Construction and counting of $2t+1$ Steiner triple

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Abstract. The methods of both constructing and enumerating Steiner triple systems of order $2t+1$ are proposed. It demonstrated the Complete trimap and descirbed the application of Complete bipartite graph in constitute design. The theorem concerning existence and construction of Steiner triple systems of order $2t+1$ is proved. 7×20 Steiner triple systems of order 19 and their entire constrction procedure are presented.

Introduction

Block design theory is an important branch of the combinatorial mathematics and plays an important role in experiment design, competition arrangement, digital communication areas. In 1850 a British mathematician Thomas P.Kirkman posed a famous 15 schoolgirls problem and solved this problem in the same year [1-2]. In 1971 D.R. Ray-Chaudhari and R.M. Wilson published a paper with topic Solution of Kirkman's schoolgirls problem to show how to construct Kirkman triple systems of order $6n+3$ [3-4]. In 1961 a Chinese mathematician Lu Jiaxi posed the decomposable condition of BIBD design [5-9].

Basic theory

Definition 1: Supposing Top Set $V(G)=\{c_1,c_2,\cdots,c_v\}$, Edge set $E(G)=\{c_1c_2,c_1c_3,\cdots,c_1c_v,c_2c_3,\cdots,c_2c_v,\cdots,c_{v-1}c_v\}$ and G is complete to K_v, if we put $|E(G)|=v(v-1)/2$ sides into triangular matrix so that it can exist the relation with any edge c_ic_j , top c_i and top c_j, then this triangular matrix called side matrix, and referred to as K'_v.

Definition 2: Supposing Top Set is $V(G)=\{c_1,c_2,\cdots,c_v\}$ and $V(G)$ include $V_i=\{c_m,c_{m+1},\cdots,c_{m+t-1}\}$, $V_j=\{c_p,c_{p+1},\cdots,c_{p+t-1}\}$, $V_k=\{c_q,c_{q+1},\cdots,c_{q+t-1}\}$ three subsets, if each top of V_i neighbor top t of V_j and top t of V_k, then graph G called complete trimap and denoted $K_{t,t,t}^{(i,j,k)}$, which three sub-graph is called complete ipartite graph, respectively $K_{t,t}^{(i,j)}$, $K_{t,t}^{(i,k)}$, $K_{t,t}^{(j,k)}$, meanwhile , $K_{t,t,t}^{(i,j,k)}=K_{t,t}^{(i,j)}\cup K_{t,t}^{(i,k)}\cup K_{t,t}^{(j,k)}$

If $t\times t$ complete graph K_3 exist in the complete trimap $K_{t,t,t}^{(i,j,k)}$ has separated , then $t\times t$ triple system matrix $K_{t,t,t}^{(i,j,k)}$ can be delieved as

$$K_{t,t,t}^{(i,j,k)}=\left\{\begin{matrix} \{m,p,q\} & \{m,p+1,q\} & \cdots & \{m,p+t-1,q+t-1\} \\ \{m+1,p+1,q\} & \{m+1,p+2,q\} & \cdots & \{m+1,p,q+t-1\} \\ \vdots & \vdots & \vdots & \vdots \\ \{m+t,p+t-1,q\} & \{m+t-1,p,q\} & \cdots & \{m+t-1,p+t-2,q+t-1\} \end{matrix}\right\}$$

Apparently, we put the fixed of minimum number from m, p, q, V_i, V_j, V_k into matrix $K_{t,t,t}^{(i,j,k)}$, then we can obtain $t \times t$ complete graph K_3, but $K_{t,t,t}^{(i,j,k)}$ has t kinds of structure rather than only one.

Theorem 1: Supposing Top Set is $V(G)=\{c_1,c_2,\cdots,c_v\}$ and $|V(G)|=v=2t+1$, t already exists Steiner triple's order, then it must be exist $v=2t+1$ order of steiner triple and this steiner triple's structure is equivalent to $v(v-1)/6$ kinds K_3 decompose from a complete graph K_v.

Proof: If K_v' is bordered matrix of $v=2t+1$ order complete graph K_v, then K_v''s i row j column and $3(v-1)/2$ side from the diagonal all can combine $(v-1)/2$ complete graph K_3, meanwhile, K_v''s i row j column and the diagonal outside can be divided into $t(t-1)/6$ triple system matrix $K_{2,2,2}^{(i,j,k)}$, which is composed by $K_{2,2}^{(i,j)}$, $K_{2,2}^{(i,k)}$, $K_{2,2}^{(j,k)}$ complete bipartite graph , therefore, it ultimate transform $v(v-1)/6$ triple system.

