

The Action of Group Object in A Topos

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Abstract. In this paper, based on the definition of group object, the definition of action of group object on arbitrary object in a topos is given, some equivalent characterizations are also obtained.

Introduction

Recall that a topos is a category which has finite limits and every object has a power object. For a fixed object A of category \mathcal{E} , the power object of A is an object PA which represents $\text{Sub}(_ \times A)$, so that $(_, PA) \simeq \text{Sub}(_ \times A)$ naturally. It means that for any arrow $B' \xrightarrow{f} B$, the following diagram commutes, where φ is the natural isomorphism.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{E}}(B, PA) & \xrightarrow{\varphi(A, B)} & \text{Sub}(B \times A) \\ \text{Hom}_{\mathcal{E}}(f, PA) \downarrow & & \downarrow \text{Sub}(f \times A) \\ \text{Hom}_{\mathcal{E}}(B', PA) & \xrightarrow{\varphi(A, B')} & \text{Sub}(B' \times A) \end{array}$$

Fig. 1

As a matter of fact, the category of sheaves of sets on a topological space is a topos. In particular, the category of sets is a topos. For details of the treatment of toposes and sheaves please see [1], [2], [3], [4]. For a general background on category theory please refers to [5], [6],[7],[8],[9],[11],[12].

Main results

Throughout this paper, we work with a fixed topos \mathcal{E} . All objects mentioned belong to the topos \mathcal{E} . We begin with some definitions.

Definition 1. A group object in \mathcal{E} is an object G of \mathcal{E} equipped with three arrows:

- 1) $e: 1 \rightarrow G$, the unit;
- 2) $m: G \times G \rightarrow G$, the product;
- 3) $i: G \rightarrow G$

And the three arrows satisfy the following diagrams.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{e \times h} & G \times G \\ \downarrow h \times e & & \downarrow h \\ G \times G & \xrightarrow{h} & G \end{array} \quad \begin{array}{ccccc} I \times G & \xrightarrow{j \times e} & G \times G & \xleftarrow{e \times j} & G \times I \\ \downarrow f & & \downarrow h & & \downarrow g \\ G & = & G & = & G \end{array}$$

Fig. 2

$$\begin{array}{ccccc}
G & \xrightarrow{t} & G \times G & \xleftarrow{e \times k} & G \times G \\
\downarrow & & & & \downarrow h \\
I & \xrightarrow{\quad} & G & &
\end{array}$$

Fig. 3

In above two figures, h is the projective morphism and $k : G \times G \rightarrow G$ is the diagonal morphism. one can express this equivalently by the familiar identities:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c; \quad a \cdot e = e \cdot a = a$$

It follows that the hom-set $\text{Hom}(X, G)$ are natural in X , it determines a group structure; conversely, a group structure on $\text{Hom}(X, G)$ gives the structure of an group object.

In topos, a morphism $X \xrightarrow{f} G$ is regarded as a generalized element of the group objects G , the generalized element is applied successfully in the partially ordered objects, please refer [8].

By the above, one can express the composite

$$fg = m \circ \langle f, g \rangle : X \rightarrow G \times G \rightarrow G$$

or an inverse

$$f^{-1} = i \circ f : X \xrightarrow{f} G \xrightarrow{i} G$$

Definition 2. Let G be a group objects and Ω any object of . An action of G on Ω is a morphism $\mu = \mu_\Omega : G \times \Omega \rightarrow \Omega$ such that the following both diagrams commute.

$$\begin{array}{ccc}
1 \times G & \xrightarrow{e \times 1} & G \times A \\
& \searrow \cong & \downarrow \mu \\
& & G
\end{array}
\quad
\begin{array}{ccc}
G \times G \times A & \xrightarrow{1 \times \mu} & G \times A \\
\downarrow m \times 1 & & \downarrow \mu \\
G \times A & \xrightarrow{\mu} & A
\end{array}$$

Fig. 4

This action can be denoted by a dot, as in $\mu(x, g) = x \cdot g$, for $X \xrightarrow{x} \Omega, X \xrightarrow{g} G$.

Definition 3. Let G be any group object and Ω any object. If the action of G on Ω is defined by $\alpha \cdot g = \alpha$ for all $\alpha \in \text{Hom}(X, \Omega)$ and $g \in \text{Hom}(X, G)$, then the action is trivial.

Definition 4. Let G be any group object and Ω any object. If the identity is the only element $g \in \text{Hom}(X, G)$ such that $\alpha \cdot g = \alpha$ for all $\alpha \in \text{Hom}(X, \Omega)$, then the action is faithful.

In general, the kernel of an action is the set of group elements that act like 1 and “fix” all $\alpha \in \text{Hom}(X, \Omega)$. (We say g fixes α if $\alpha \cdot g = \alpha$)

The most useful actions of finite group objects are usefully internal (in some sense) to the group structure. There are, two important ways in which a group object G can act on itself. (In other words, we can take $\Omega = G$.) The first of these is the regular action defined by $x \cdot g = xg$ for all $x \in \text{Hom}(X, G)$ and $g \in \text{Hom}(X, G)$. The other important action of G on itself is the conjugation action, where we define $x \cdot g = x^g = g^{-1}xg$.

If $X \subseteq G$ is any subobject of G and $g \in \text{Hom}(X, G)$, then as usual we define the product $Xg = \{xg \mid x \in X\}$. This can be used to define an action of G on the set of all subsets of G by setting $X \cdot g = Xg$.

