The Action of Group Object in A Topos

Tao Lu 1, a and Zhenzhen Zhu 2,b

¹School of Mathematics Science, HuaiBei Normal University, HuaiBei, AnHui, China ²School of Mathematics Science, HuaiBei Normal University, HuaiBei, AnHui, China ^a lutao7@live.com, ^b995600216@gg.com

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Abstract. In this paper, based on the definition of group object, the definition of action of group object on arbitrary object in a topos is given, some equivalent characterizations are also obtained.

Introduction

Recall that a topos is a category which has finite limits and every object of has a power object. For a fixed object A of category, the power object of A is an object PA which represents $^{Sub}(_- \times ^A)$, so that $(_-, ^{PA}) \cong ^{Sub}(_- \times ^A)$ naturally. It means that for any arrow $^{B'} \xrightarrow{f} ^{B}$, the following diagram commutes, where $^{\Phi}$ is the natural isomorphism.

$$\begin{array}{c|c} \operatorname{Hom}_{\mathcal{E}}(B,PA) & \xrightarrow{\varphi(A,B)} & \operatorname{Sub}(B \times A) \\ \operatorname{Hom}_{\mathcal{E}}(f,PA) \downarrow & & & & & & \\ \operatorname{Hom}_{\mathcal{E}}(B',PA) & \xrightarrow{\varphi(A,B')} & \operatorname{Sub}(B' \times A) \end{array}$$

As a matter of fact, the category of sheaves of sets on a topological space is a topos. In particular, the category of sets is a topos. For details of the treatment of toposes and sheaves please see [1], [2], [3], [4]. For a general background on category theory please refers to [5], [6],[7],[8],[9],[11],[12].

Main results

Throughout this paper, we work with a fixed topos , All objects mentioned belong to the topos . We begin with some definitions.

Definition 1. A group object in is an object G of equipped with three arrows:

- 1) $e:1 \rightarrow G$, the unit;
- 2) $m: G \times G \rightarrow G$, the product;
- 3) $i: G \rightarrow G$

And the three arrows satisfy the following diagrams.

Fig. 2

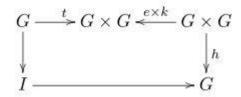


Fig. 3

In above two figures, h is the projective morphism and $k: G \times G \to G$ is the diagonal morphism. one can express this equivalently by the familiar identities:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c; \quad a \cdot e = e \cdot a = a$$

It follows that the hom-set Hom (X, G) are natural in X, it determines a group structure; conversely, a group structure on HomE (X,G) gives the structure of an group object.

In topos , a morphism $X \xrightarrow{f} G$ is regarded as a generalized element of the group objects G, the generalized element is applied successfully in the patially ordered objects, please refer [8].

By the above, one can express the composite

$$fg = m \circ \langle f, g \rangle : X \to G \times G \to G$$

or an inverse

$$f^{-1} = i \circ f : X \xrightarrow{f} G \xrightarrow{i} G$$

Definition 2. Let G be a group objects and Ω any object of Ω . An action of Ω is a morphism $\mu = \mu_{\Omega}: G \times \Omega \to \Omega$ such that the following both diagrams commute.

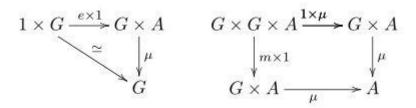


Fig. 4

This action can be denoted by a dot, as in $\mu(x,g) = x \cdot g$, for $X \xrightarrow{x} \Omega, X \xrightarrow{g} G$.

Definition 3. Let G be any group object and Ω any object. If the action of G on Ω is defined by $\alpha \cdot g = \alpha$ for all $\alpha \in \text{Hom}$ (X,Ω) and $g \in \text{Hom}$ (X,G), then the action is trivial.

Definition 4. Let G be any group object and Ω any object. If the identity is the only element $g \in G$ Hom (X, G) such that $\alpha \cdot g = \alpha$ for all $\alpha \in G$ Hom (X, Ω) , then the action is faithful.

In general, the kernel of an action is the set of group elements that act like 1 and "fix" all $\alpha \in \text{Hom}$ (X, Ω) . (We say g fixs α if $\alpha \cdot g = \alpha$)

The most useful actions of finite group objects are usefully internal (in some sense) to the group structure. There are, two important ways in which a group object G can act on itself. (In other words, we can take $\Omega = G$.) The first of these is the regular action defined by $x \cdot g = xg$ for all $x \in Hom$ (X, G) and $g \in Hom$ (X,G). The other important action of G on itself is the conjugation action, where we define $x \cdot g = x^g = g^{-1}xg$.

If $X \subseteq G$ is any subobject of G and $g \in Hom$ (X, G), then as usual we define the product $Xg = \{xg \mid x \in X\}$. This can be used to define an action of G on the set of all subsets of G by setting $X \cdot g = Xg$

Lemma 1. Let G be any group object and G any object and G act on G. For each $G \in G$ is a homomorphism whose kernel is equal to the kernel of the action. (X, G), define $G \in G$ is a homomorphism whose kernel is equal to the kernel of the action.