Theorem 2: Supposing Top Set is $V(G)=\{c_1,c_2,\cdots,c_v\}$ and $|V(G)|=v=2t+1$, t already exists steiner triple's order, G is complete to K_v, K_v' is bordered matrix of complete graph K_v, then K_v''s $t(t-1)/2$ sub-matrix with 2×2 dimension classification scheme such as $P_1,P_2,\cdots,P_{v+1}$, total $v+1$. Where the program P_i empressed: A section , consist of i row 1 column and $3(v-1)/2$ sides on the diagonal matrix and the section outside A can be divide into 2×2 dimension sub-matrix.

For example one , we set $t=3$, $v=2t+1=7$, then K_7''s 2×2 dimension sub-matrix programs have altogether $v+1=8$, which $P_1,P_2,\cdots,P_8$. After the division of the sub-matrix, we can obtain bordered matrix $K_7^{1(1)},K_7^{1(2)},\cdots,K_7^{1(8)}$, which

$$K_7^{1(1)}=\begin{array}{|cccccc|c} c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \\ 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & c_1 \\ & 2_3 & 2_4 & 2_5 & 2_6 & 2_7 & c_2 \\ & & 3_4 & 3_5 & 3_6 & 3_7 & c_3 \\ & & & 4_5 & 4_6 & 4_7 & c_4 \\ & & & & 5_6 & 5_7 & c_5 \\ & & & & & 6_7 & c_6 \end{array}, \quad \ldots, \quad K_7^{1(8)}=\begin{array}{|cccccc|c} c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & \\ 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & c_1 \\ & 2_3 & 2_4 & 2_5 & 2_6 & 2_7 & c_2 \\ & & 3_4 & 3_5 & 3_6 & 3_7 & c_3 \\ & & & 4_5 & 4_6 & 4_7 & c_4 \\ & & & & 5_6 & 5_7 & c_5 \\ & & & & & 6_7 & c_6 \end{array}$$

For example two, when t=9, then $v=2t+1=19$, the triple system of 19 order has 7 different configurations and disjoint triple system of 9 order. Details are as follows:

$$\begin{array}{c} KT_1(9) \\ \{1,2,3\}\{1,4,7\}\{1,5,6\}\{1,8,9\} \\ \{4,6,9\}\{2,6,8\}\{2,7,9\}\{2,4,5\} \\ \{5,7,8\}\{3,5,9\}\{3,4,8\}\{3,6,7\} \end{array}, \quad \ldots, \quad \begin{array}{c} KT_7(7) \\ \{1,2,9\}\{1,3,8\}\{1,4,6\}\{1,5,7\} \\ \{3,4,7\}\{2,6,7\}\{2,3,5\}\{2,4,8\} \\ \{5,6,8\}\{4,5,9\}\{7,8,9\}\{3,6,9\} \end{array}$$

When bordered matrix of complete graph K_{19} has emerged K_{19}', we can based on program P_1 make K_{19}' divided into bordered matrix of complete bipartite graph $K_{2,2}^{(i,j)}$, $K_{2,2}^{(i,k)}$, $K_{2,2}^{(j,k)}$, total $t(t-1)/2=36$, then we based on $KT_1(9)$ make $t(t-1)/2$ complete bipartite graph $K_{2,1}^{(i,j)}$, $K_{2,1}^{(i,k)}$, $K_{2,2}^{(j,k)}$ combine to $t(t-1)/6=12$ triple system matrix $K_{2,2,2}^{(i,j,k)}$ with 2×2 dimension and we will put 1 row , 1 column and the $3(v-1)/2$ sides on the diagonal of K_v' combine to $(v-1)/2=18$ complete graph K_3.

Finally , we obtain P_1 kinds $ST_{P_1}^{(1)}(19)$ of Steiner triple system with 19 order, then we will successively decompose $v(v-1)/6$ complete graph of K_3 base on $KT_2(9),\cdots,KT_7(9)$, this way we can obtain the 6 remaining of triple systems, they are $ST_{P_1}^{(2)}(19), ST_{P_1}^{(3)}(19),\cdots,ST_{P_1}^{(7)}(19)$.

Similarly, we can get $P_2 \ \ldots, P_{19}$ kinds of Steiner triple system

Counting of Steiner triple system with 19 order

We can find from the actual construction of Steiner triple system with $2t+1$, its number N is decided to the number $N^{(1)}$ of Steiner triple system with t order. 2×2 dimension sub-matrix of bordered matrix K_v' have $N^{(2)} = v+1$ kinds of classification program, which is $P_1, P_2, \cdots, P_{v+1}$ and the number of triple system matrix with 2×2 dimension of construct program is $N^{(3)} = 2$. According to the multiplication rule, the number of Steiner triple system with 19 order is $N = N^{(1)} \times N^{(2)} \times N^{(3)} = N^{(2)} \times 2(v+1) = 105\times 20\times 2$ 。

Basing on the Steiner triple system construct method of sub-matrix factorization of bordered matrix can apply to the construction of $v \equiv 1(\bmod t)$ Steiner triple system with v order.

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