Multi-symplectic integration for the Camassa-Holm- γ equation

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Abstract. In this paper, we construct a multi-symplectic Hamiltonian form of the Camassa-Holm-γequation(CH-γ) and use the multi-symplectic Fourier pseudo spectral Method(MSFP) for the discretization of the CH-γequation. peaked periodic solution with periodic boundary condition is taken to confirm the accuracy and the good ability of conserving the invariants of the MSFP method. Then the collision of two peaked periodic solutions is shown.

Introduction

In this paper, we aim to study the Camassa-Holm-γ equation(CH-γ):

$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx}$,(1)

Which is derived by dullin et al.[1-3] in 2001 as a model for the unidirectional propagation of nonlinearly dispersive shallow wave equation. Here $u(x,t)$ is the fluid velocity, $m = u - a u_{xx}$ is the momentum density, c_0 , a and γ are constants.

CH-γcan be rewritten in an equivalent manner as the following form

$u_t + c_0 u_x + 3 u u_x - a(u_{xxt} + 2 u_x u_{xx} + u u_{xxx}) + \gamma u_{xxx} = 0$. (2)

Guo and Liu present some explicit expressions of peaked solitary wave solutions and peaked periodic wave solutions in Ref.[4]. Many analytical solutions including smooth and peaked periodic solutions have been given in Refs.[5,6,7].Kang studied the admitted symmetries and conservation laws of the CH-γ equation in Ref.[8]. As we know, there is little numerical method for the CH-γequation in the existing literatures.

We use Multi-symplectic method to deal with it in this paper. Multi-symplectic methods for other PDEs including nonlinear Schrodinger equations [10,11], Coupled-Schrodinger-KdVequations [12],CHequations [13],quasi-Degasperis-Procesi equation [14] have shown their advantages over other methods in local conservation properties,invariants preserving and long-term numerical simulation. The multi-symplectic structure for the Hamiltonian form of the CH-γ equation has not been proposed, therefore it'sof great value to construct a multi-symplectic scheme for the CH-γequation.

Multi-symplectic schemes of the CH-γ equation

Introducing the following canonical momenta

$$\begin{cases} u_t = -w_x \\ \varphi_x = u \\ u_x = v \\ -\dfrac{a}{2} u_t = a u v + a\psi - \gamma v \end{cases} , (3)$$

Eq.2can be rewritten as the first-order scheme as follows:

$$\frac{1}{2}\varphi_t - \frac{a}{2} v_t + a\psi_x = \frac{1}{2} w - \frac{a}{2} v^2 - c_0 u - \frac{3}{2} u^2 . \ (4)$$

Introducing the state variable $z = [u, \varphi, w, \psi, v]^T$,we can get a multi-symplectic scheme of the

CH-γ equation

$Mz_t + Kz_x = \nabla_z S(z)$, (5)

in which $M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{a}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{a}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 0 & 0 & a & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

the Hamiltonian function $S(z) = -\frac{1}{2}u^3 - \frac{c_0}{2}u^2 - \frac{a}{2}v^2u + \frac{1}{2}wu - \frac{\gamma}{2}v^2 - av\psi$.

The multi-symplectic structure (5) has several basic local conservation laws including

(1) a local multi-symplectic conservation law

$\partial_t \omega + +\partial_x \kappa = 0$, (6)

where $\omega = du \wedge d\varphi + adv \wedge du$, $\kappa = 2adu \wedge d\psi + dw \wedge d\varphi$

(2) a local energy conservation law

$\partial_t E + +\partial_x F = 0$ (7)

where the energy density $E = S(z) - \frac{1}{2}z^T Kz_x = S(z) - \frac{1}{4}(w\varphi_x - \varphi w_x + 2a\psi_x u - 2a\psi u_x)$

the energy flux $F = \frac{1}{2}z^T Kz_t = \frac{1}{4}(w\varphi_t - \varphi w_t + 2a\psi_t u - 2a\psi u_t)$.

