

Application of Convex Functions and Jensen Inequality

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Abstract. Functions are the important fundamental concepts in mathematics and the key research object in mathematical analysis. Convex function is a class of very important functions which is widely used in pure mathematics, functional analysis, optimization theory, mathematical economics. This paper firstly introduces a definition of convex function, and then gives properties of convex function and its judging theorem, finally, gives examples which show how to prove more complex inequalities using Jensen inequality of discrete form.

Introduction

Since the beginning of twenty-first Century the establishment of the theory of convex functions, the concept of convex function has been widely used in many branches of mathematics, for example in mathematical analysis, function theory, functional analysis, optimization theory and so on. Many authors [1-11] studied properties and applications of convex function.

Definition (see [1]) Let $f(x)$ be a function defined on interval I . If

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2), \forall x_1, x_2 \in I, \lambda \in (0,1). \quad (1)$$

Then, $f(x)$ is called a convex function on interval I , and $f(x)$ is called a strict convex function on interval I when the inequality (1) is the strict inequality.

Lemma 1 (see [2]) Suppose that $f(x)$ is a continuous function defined on the closed interval $[a, b]$, and is differential on the open interval (a, b) . Then $f(x)$ is a convex function on the closed interval $[a, b]$ when and only when

$$f''(x) \geq 0, x \in (a, b). \quad (2)$$

$f(x)$ is a strict convex function when and only when

$$f''(x) > 0, x \in (a, b). \quad (3)$$

Lemma 2 (see [3]) (Jensen inequality of discrete form) Suppose that $f(x)$ is a convex function on interval $[a, b]$. Then, for any $\forall x_i \in [a, b], i = 1, 2, \dots, n, \lambda_i \in (0,1), \sum_{i=1}^n \lambda_i = 1$, the inequality

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i), \quad (4)$$

holds.

This paper discusses how to prove more complex inequalities using Jensen inequality of discrete form.

Application of Jensen inequality of discrete form

In many problems, we often encountered in the proof of some inequalities. Proof of inequality often requires some high skills. Proof process can be made to be simple and clear using Jensen inequality of the discrete form of a convex function. But the key of this method is constructing appropriate convex functions

Example 1 (see [4]) Prove the inequality

$$\frac{e^x + e^y}{2} > e^{\frac{x+y}{2}}. \quad (5)$$

Proof: Let $f(t) = e^t$, we have $f''(t) = e^t > 0$, then

$f(t)$ is a strict convex function on interval $(-\infty, +\infty)$. From (1), we have

$$\frac{e^x + e^y}{2} > e^{\frac{x+y}{2}}.$$

Example 2(see [1]) Prove the inequality

$$(abc)^{\frac{a+b+c}{3}} < a^a b^b c^c. \quad (6)$$

where a, b, c are positive real numbers.

Proof: Let $f(x) = x \ln x, x > 0$, then

$$f'(x) = 1 + \ln x, f''(x) = \frac{1}{x} > 0.$$

By Lemma 1, we see that $f(x)$ is a convex function on interval $(0, +\infty)$. From (4) in Lemma 2, we have

$$f\left(\frac{a+b+c}{3}\right) < \frac{1}{3}(f(a) + f(b) + f(c)),$$

i.e.

$$\frac{a+b+c}{3} \ln \frac{a+b+c}{3} < \frac{1}{3}(a \ln a + b \ln b + c \ln c).$$

It implies that $\left(\frac{a+b+c}{3}\right)^{a+b+c} < a^a b^b c^c$.

Since $\sqrt[3]{abc} < \frac{a+b+c}{3}$, we have

$$(abc)^{\frac{a+b+c}{3}} < a^a b^b c^c.$$

Example 3(see [5]) Prove the following inequalities

$$(a) \arctan(\lambda_1 a_1 + \dots + \lambda_n a_n) \geq \lambda_1 \arctan a_1 + \dots + \lambda_n \arctan a_n; \quad (7)$$

$$(b) \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n \geq a_1^{\lambda_1} \cdot a_2^{\lambda_2} \cdot \dots \cdot a_n^{\lambda_n}. \quad (8)$$

where $a_j \geq 0, \lambda_j \geq 0, j = 1, 2, \dots, n, \lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.

Proof: (a) Let $f(x) = -\arctan x, x \in (0, +\infty)$, then

$$f'(x) = -\frac{1}{1+x^2}, f''(x) = \frac{2x}{(1+x^2)^2} > 0, x \in (0, +\infty).$$

Using Lemma 1, we see that $f(x)$ is a strict convex function on interval $(0, +\infty)$. From Jensen inequality (4), we get

$$-\arctan(\lambda_1 a_1 + \dots + \lambda_n a_n) \leq -(\lambda_1 \arctan a_1 + \dots + \lambda_n \arctan a_n).$$

It implies the required inequality $\arctan(\lambda_1 a_1 + \dots + \lambda_n a_n) \geq \lambda_1 \arctan a_1 + \dots + \lambda_n \arctan a_n$.

