_0^a \varphi(t) dt\right).$$

Example 2 (see [3]) Let $a > 0, b > 0$, prove the following inequality

$$\sqrt{ab} \leq \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \leq \frac{a+b}{2}. \quad (7)$$

Proof. Let $f(x) = -\ln x$, then $f''(x) = \frac{1}{x^2} > 0, x \in (0, +\infty)$. By Lemma 1, we see that $f(x)$ is a convex function on $(0, +\infty)$. By hadamard inequality (5) in Lemma 3, for two positive real numbers a, b , we have

$$\frac{\ln b + \ln a}{2} \leq \frac{\int_a^b \ln x dx}{b-a} \leq \ln \frac{a+b}{2},$$

i.e.

$$\ln \sqrt{ab} \leq \ln \left(\frac{1}{e} \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \right) \leq \ln \frac{a+b}{2}.$$

It implies that

$$\sqrt{ab} \leq \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \leq \frac{a+b}{2}.$$

Example 3 (see [4]) Let $x \in \left(0, \frac{\pi}{2}\right)$, prove the inequality

$$(\sin x)^{1-\cos 2x} + (\cos x)^{1+\cos 2x} \geq \sqrt{2}. \quad (8)$$

Proof. Since $f(x) = x^x$ is a convex function on $(0, +\infty)$, for $a > 0, b > 0$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}[f(a) + f(b)].$$

Let $a = \sin^2 x, b = \cos^2 x$, then

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{\sin^2 x + \cos^2 x}{2}\right) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{2}.$$

Since $\frac{1}{2}[f(a) + f(b)] = \frac{1}{2}\left[(\sin^2 x)^{\sin^2 x} + (\cos^2 x)^{\cos^2 x}\right]$, we have

$$(\sin^2 x)^{\sin^2 x} + (\cos^2 x)^{\cos^2 x} \geq \sqrt{2}.$$

It implies that

$$(\sin x)^{1-\cos 2x} + (\cos x)^{1+\cos 2x} \geq \sqrt{2}.$$

Example 4 (see [5]) Suppose that $f(x)$ is a convex function on $[N, +\infty)$, and $f'(+\infty)$ is bounded, then $\sum_{i=1}^{\infty} a_k$ converges, where

$$a_k = \frac{f(k) + f(k+1)}{2} - \int_k^{k+1} f(x) dx.$$

Proof. Since $f(x)$ is a convex function on $[N, +\infty)$, by Lemma 2, we see that $f'(x)$ is increasing on $[N, +\infty)$ and $f(x) \geq f(k) + f'(k)(x-k)$ holds. So

$$\begin{aligned} a_k &= \frac{1}{2} \left(f(k) + f(k+1) - 2 \int_k^{k+1} f(x) dx \right) \\ &\leq \frac{1}{2} \left(f(k) + f(k+1) - 2 \int_k^{k+1} [f(k) + (x-k)f'(k)] dx \right) \\ &= \frac{1}{2} (f(k) + f(k+1) - 2f(k) - (2k+1)f'(k) + 2kf'(k)) \\ &= \frac{1}{2} (f(k+1) - f(k) - f'(k)) = \frac{1}{2} (f'(\xi) - f'(k)) \\ &\leq \frac{1}{2} (f'(k+1) - f'(k)), \end{aligned}$$

where $\xi \in (k, k+1)$. By Lemma 3 we see that $\frac{f(k) + f(k+1)}{2} \geq \int_k^{k+1} f(x) dx$ holds, So $\sum_{i=1}^{\infty} a_k$ is

series with positive term. Since $\sum_{k=n}^n a_k \leq \frac{1}{2} (f'(n+1) - f'(N)) \leq \frac{1}{2} (f'(+\infty) - f'(N))$, we see that positive

term series $\sum_{i=1}^{\infty} a_k$ converges.

Example 5 (see [5]) Suppose that $f(x)$ is a convex function on $[1, +\infty)$, and $f'(+\infty)$ is bounded, then

the convergence of series $\sum_{k=1}^{\infty} f(k)$ is same as integral $\int_1^{+\infty} f(x)dx$.

Proof. Since $f(x)$ is a convex function, by Lemma 2, we know that $f''(x) \geq 0$ and $f'(x)$ is increasing, by Lemma 3 we see that $\frac{f(k)+f(k+1)}{2} \geq \int_k^{k+1} f(x)dx$ holds. By Example 4, we see that the series with positive term

$$\sum_{k=1}^{\infty} \left(\frac{f(k)+f(k+1)}{2} - \int_k^{k+1} f(x)dx \right)$$

converges. Since

$$\sum_{k=1}^{\infty} \left(\frac{f(k)+f(k+1)}{2} \right) - \int_1^{+\infty} f(x)dx = \sum_{k=1}^{\infty} \left(\frac{f(k)+f(k+1)}{2} - \int_k^{k+1} f(x)dx \right),$$

We see that the convergence of series $\sum_{k=1}^{\infty} \frac{f(k)+f(k+1)}{2}$ is same as the integral $\int_1^{+\infty} f(x)dx$. Because $f(x)$ is a convex function, we know that there is a $M > 0$ such that $f(x)$ is Invariant sign when $x > M$. Thus, the convergence of series $\sum_{k=1}^{\infty} \frac{f(k)+f(k+1)}{2}$ is same as $\sum_{k=1}^{\infty} f(k)$. In summary, the convergence of series $\sum_{k=1}^{\infty} f(k)$ is same as $\int_1^{+\infty} f(x)dx$.

Summary

In many problems, we often encountered in the proof of some inequalities. Proof of inequality often requires some high skills. Proof process can be made to be simple and clear using properties of a convex function. But the key of this method is constructing appropriate convex functions. This paper discusses applications of convex function in proving inequalities and verifying astringency of series.

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