Eq.7 can be rewritten in a specific form:

$\partial_t(-\frac{u^3}{2} - \frac{c_0u^2}{2} - \frac{auv^2}{2} + \frac{\gamma v^2}{2}) + \partial_x(\frac{1}{2}w\varphi_t - au_t\psi) = 0$ (8)

(3) a momentum conservation law

$\partial_t I + +\partial_x G = 0$ (9)

where the momentum density $I = \frac{1}{2}z^T Mz_x = \frac{1}{4}(u\varphi_x - \varphi u_x + avu_x - auv_x)$

the momentum flux $G = S(z) - \frac{1}{2}z^T Mz_t = S(z) - \frac{1}{4}(u\varphi_t - \varphi u_t + avu_t - auv_t)$.

Eq.9 can be rewritten in a specific form:

$\partial_t(\varphi_x u + au_x v) + \partial_x(S(z) - \frac{\varphi_t u}{2} - \frac{au_t\varphi}{2}) = 0$ (10)

with the periodic boundary conditions, the CH-γ equation has two Hamiltonian invariants, global energy invarian H_1 and global momentum invariant H_2 :

$\mathrm{H}_1 = \int E(x,t)dx = -\frac{1}{2}\int(u^3 + c_0u^2 + auv^2 - \gamma v^2)dx$ (11)

$\mathrm{H}_2 = \int I(x,t)dx = \frac{1}{2}\int(u^2 + au_x^2)dx$ (12)

which satisfies $\frac{d\mathrm{H}_1}{dt} = 0, \frac{d\mathrm{H}_2}{dt} = 0$.

Those two Hamiltonain invariants can usually used as the evaluation critria for the multi-symplectic method. The discretizaiton of H_1 and H_2 are defined as follows

$$\mathrm{H}_1^n = \frac{1}{2}\Delta x \sum_{j=1}^{N} ((u_j^n)^3 + c_0 (u_j^n)^2 + a u_j^n (D_1 u_j^n)^2 - \gamma (u_j^n)^2),$$
$$\mathrm{H}_2^n = \frac{1}{2}\Delta x \sum_{j=1}^{N} ((u_j^n)^2 + a (D_1 u_j^n)^2).$$

Imagine N is the configuration point number in space, L represents the spatial period, $U = \left[u_1, u_2, \cdots u_N\right]^T$, $W = \left[w_1, w_2, \cdots w_N\right]^T$, $\Psi = \left[\psi_1, \psi_2, \cdots \psi_N\right]^T$, $V = \left[v_1, v_2, \cdots v_N\right]^T$. $U^n = \left[u_1^n, u_2^n, \cdots u_N^n\right]^T$ where $u_i^j = u(x_j, t_i)$ and others are similar.Set the time step Δt ,the first order difference operator $D_t U^n = {(U^{n+1} - U^n)}/{\Delta t}$,the average operator $A_t U^n = (U^{n+1} + U^n)/2$.

Fourier pseudo spectral mothed applied to discrete the space direction and implicit midpoint method for time discretization,we obtain the discrete scheme as follows:

$$\begin{cases} \frac{1}{2} D_t \Phi^n - \frac{a}{2} D_t V^n + a D_1 A_t \Psi = \frac{1}{2} A_t W^n - \frac{a}{2} (A_t V^n)^2 - c_0 A_t U^n - \frac{3}{2} (A_t U^n)^2 \\ D_t U^n = -A_t D_1 W \\ D_1 A_t \Phi = A_t U^n \\ D_1 A_t U = A_t V^n \\ -\frac{a}{2} D_t U^n = a (A_t U^n)(A_t V^n) + a A_t \Psi^n - \gamma A_t V^n \end{cases} \tag{13}$$

where D_1 is the first order Fourier spectrum differential matrix $(D_1)_{j,l} = \begin{cases} \frac{1}{2} (-1)^{j+l} \frac{2\pi}{L} \cot(\frac{2\pi}{L} \frac{x_j - x_l}{2}), & j \neq l \\ 0, & j = l \end{cases}$ $(1 \le j, l \le N)$, $x_j = \frac{L}{N}(j-1) (j = 1, 2, \cdots N)$

the vector operation is defined by $(U^n)\bullet(V^n) = [u_1^n v_1^n, u_2^n v_2^n, \cdots, u_N^n v_N^n]$.

