Construction and Enumeration of Different Configuration

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Abstract. This paper clarifies the basic idea of block design on different configuration. Clarify the properties of (v,k,λ) derived design and $(v,v-k,v-2r+\lambda)$ complement design. Introduce the nature of (v,k,λ) derived design and $(v,v-k,v-2r+\lambda)$ complementary design. The method of constructing (v,k,λ) derived design and $(v,v-k,v-2r+\lambda)$ complementary design is given. The entire procedure of constructing (16,6,2) design (16,10,6) design is also presented completely. 11 disjoint (16,6,2) design and 11 disjoint (16,10,6) design are obtained.

Introduction

Block design theory is an important branch of combinatorial mathematics. It plays an important role in the field of experimental design, competition arrangements and digital communication. In 1850, England mathematician Kirkman proposed an interesting problem '15 girls question' and solve it in the same year. In 1971, D. K. Ray-Chaudhuri and R. M. Wilson published "the solution of Kirkman 15 girls question", in order to clarify the structure of Kirkman triple system. More than a century, whether there always exists $6n+3$ order of Kirkman triple system for each $n = 0,1,2,\cdots$ has been a difficult problem [1-5]. In 2003, the author proposes a method for the construction of high order Steiner triple system, and structures $v = 27,45,81,135$ order of Kirkman triple system [6, 7].

Basic idea

Let there be two multiple-element systems with v order, one is (v,k,λ) design with the block capacity of k $S' = \{B_1', B_2', \cdots, B_v'\}$ on $X = \{1,2,\cdots,v\}$; The other one is $(v,v-k,v-2r+\lambda)$ design with the block capacity of $v-k$ $S'' = \{B_1'', B_2'', \cdots, B_v''\}$ on $X = \{1,2,\cdots,v\}$. If there exists the following complementary relationship between v blocks $B_1^{(1)}.B_2^{(1)},\cdots,B_v^{(1)}$ of (v,k,λ) design $S' = \{B_1', B_2', \cdots, B_v'\}$ and v blocks $B_1^{(3)}, B_2^{(3)}, \cdots, B_v^{(3)}$ of $(v,v-k,v-2r+\lambda)$ design $S'' = \{B_1'', B_2'', \cdots, B_v''\}$:

$$B_1^{(3)} = X \cap B_1^{(1)}, B_2^{(3)} = X \cap B_2^{(1)},\cdots,B_v^{(3)} = X \cap B_v^{(1)}$$

Then the $S' = \{B_1', B_2', \cdots, B_v'\}$ design of (v,k,λ) is called derived design; the $S'' = \{B_1'', B_2'', \cdots, B_v''\}$ design of $(v,v-k,v-2r+\lambda)$ is called complementary design.

The derived design $S' = \{B_1', B_2', \cdots, B_v'\}$ of (v,k,λ) and the complementary design $S'' = \{B_1'', B_2'', \cdots, B_v''\}$ of $(v,v-k,v-2r+\lambda)$ has the following properties.

(1) The order of the derived design S' of (v,k,λ) equals to the order of the complementary design S'' of $(v,v-k,v-2r+\lambda)$.

(2) The block capacity of the derived design $S^{'}$ of (v,k,λ) is k, the block capacity of the complementary design $S^{'}$ of $(v,v-k,v-2r+\lambda)$ is $v-k$.

(3) There are complementary relationships between derived design $S^{'}$ of (v,k,λ) and the complementary design of $(v,v-k,v-2r+\lambda)$ as follows:

$$B_1^{''} = X \cap B_1^{'}, \quad B_2^{''} = X \cap B_2^{'}, \quad \ldots, \quad B_v^{''} = X \cap B_v^{'}$$

(4) The number of disjoint in the derived design $S^{'}$ of (v,k,λ) is the same with the complementary design $S^{'}$ of $(v,v-k,v-2r+\lambda)$.

(5) If the correlation matrix for the derived design $S^{'}$ of (v,k,λ) is A, then the correlation matrix for the complementary design $S^{'}$ of $(v,v-k,v-2r+\lambda)$ is $J_{v\times k} - A$.

3. The design and construction of (16, 6, 2)

The constructed design can be summarized as following step:

(1)Using the subset $X_1 = \{1,2,3,4,5,6\}$ of $X = \{1,2,3,\cdots,16\}$ as a variable element set of design $(15,6,5,2,1)$ and the other subset $X_2 = \{7,8,9,\cdots,16\}$ as a variable element set of design $(15,10,6,4,2)$.

