# On Generalized Order of Vector Dirichlet Series of Fast Growth

Wanchun Lu
Department of Mathematics, Pingxiang University, Pingxiang, China
Luwanchun540@163.com

***Abstract*—The concept of vector valued Dirichlet series was introduced by B. L. Srivastava [2] who characterized the growth of entire functions represented by these series. In this paper we introduce the generalized order of analysis functions fast growth.**

***Keywords-Vector valued dirichlet series; Analysis functions; Generalized order; Fast growth.***

## I. INTRODUCTION

Let

$$f(s)=\sum_{n=0}^{+\infty}a_n e^{-\lambda_n s},\ (s=\sigma+it,\sigma,t\in\mathbb{R}) \qquad (1)$$

Where $a_n$'s belong to a complex commutative Banach algebra $B$ with identity element $\|\omega\|=1$ and $\lambda_n$'s $\in\mathbb{R}$ satisfy the conditions $0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_n\uparrow+\infty$

$$\overline{\lim_{n\to+\infty}}\frac{\log\|a_n\|}{\lambda_n}=0,\ \overline{\lim_{n\to+\infty}}\frac{n}{\lambda_n}=D<+\infty, \qquad (2)$$

Then, the vector valued Dirichlet series in (1) represents an analytic function $f(s)$ in right plane (see [1]). For the vector valued analytic function $f(s)$ defined as above by (1) the maximum modulus, the maximum term and the index of maximum term are defined as

$$M(\sigma,f)=\sup_{-\infty<t<+\infty}\{\|f(\sigma+it)\|\}$$

$$m(\sigma,f)=\max_{n\in N}\{\|a_n\|e^{-\lambda_n\sigma}\}.$$

The order $\rho$ of $f(s)$ is defined as

$$\rho=\overline{\lim_{\sigma\to 0}}\frac{\log^+\log^+M(\sigma,f)}{-\log\sigma}.$$

We shall call the vector valued analytic function $f(s)$ to be of fast growth if the order $\rho=\infty$. We obtain the characterization of growth parameters in the context of generalized order of vector valued Dirichlet series of fast growth.

Let $\Delta_0$ be the class of all functions $\beta$ satisfying the following two conditions:

(i) $\beta(x)$ is defined on $[a,\infty)$, $a>0$, and is positive, strictly increasing, differentiable and tends to $\infty$ as $x\to\infty$;

(ii) $\dfrac{d\beta(x)}{d\log x}=o(1)$ as $x\to\infty$.

For a vector valued analytic function $f(s)$ given by (1) and $\beta(x)\in\Delta_0$, set

$$\rho(\beta,f)=\overline{\lim_{\sigma\to 0}}\frac{\beta(\log^+M(\sigma,f))}{-\log\sigma}$$

Then $\rho(\beta,f)$ will be called, respectively, $\beta$-order of $f(s)$. To avoid some trivial cases we shall assume throughout that $M(\sigma,f)\to\infty$ as $\sigma\to 0$.

## II. MAIN RESULTS

Lemma 2.1 If the vector valued Dirichlet series given by (1) satisfies (2), then

$$m(\sigma,f)\le M(\sigma,f)\le K(\varepsilon)m(\sigma(1-\varepsilon),f)\frac{1}{\sigma},$$

Where $K(\varepsilon)$ is a positive number of $\varepsilon$ and $f(s)$.

Proof: From the second equation of (2), for given $\varepsilon>0$, there exists an $n_0=n_0(\varepsilon)$, such that for $n>n_0$, $n<(D+\varepsilon)\lambda_n$. Taken $\sigma<0$, $\varepsilon\in(0,-\sigma)$, we have

$$M(\sigma,f)\le\sum_{n=1}^{n_0}\|a_n\|e^{-\lambda_n\sigma}+\sum_{n=n_0}^{\infty}\|a_n\|e^{-\lambda_n(1-\varepsilon)\sigma}e^{-\lambda_n\varepsilon\sigma}$$

$$\le n_0m(\sigma,f)+m((1-\varepsilon)\sigma,f)\sum_{n=n_0+1}^{\infty}e^{-n\varepsilon\sigma/(D+\varepsilon)}$$

$$<n_0m(\sigma,f)+\frac{m((1-\varepsilon)\sigma,f)}{1-e^{-\varepsilon\sigma/(D+\varepsilon)}}$$

The lemma now follows from above and the well known relation $m(\sigma,f)\le M(\sigma,f)$