The Fourier pseudo spectral scheme of Halmitonian PDEs, has been proved as one of the typical discrete methods that leads to multi-symplectic[15,16].

Eliminating the canonical variables φ, w, ψ and v , we can get a three-level scheme

$$\begin{aligned} & a D_1^2 A_t D_t U^n - c_0 A_t D_1 A_t U^n - \gamma D_1^3 A_t A_t U^n - D_t A_t U^n \\ & = \frac{3}{2} D_1 A_t (A_t U^n)^2 + \frac{a}{2} A_t D_1 (D_1 A_t U^n)^2 - a D_1^2 A_t [(A_t U^n)(D_1 A_t U^n)] \end{aligned} \tag{14}$$

Notice that the derivatives of $\varphi_x = u$ and $u_x = v$ in Eqs.(5) have no relationship with time, we can replace $D_1 A_t \Phi = A_t U^n$ and $D_1 A_t U = A_t V^n$ in Eqs.13 with $D_1 A_t \Phi = U^n$ and $D_1 A_t U = V^n$,a two-level discrete scheme can be obtained :

$$\begin{aligned} & D_t U^n - a D_1^2 D_t U^n + c_0 D_1 A_t U^n + \gamma D_1^3 A_t U^n \\ & = -\frac{3}{2} D_1 (A_t U^n)^2 - \frac{a}{2} D_1 (D_1 A_t U^n)^2 + a D_1^2 [(A_t U^n)(D_1 A_t U^n)] \end{aligned} \tag{15}$$

When we use Scheme(14) for numerical simulation, Scheme (15) can be applied to set the initial value on the second time level.

Numerical Experiments

In this section , we firstly bring in a peak periodic solution for numerical experiment.

If $a > 0$, in order to stay the same with the description in Ref.4,the parameter a^2 is used to replace a .The CH-γ equation has an periodic peaked solution as follows[4]:

$u_2(x,t) = u_0(\xi - 2nT_2)$

where $n = 0, \pm 1, \pm 2, \cdots$, $x - ct = \xi \in [(2n-1)T_2, (2n+1)T_2]$ and

$$u_0(\xi) = \frac{\delta_0}{4} e^{-|\xi|/\sqrt{a}} + \frac{\delta_1^2}{4\delta_0} e^{-|\xi|/\sqrt{a}} - \frac{c_o + \frac{\gamma}{a}}{2}$$

in which $\delta_0 = c_0 + 2c + \frac{3\gamma}{a} + 2\sqrt{(c + \frac{\gamma}{a})(c_0 - 4c - \frac{3\gamma}{a})}$ $\delta_1 = \sqrt{(c_0 + \frac{\gamma}{a})(c_0 - 4c - \frac{3\gamma}{a})}$ T_2 is given by $\sqrt{a} \ln|\delta_0 / \delta_1|$

Then take the initial condition and the boundary condition as follows:

$$\begin{cases} u_2(x,0) = \dfrac{\delta_0}{4} e^{-|x|/\sqrt{a}} + \dfrac{\delta_1^2}{4\delta_0} e^{-|x|/\sqrt{a}} - \dfrac{c_o + \frac{\gamma}{a}}{2} \quad (16) \\ u_2(-T_2,t) = u_2(T_2,t) \end{cases}$$

Set the parameter $a = 0.9$ $\gamma = -1$ $c_o = 1$ and $c = 2$.

In the following numerical experiments, we let the time step length $\Delta t = 2 \times 10^{-4}$, the spatial interval $[-T_2, T_2]$, $\Delta x = T_2/500$

In Fig.1, the numerical solutions obtained by the Multi-symplectic Fourier pseudo spectral method are presented at $t = 4, 8, 12$ .The error with the exact solution is shown in Fig.2. Fig.3 shows the global error of the two Hamiltonian invariants and L_2-error of the solution.