(2)Construct 15 blocks of design $(15,6,5,2,1)$ on $X_1 = \{1,2,3,4,5,6\}$:

$$B_1 = (1,2), B_2 = (1,3), B_3 = (1,3), B_4 = (1,4),\ldots, B_{15} = (5,6).$$

(3)Construct 15 blocks of design $(15,10,6,4,2)$ on $X_2 = \{7,8,9,\cdots,16\}$:

$$B_1 = (7,8,9,10) \quad B_2 = (7,11,12,13), \quad B_3 = (8,11,14,15), \quad \ldots, \quad B_{15} = (7,8,13,14).$$

(4)Give the union of the block $B_1^{'},B_2^{'},\cdots,B_{15}^{'}$ from design $(15,6,5,2,1)$ and $B_1^{''},B_2^{''},\cdots,B_{15}^{''}$ from design $(15,10,6,4,2)$. And let $B_1 = X_1 = \{1,2,3,4,5,6\}$ and other blocks $B_2 = B_1^{'} \cup B_1^{''}$, $B_3 = B_2^{'} \cup B_2^{''}$, $B_4 = B_3^{'} \cup B_3^{''}$, $\ldots$, $B_{16} = B_{15}^{'} \cup B_{15}^{''}$. Then we can obtain the design $S^{'} = \{B_1^{'},B_2^{'},\cdots,B_{16}^{'}\}$ of $(16,6,2)$ as follows:

$S_1^{'}=\{$ $(1,2,3,4,5,6)$, $(1,2,7,8,9,10)$, $(1,3,7,11,12,13)$, $(1,4,8,11,14,15)$, $(1,5,9,12,14,16)$, $(1,6,10,13,15,16)$, $(2,3,7,14,15,16)$, $(2,4,8,12,13,16)$, $(2,5,9,11,13,15)$, $(2,6,10,11,12,14)$, $(3,4,9,10,13,14)$, $(3,5,8,10,11,16)$, $(3,6,8,9,12,15)$, $(4,5,7,10,12,15)$, $(4,6,7,9,11,16)$, $(5,6,7,8,13,14)$ $\}$。

Perform the above construction steps and obtain other ten different configurations of design $(16,6,2)$ $S_2^{'},S_3^{'},\ldots, S_{11}^{'}$,as follows:

$S_2^{'}=\{$ $(1,2,5,6,7,8)$, $(1,2,3,4,9,10)$, $(1,3,5,11,12,13)$, $(1,4,6,11,14,15)$, $(1,7,9,12,14,16)$, $(1,8,10,13,15,16)$, $(2,3,5,14,15,16)$, $(2,4,6,12,13,16)$, $(2,7,9,11,13,15)$, $(2,8,10,11,12,14)$, $(3,4,7,8,9,15)$, $(3,6,7,12,13,14)$, $(3,6,8,9,11,16)$, $(4,5,7,10,13,16)$, $(4,5,8,11,12,14)$, $(5,6,9,10,13,14)$ $\}$, $\ldots,$

$S_{11}^{'}=\{$ $(1,2,3,6,7,10)$, $(1,2,4,5,8,9)$, $(1,3,4,11,12,13)$, $(1,5,6,11,14,15)$, $(1,7,8,12,14,16)$, $(1,9,10,13,15,16)$, $(2,3,4,14,15,16)$, $(2,5,6,12,13,16)$, $(2,7,8,11,13,15)$, $(2,9,10,11,12,14)$, $(3,6,8,9,13,14)$, $(3,7,9,10,11,16)$, $(3,5,8,10,12,15)$, $(4,6,7,9,12,15)$, $(4,6,8,10,11,16)$, $(5,5,7,10,13,14)$ $\}$。

The design and construction of (16, 10, 6)

As the number of different configurations for design $(16,10,6)$ equals $S'_{(i)} = \{B_1^{(1)}, B_2^{(1)}, \cdots, B_{16}^{(1)}\}$ of design $(16,6,2)$, so the steps for constructing other design of $(16,10,6)$ S''_1, S''_2, …, S''_{11} are as follows:

(1) Give the 16 blocks of first design $S'_1 = \{B_1^{(1)}, B_2^{(1)}, \cdots, B_{16}^{(1)}\}$ from $(16,6,2)$, then use complementary relationship between the blocks of residual design $S'' = \{B_1^{(2)}, B_2^{(2)}, \cdots, B_{16}^{(2)}\}$ from $(16,10,6)$ and blocks of export design $S' = \{B_1^{(1)}, B_2^{(1)}, \cdots, B_{16}^{(1)}\}$ from $(16,6,2)$:

$B_1^{(2)} = X \cap B_1^{(1)}$, $B_2^{(2)} = X \cap B_2^{(1)}$, … , $B_{16}^{(2)} = X \cap B_{16}^{(1)}$, get the first residual design $S''_1 = \{B_1^{(2)}, B_2^{(2)}, \cdots, B_{16}^{(2)}\}$ as follows:

$S''_1 = \{ (7,8,9,10,11,12,13,14,15,16)$, $(3,4,5,6,11,12,13,14,15,16)$,
$(2,4,5,6,8,9,10,14,15,16)$, $(2,3,5,6,7,9,10,12,13,16)$, $(2,3,4,6,7,8,10,11,13,15)$,
$(2,3,4,5,7,8,9,11,12,14)$, $(1,4,5,6,8,9,10,11,12,13)$, $(1,3,5,6,7,9,10,11,14,15)$,
$(1,3,4,6,7,8,10,12,14,16)$, $(1,3,4,5,7,8,9,13,15,16)$, $(1,2,5,6,7,8,11,12,15,16)$,
$(1,2,4,6,7,9,12,13,14,15)$, $(1,2,4,5,7,10,11,13,14,16)$, $(1,2,3,6,8,9,11,13,14,16)$,
$(1,2,3,5,8,10,12,13,14,15)$, $(1,2,3,4,9,10,11,12,15,16) \}$

(2) List the complementary relationship in turn between the second, the third design from $(16,10,6)$ and 16 blocks from design $(16,10,6)$ as follows:

$S''_2 = \{ (3,4,9,10,11,12,13,14,15,16)$, $(5,6,7,8,11,12,13,14,15,16)$,
$(2,4,6,7,8,9,10,14,15,16)$, $(2,3,5,7,8,9,10,12,13,16)$, $(2,3,4,5,6,8,10,11,13,15)$,
$(2,3,4,5,6,7,9,11,12,14)$, $(1,4,6,7,8,9,10,11,12,13)$, $(1,3,5,7,8,9,10,11,14,15)$,
$(1,3,4,5,6,8,10,12,14,16)$, $(1,3,4,5,6,7,9,13,15,16)$, $(1,2,5,6,10,11,12,13,14,16)$,
$(1,2,4,5,8,9,10,11,13,16)$, $(1,2,4,5,7,10,12,13,14,15)$, $(1,2,3,6,8,9,11,12,14,15)$,
$(1,2,3,6,7,9,10,13,15,16)$, $(1,2,3,4,7,8,11,15,16) \}$, … ,
$S''_{11} = \{ (4,5,8,9,11,12,13,14,15,16)$, $(3,6,7,10,11,12,13,14,15,16)$,
$(2,5,6,7,8,9,10,14,15,16)$, $(2,3,4,7,8,9,10,12,13,16)$, $(2,3,4,5,6,9,10,11,13,15)$,
$(2,3,4,5,6,7,8,11,12,14)$, $(1,5,6,7,8,9,10,11,12,13)$, $(1,3,4,7,8,9,10,11,14,15)$,
$(1,3,4,5,6,9,10,12,14,16)$, $(1,3,4,5,6,7,8,13,15,16)$, $(1,2,4,5,7,10,11,12,15,16)$,
$(1,2,4,6,8,10,12,13,14,15)$, $(1,2,4,6,7,9,11,13,14,16)$, $(1,2,3,5,8,10,11,13,14,16)$,
$(1,2,3,5,7,9,12,13,14,15)$, $(1,2,3,6,8,9,11,12,15,16) \}$, … 。

Conclusions

Main contribution of this paper: (1) Illustrates the complementary relationship and nature of the export design (v,k,λ) and the residual design $(v,v-k,v-2r+\lambda)$. (2) Propose the design and construction method of (v,k,λ) and $(v,v-k,v-2r+\lambda)$; (3) Construct design S'_1, S'_2, ……, S'_{11} from $(16,6,2)$ and 11 different configured design S''_1, S''_2, ……, S''_{11} from $(16,10,6)$.

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