Proof. If $g, h \in \text{Hom}$ (X, G) and $\alpha \in \text{Hom}$ (X, G), then $(\alpha)\pi_g\pi_h = (\alpha \cdot g)\pi_h = \alpha(gh) = (\alpha)\pi_{gh}$

and so $\pi_g \pi_h = \pi_{gh}$ for all $g, h \in G$. Also, by definition 2, $(\alpha)\pi_1 = \alpha \cdot 1 = \alpha$, and so π_1 is the identity function i_{Ω} on Ω .

Now for $g \in \text{Hom}$ (X, G), we have $\pi_g \pi_{g^{-1}} = \pi_1 = \pi_{g^{-1}} \pi_g$, thus π_g is an element of Sym (Ω) .

We have $\theta(g)\theta(h) = \pi_g \pi_h = \pi_{gh} = \theta(gh)$ and θ is a homomorphism. An element $g \in G$ lies in $\ker^{(\theta)}$ iff $\pi_g = i_{\Omega}$, and this is equivalent to saying that $\alpha \cdot g = \alpha$ for all $\alpha \in Hom^{(X,\Omega)}$; that is, g is in the kernel of the action.

Group actions can also be used to produce subgroup objects. If G acts on Ω and $\alpha \in \operatorname{Hom}(X,G)$, we write $G_{\alpha} = \{g \in \operatorname{Hom}(X,G) | \alpha \cdot g = \alpha\}$. This is called the stabilizer of α in $\operatorname{Hom}(X,G)$, and it is routine to check that G_{α} is always a subgroup object of G. We consider some examples.

Let G act on itself via conjugation. If $x \in G$, then $G_x = \{g \in G \mid x^g = x\}$, and since $x^g = x$, we can see that the stabilizer in G of $x \in Hom$ (X,G) under conjugation is just $G_G(x)$.

We return now to the general case of a group object G acting on an object Ω .

Definition 4. The action is transitive if for every two elements $\alpha, \beta \in \text{Hom}$ (X, Ω) , there exists an element $g \in \text{Hom}$ (X,G) with $\alpha \cdot g = \beta$.

For instance, the regular action of G and the usual action on the right cosets of a subgroup object are transitive. In general conjugation action of G on itself is not transitive, since if $x, y \in Hom$ (X, G) have different orders, then there can exist no $g \in Hom$ (X, G) with $x^g = y$.

In general, if G acts on Ω , then the orbits of this action are the sets of the form $\{\alpha \cdot g \mid g \in Hom\ (X,G)\}\subseteq \Omega$.

Lemma 2. Let G acts on Ω . Then the orbits partition Ω . This means

a. Ω is the union of the orbits and

b. any two different orbits are disjoint.

Proof. Write $O_{\alpha} = \{\alpha \cdot g \mid g \in \text{Hom } (X, G)\}$. Since $\alpha \cdot 1 = \alpha$, we have $\alpha \in O_{\alpha}$ and thus $\Omega = \bigcup_{\alpha \in \Omega} O_{\alpha}$

proving part (a).

We show now that if $\gamma \in O_{\alpha}$, then $O_{\gamma} = O_{\alpha}$. We have $\gamma = \alpha \cdot x$ for some $x \in Hom$ (X, G), and thus

$$\gamma \cdot g = (\alpha \cdot x) \cdot g = \alpha \cdot xg \in O_{\alpha}$$

This yields $O_{\gamma} \subseteq O_{\alpha}$. Also, $\alpha = \gamma \cdot x^{-1}$, so that $\alpha \in O_{\gamma}$ and hence the above argument yields $O_{\alpha} \subseteq O_{\gamma}$. We have shown that $O_{\alpha} = O_{\gamma}$, as claimed.

Finally, if $O_{\alpha} \cap O_{\beta} \neq \emptyset$, choose $\gamma \in O_{\alpha} \cap O_{\beta}$. Then $O_{\alpha} = O_{\gamma} = O_{\beta}$. And part (b) is proved.

The partition of Ω by the orbits of an action is analogous to the partition of a group by the cosets of a subgroup. This is not entirely accidental, since if $H \subseteq G$, we can let H act on G by right multiplication. In this case, the orbit containing $g \in G$ is exactly the left cosets gH.

One of the major applications of actions is for counting. The key to this is the following theorem.

Theorem Let G act on Ω and let G be an orbit of this action. Let $\alpha \in G$ and write G and write G stabilizer. Then there exists a bijection $G \leftrightarrow \{Hx \mid x \in G\}$.

Proof. We construct a map $f: O \leftrightarrow \{Hx \mid x \in G\}$ as follows. If $\beta \in O$, choose $x \in G$ with $\beta = \alpha \cdot x$, and set $f(\beta) = Hx$. We need to check that this is well defined. In other words, if also $\beta = \alpha \cdot y$, we must establish that Hx = Hy, as required.

It is clear that f maps onto $\{Hx \mid x \in G\}$, since for any x, we have $Hx = f(\alpha \cdot x)$. Finally, to show that f is injective, suppose that $f(\beta) = f(\gamma)$. Then $\beta = \alpha \cdot x$ and $\gamma = \alpha \cdot y$ with Hx = Hy. This yields y = hx for some $h \in H$, and hence $\gamma = \alpha \cdot y = (\alpha \cdot h) \cdot x = \alpha \cdot x = \beta$, where the third equality holds since $h \in H = G_{\alpha}$

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