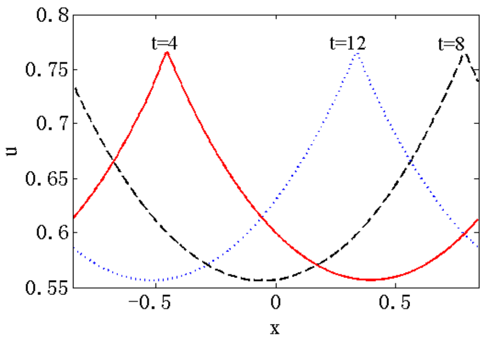

Fig.1 the numerical solutions with initial condition(16) at $t = 4, 8, 12$ $t = 4, 8, 12$

$\Delta t = 2 \times 10^{-4}$ $\Delta x = T_2/500$ $a = 0.9$ $\gamma = -1$ $c_o = 1$ $c = 2$

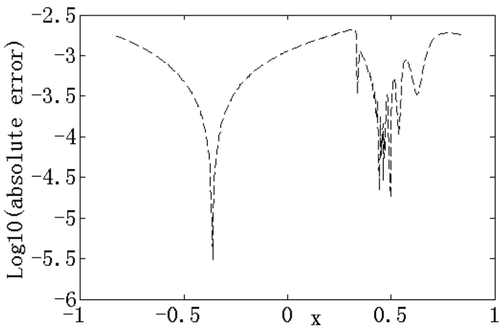

Fig.2 The error with the exact solution with initial condition(16)at $t = 4, 8, 12$

$\Delta t = 2 \times 10^{-4}$ $\Delta x = T_2/500$ $a = 0.9$ $\gamma = -1$ $c_o = 1$ $c = 2$

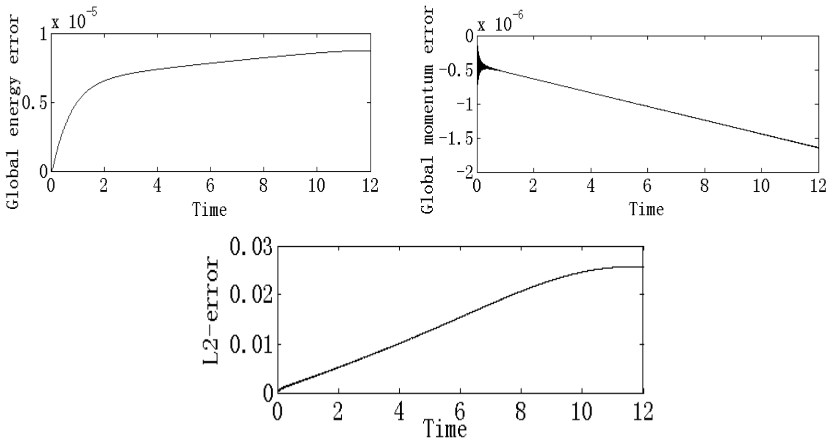

Fig.3the global error of the two Hamiltonian invariants and L_2-error

The numerical results show that the MSFP method conserves the global error of the two Hamiltonian invariants and solves the CH-γ equation with peaked periodic boundary condition quite exactly.

Now we use the MSFP method to simulate the collision of two peaked waves with the following initial condition and boundary condition:

$$\begin{cases} u_{c_1c_2}(x,0) = u_{c_1} + u_{c_2} \\ u_{c_1}(x,0) = e^{-\left|\frac{x-3}{a}\right|} \\ u_{c_2}(x,t) = e^{-\left|\frac{x+3}{a}\right|} \\ u_{c_1c_2}(x,-T_{c_1c_2}) = u_{c_1c_2}(x,-T_{c_1c_2}) \end{cases} \qquad (17)$$

Where $c_1 = 2, c_2 = 4$.

We set $\alpha = 1-10^{-9}$ $\gamma = -1$ $c_o = 1, c = 2$, the period $T_{c_1c_2} = 10.3$

$\Delta x = {2T_{c_1c_2}}/{1600}$ $\Delta t = 2\times 10^{-4}$

The numerical results of the collision are shown in Fig.4, the collision is simulated very well. The global error of the two Hamiltonian invariants are shown in Fig.5, the Hamiltonian invariants is well conserved by using our method.