Lemma 1. Let G be any group object and Ω any object and G act on Ω . For each $g \in \text{Hom}(X, G)$, define $\pi_g : \Omega \rightarrow \Omega$ by $(\alpha)\pi_g = \alpha \cdot g$. Then $\pi_g \in \text{Sym}(\Omega)$ and the map $\theta : G \rightarrow \text{Sym}(\Omega)$ defined by $\theta(g) = \pi_g$ is a homomorphism whose kernel is equal to the kernel of the action.

Proof. If $g, h \in \text{Hom}(X, G)$ and $\alpha \in \text{Hom}(X, \Omega)$, then

$$(\alpha)\pi_g\pi_h = (\alpha \cdot g)\pi_h = \alpha(gh) = (\alpha)\pi_{gh},$$

and so $\pi_g\pi_h = \pi_{gh}$ for all $g, h \in G$. Also, by definition 2, $(\alpha)\pi_1 = \alpha \cdot 1 = \alpha$, and so π_1 is the identity function i_Ω on Ω .

Now for $g \in \text{Hom}(X, G)$, we have $\pi_g\pi_{g^{-1}} = \pi_1 = \pi_{g^{-1}}\pi_g$, thus π_g is an element of $\text{Sym}(\Omega)$.

We have $\theta(g)\theta(h) = \pi_g\pi_h = \pi_{gh} = \theta(gh)$ and θ is a homomorphism. An element $g \in G$ lies in $\ker(\theta)$ iff $\pi_g = i_\Omega$, and this is equivalent to saying that $\alpha \cdot g = \alpha$ for all $\alpha \in \text{Hom}(X, \Omega)$; that is, g is in the kernel of the action.

Group actions can also be used to produce subgroup objects. If G acts on Ω and $\alpha \in \text{Hom}(X, \Omega)$, we write $G_\alpha = \{g \in \text{Hom}(X, G) \mid \alpha \cdot g = \alpha\}$. This is called the stabilizer of α in $\text{Hom}(X, G)$, and it is routine to check that G_α is always a subgroup object of G . We consider some examples.

Let G act on itself via conjugation. If $x \in G$, then $G_x = \{g \in G \mid x^g = x\}$, and since $x^x = x$, we can see that the stabilizer in G of $x \in \text{Hom}(X, G)$ under conjugation is just $C_G(x)$.

We return now to the general case of a group object G acting on an object Ω .

Definition 4. The action is transitive if for every two elements $\alpha, \beta \in \text{Hom}(X, \Omega)$, there exists an element $g \in \text{Hom}(X, G)$ with $\alpha \cdot g = \beta$.

For instance, the regular action of G and the usual action on the right cosets of a subgroup object are transitive. In general conjugation action of G on itself is not transitive, since if $x, y \in \text{Hom}(X, G)$ have different orders, then there can exist no $g \in \text{Hom}(X, G)$ with $x^g = y$.

In general, if G acts on Ω , then the orbits of this action are the sets of the form $\{\alpha \cdot g \mid g \in \text{Hom}(X, G)\} \subseteq \Omega$.

Lemma 2. Let G acts on Ω . Then the orbits partition Ω . This means

- Ω is the union of the orbits and
- any two different orbits are disjoint.

Proof. Write $O_\alpha = \{\alpha \cdot g \mid g \in \text{Hom}(X, G)\}$. Since $\alpha \cdot 1 = \alpha$, we have $\alpha \in O_\alpha$ and thus $\Omega = \bigcup_{\alpha \in \Omega} O_\alpha$

proving part (a).

We show now that if $\gamma \in O_\alpha$, then $O_\gamma = O_\alpha$. We have $\gamma = \alpha \cdot x$ for some $x \in \text{Hom}(X, G)$, and thus

$$\gamma \cdot g = (\alpha \cdot x) \cdot g = \alpha \cdot xg \in O_\alpha$$

This yields $O_\gamma \subseteq O_\alpha$. Also, $\alpha = \gamma \cdot x^{-1}$, so that $\alpha \in O_\gamma$ and hence the above argument yields $O_\alpha \subseteq O_\gamma$. We have shown that $O_\alpha = O_\gamma$, as claimed.

Finally, if $O_\alpha \cap O_\beta \neq \emptyset$, choose $\gamma \in O_\alpha \cap O_\beta$. Then $O_\alpha = O_\gamma = O_\beta$. And part (b) is proved.

The partition of Ω by the orbits of an action is analogous to the partition of a group by the cosets of a subgroup. This is not entirely accidental, since if $H \subseteq G$, we can let H act on G by right multiplication. In this case, the orbit containing $g \in G$ is exactly the left cosets gH .

One of the major applications of actions is for counting. The key to this is the following theorem.

Theorem Let G act on Ω and let O be an orbit of this action. Let $\alpha \in O$ and write $H = G_\alpha$, the stabilizer. Then there exists a bijection $O \leftrightarrow \{Hx \mid x \in G\}$.

Proof. We construct a map $f : O \leftrightarrow \{Hx \mid x \in G\}$ as follows. If $\beta \in O$, choose $x \in G$ with $\beta = \alpha \cdot x$, and set $f(\beta) = Hx$. We need to check that this is well defined. In other words, if also $\beta = \alpha \cdot y$, we must establish that $Hx = Hy$, as required.

It is clear that f maps onto $\{Hx \mid x \in G\}$, since for any x , we have $Hx = f(\alpha \cdot x)$. Finally, to show that f is injective, suppose that $f(\beta) = f(\gamma)$. Then $\beta = \alpha \cdot x$ and $\gamma = \alpha \cdot y$ with $Hx = Hy$. This yields $y = hx$ for some $h \in H$, and hence $\gamma = \alpha \cdot y = (\alpha \cdot h) \cdot x = \alpha \cdot x = \beta$, where the third equality holds since $h \in H = G_\alpha$.

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