Theorem 2.1 If the vector valued Dirichlet series given by (1) satisfies (2), then

$$\rho(\beta,f)=\overline{\lim_{\sigma\to 0}}\frac{\beta(\log^+m(\sigma,f))}{-\log\sigma}$$

 

Proof: By the first inequality of Lemma 2.1, this gives, since $\beta \in \Delta_0$, that

$$\overline{\lim_{\sigma\to 0}}\frac{\beta(\log^+ m(\sigma,f))}{-\log\sigma} \le \overline{\lim_{\sigma\to 0}}\frac{\beta(\log^+ M(\sigma,f))}{-\log\sigma} \quad (3)$$

By the second inequality of Lemma 2.1, we have

$$\log^+ M_u(\sigma,F)$$

$$\le \log^+ m(\sigma(1-\varepsilon),f) - \log\sigma + O(1)$$

$$\le C\log^+ m(\sigma(1-\varepsilon),f)\cdot(-\log\sigma) \quad (4)$$

For all $\sigma(\sigma>0)$ sufficiently close to 0. Here $C$ is a constant. Now, (4) gives

$$\beta(\log^+ M(\sigma,f))$$

$$\le \beta(\log^+ m(\sigma(1-\varepsilon),m)) + \log((-\log\sigma)^C)\cdot \left.\frac{d\beta(x)}{d\log x}\right|_{x=x^*(\sigma)}$$

Where $\log^+ \mu(\sigma(1-\varepsilon),F) < x^*(\sigma) < C\log^+ \mu(\sigma(1-\varepsilon),F)\bullet$. $(-\log\sigma)$. This easily gives

$$\overline{\lim_{\sigma\to 0}}\frac{\beta(\log^+ M(\sigma,f))}{-\log\sigma} \le \overline{\lim_{\sigma\to 0}}\frac{\beta(\log^+ m(\sigma,f))}{-\log\sigma} \quad (5)$$

The theorem follows from (3) and (5).

Theorem 2.2 If the vector valued Dirichlet series given by (1) satisfies (2) and has $\beta$-order $\rho(\beta,f)$, then

$$\rho(\beta,f) = \overline{\lim_{n\to\infty}}\frac{\beta(\lambda_n)}{\log\lambda_n - \log^+\log\|a_n\|}$$

Proof: Let $\overline{\lim\limits_{n\to\infty}}\dfrac{\beta(\lambda_n)}{\log\lambda_n - \log^+\log\|a_n\|} = \theta$. Clearly $0 \le \theta \le \infty$. First let $0<\theta<\infty$. Then, for $0<\varepsilon<\theta$ there exist a sequence $\{n_k\}\subset\mathbb{N}$ such that

$$\log\|a_{n_k}\| > \lambda_{n_k}\exp\{-\frac{\beta(\lambda_{n_k})}{\theta-\varepsilon}\},\ k=1,2,3,\cdots.$$

By Lemma 2.1, we have

$$\log^+ M(\sigma,f) \ge \log^+ m(\sigma,f) \ge \log\|a_{n_k}\| - \sigma\lambda_{n_k}$$

$$> \lambda_{n_k}\exp\{-\frac{\beta(\lambda_{n_k})}{\theta-\varepsilon}\} - \sigma\lambda_{n_k} \quad (6)$$

For $k=1,2,3,\cdots$, set $\sigma_k = \frac{1}{2}\exp\{-\frac{\beta(\lambda_{n_k})}{\theta-\varepsilon}\}$. Putting, in particular, $\sigma=\sigma_k$ in (6) ,we get

$$\log^+ M(\sigma_k,f) > \frac{1}{2}\lambda_{n_k}\exp\{-\frac{\beta(\lambda_{n_k})}{\theta-\varepsilon}\} = \lambda_{n_k}\sigma_{n_k},$$

Or

$$\beta(\frac{1}{\sigma_k}\log^+ M(\sigma_k,f)) > \beta(\lambda_{n_k}) = (\theta-\varepsilon)\log(\frac{1}{2\sigma_k}).$$

Since $\beta\in\Delta_0$, we have

$$\beta(\frac{1}{\sigma_k}\log M(\sigma_k,f))$$

$$= o(1)(\log(\frac{1}{\sigma_k}\log M(\sigma_k,f))$$

$$= o(1)\log(\frac{1}{\sigma_k}) + o(1)\log\log M(\sigma_k,f)$$

$$= \log(\frac{1}{\sigma_k})\left.\frac{d\beta(x)}{d\log x}\right|_{x=x^*(\sigma_k)} + \beta(\log M(\sigma_k,f)) \quad (7)$$

where $\log M(\sigma_k,f) < x^*(\sigma_k) < \frac{1}{\sigma_k}\log M(\sigma_k,f)$.