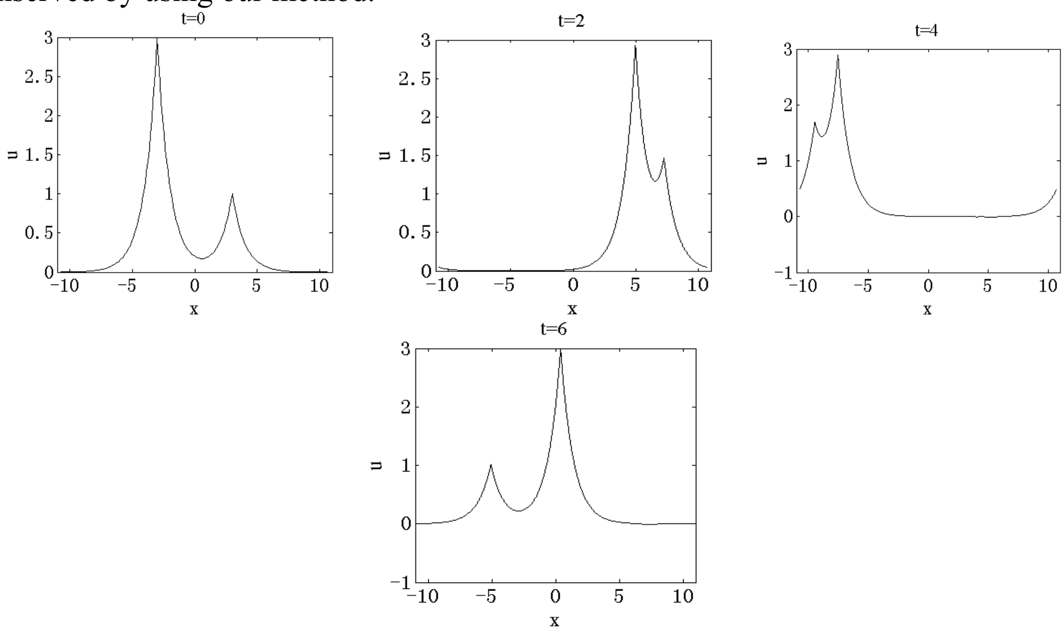

Fig.4 the numerical results of the collision at $t = 0, 2, 4, 6$, $\alpha = 1-10^{-9}$ $\gamma = -1$

$c_o = 1, c = 2$ $\Delta x = {2T_{c_1 c_2}}/{1600}$ $\Delta t = 2\times 10^{-4}$

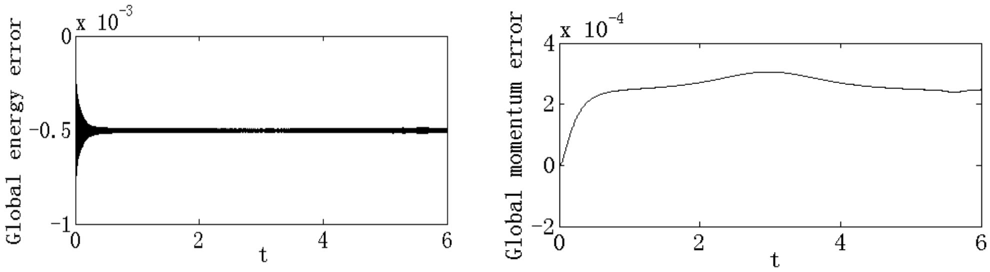

Fig.5 the global error of the two Hamiltonian invariants of the colllision from $t = 0$ to $t = 6$ $\alpha = 1 - 10^{-9}$ $\gamma = -1$ $c_o = 1, c = 2$ $\Delta x = {2T_{c_1 c_2}}/{1600}$ $\Delta t = 2\times 10^{-4}$

Conclusions

In this paper, we first construct a Multi-symplectic Hamiltonian form for the Camassa-Holm-γequation. Numerical results have shown high accuracy and good ability of conserving the invariants. The collision of two peaked waves is well simulated by the our method.

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