(b) Let $f(x) = -\ln x, x \in (0, +\infty)$, then

$$f'(x) = -\frac{1}{x}, f''(x) = \frac{1}{x^2} > 0, x \in (0, +\infty).$$

We see that $f(x)$ is a strict convex function on interval $(0, +\infty)$. By Jensen inequality (4), we obtain

$$-\ln(\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n) \leq -(\lambda_1 \ln a_1 + \lambda_2 \ln a_2 + \dots + \lambda_n \ln a_n) = -\ln(a_1^{\lambda_1} \cdot a_2^{\lambda_2} \cdot \dots \cdot a_n^{\lambda_n}),$$

i.e. $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n \geq a_1^{\lambda_1} \cdot a_2^{\lambda_2} \cdot \dots \cdot a_n^{\lambda_n}$.

Example 4(see [6]) Suppose that $a_i, b_i \in R^+ (i=1,2,\dots,n)$, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, then

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \geq \frac{1}{2} \sum_{i=1}^n a_i. \quad (9)$$

Proof: Let $S = \sum_{i=1}^n a_i$, then $S = \sum_{i=1}^n b_i$, $\sum_{i=1}^n \frac{a_i}{S} = 1$. Let $f(x) = \frac{1}{1+x}$, $x \in (0, \infty)$,

then $f'(x) = -\frac{1}{(1+x)^2}$, $f''(x) = \frac{2}{(1+x)^3} > 0$, $x \in (0, \infty)$. So $f(x)$ is a strict convex function on

interval $(0, +\infty)$. Let $x_i = \frac{b_i}{a_i}$, $i=1,2,\dots,n$. Since $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = S$, we have

$$\begin{aligned} \sum_{i=1}^n \frac{a_i^2}{a_i + b_i} &= S \sum_{i=1}^n \frac{a_i}{S} \cdot \frac{1}{1+x_i} = S \sum_{i=1}^n \frac{a_i}{S} f(x_i) > S f\left(\sum_{i=1}^n \frac{a_i}{S} x_i\right) \\ &= S \frac{1}{1 + \sum_{i=1}^n \frac{a_i}{S} x_i} = \frac{S}{1 + \sum_{i=1}^n \frac{b_i}{S}} = \frac{S}{2} = \frac{1}{2} \sum_{i=1}^n a_i, \end{aligned}$$

i.e.

$$\sum_{i=1}^n \frac{a_i^2}{a_i + b_i} \geq \frac{1}{2} \sum_{i=1}^n a_i.$$

Example 5(see [7]) Suppose that $g(x)$ is a integrable function on interval $[a, b]$ with $c \leq g(x) \leq d$, $f(y)$ is a convex function on interval $[c, d]$, then

$$f\left(\frac{1}{b-a} \int_a^b g(x) dx\right) \leq \frac{1}{b-a} \int_a^b f[g(x)] dx. \quad (10)$$

Proof: Since $f(y)$ (or $g(x)$) is integrable function on interval $[c, d]$ (or on interval $[a, b]$), we have

$$f\left(\frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n g\left(a + \frac{b-a}{n} i\right) \frac{b-a}{n}\right) = f\left(\frac{1}{b-a} \int_a^b g(x) dx\right),$$

$$\frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(g\left(a + \frac{b-a}{n} i\right)\right) \frac{b-a}{n} = \frac{1}{b-a} \int_a^b f[g(x)] dx.$$

Let $y_i = g\left(a + \frac{b-a}{n} i\right)$, $i=1,2,\dots,n$. Since $f(y)$ is a convex function on interval $[c, d]$, using Jensen inequality

(4), we have

$$\begin{aligned} f\left(\frac{1}{b-a} \sum_{i=1}^n g\left(a + \frac{b-a}{n} i\right) \frac{b-a}{n}\right) &= f\left(\frac{1}{n} \sum_{i=1}^n y_i\right) \leq \frac{1}{n} \sum_{i=1}^n f(y_i) \\ &= \frac{1}{b-a} \sum_{i=1}^n f\left(g\left(a + \frac{b-a}{n} i\right)\right) \frac{b-a}{n}. \end{aligned}$$

Limiting on both sides of the above inequality, we have

$$\lim_{n \rightarrow \infty} f\left(\frac{1}{b-a} \sum_{i=1}^n g\left(a + \frac{b-a}{n} i\right) \frac{b-a}{n}\right) \leq \lim_{n \rightarrow \infty} \frac{1}{b-a} \sum_{i=1}^n f\left(g\left(a + \frac{b-a}{n} i\right)\right) \frac{b-a}{n}.$$

Since $f(y)$ is a continuous function, we obtain

$$f\left(\frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n g\left(a + \frac{b-a}{n} i\right) \frac{b-a}{n}\right) \leq \frac{1}{b-a} \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(g\left(a + \frac{b-a}{n} i\right)\right) \frac{b-a}{n}.$$

By the Integrable property of f and $f \circ g$, we have proved the required inequality

$$f\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \leq \frac{1}{b-a}\int_a^b f[g(x)]dx.$$

Summary

In many problems, we often encountered in the proof of some inequalities. Proof of inequality often requires some high skills. Proof process can be made to be simple and clear using Jensen inequality of the discrete form of a convex function. But the key of this method is constructing appropriate convex functions. This paper gives examples which show how to prove more complex inequalities using Jensen inequality of discrete form.

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