By (7), we have

$$\beta(\log^+ M(\sigma_k,f)) + \log(\frac{1}{\sigma_k})\cdot\left.\frac{d\beta(x)}{d\log x}\right|_{x=x^*(\sigma_k)}$$

$$> (\theta-\varepsilon)\log(\frac{1}{2\sigma_k})$$

Since $\beta\in\Delta_0$, dividing by $\log(\frac{1}{\sigma_k})$ and passing to limits, we get

$$\rho(\beta,f) \ge \theta \quad (8)$$

(8) is obvious for $\theta=0$. For $\theta=\infty$, the above arguments with an arbitrarily large number in place of $\theta-\varepsilon$ give $\rho(\beta,F)=\infty$.

To prove the reveres inequality, since there is nothing to prove if $\theta=\infty$, we may assume that $\theta<\infty$. Then, given $\varepsilon>0$ and for all $n>n_1=n_1(\varepsilon)$ we have

$$\log \| a_n \| < \lambda_n \exp(-\frac{\beta(\lambda_n)}{\theta+\varepsilon}) \qquad (9)$$

Now the second equation of (2) holds, we have $n < \overline{D}\lambda_n$ for all $n > n_2 = n_2(\overline{D})$, where $\overline{D} > D$ is a fixed constant. Let $n_3 = \max\{n_1, n_2\}$, then from (9), we have

$$M(\sigma, f) \le \sum_{n=1}^{n_3} \exp\{\lambda_n \exp(-\frac{\beta(\lambda_n)}{\theta+\varepsilon}) - \lambda_n\sigma\}$$

$$+ \sum_{n=n_3+1}^{\infty} \exp\{\lambda_n \exp(-\frac{\beta(\lambda_n)}{\theta+\varepsilon}) - \lambda_n\sigma\}.$$

For every $\sigma(\sigma > 0)$ we define $n(\sigma)$ as

$$\lambda_{n(\sigma)} \le \beta^{-1}(-(\theta+\varepsilon)\log(\frac{\sigma}{2})) < \lambda_{n(\sigma)+1}$$

For $\sigma(\sigma > 0)$ is sufficiently close to 0, we have $n(\sigma) > n_3$. Thus, we have

$$\sum_{n=n(\sigma)+1}^{\infty} \exp\{\lambda_n \exp(-\frac{\beta(\lambda_n)}{\theta+\varepsilon}) - \sigma\lambda_n\}$$

$$< \sum_{n=n(\sigma)+1}^{\infty} \exp\{\frac{-\sigma\lambda_n}{2}\} < \sum_{n=n(\sigma)+1}^{\infty} \exp\{\frac{-\sigma n}{2\overline{D}}\}$$

$$< \frac{\exp\{\frac{-\sigma(n(\sigma)+1)}{2\overline{D}}\}}{1-\exp\{\frac{-\sigma}{2\overline{D}}\}} = H(n(\sigma)).$$

Now,

$$\log H(n(\sigma)) = \frac{-\sigma(n(\sigma)+1)}{2\overline{D}} + \log\frac{2}{\sigma} + O(1)$$

$$< \frac{-1}{x(\sigma)}\beta^{-1}((\theta+\varepsilon)\log x(\sigma)) + \log x(\sigma) + o(1),$$

Where $x(\sigma) = \frac{2}{\sigma}$. Clearly $x(\sigma) \to \infty$ as $\sigma \to 0$. Since $\beta \in \Delta_0$, it follows that $\beta^{-1}(\overline{\theta}\log x(\sigma)) > (x(\sigma))^2$ for all $\sigma$ sufficiently close to 0. This shows that

$$\log H(n(\sigma)) \to \infty \text{ as } \sigma \to 0$$

or

$$H(n(\sigma)) \to 0 \text{ as } \sigma \to 0. \qquad (10)$$

Consider the function $\varphi(x) = x\exp\{-\frac{\beta(x)}{\theta+\varepsilon}\} - x\sigma$. Taking derivative of $\varphi(x)$ and setting it equal to 0 we get

$$\varphi'(x) = \exp\{-\frac{\beta(x)}{\theta+\varepsilon}\}(1 - \frac{1}{\theta+\varepsilon}\frac{\mathrm{d}\beta(x)}{\mathrm{d}\log x}) - \sigma = 0$$

Since $\frac{\mathrm{d}\beta(x)}{\mathrm{d}\log x} = o(1)$ as $x \to \infty$, it follows that

$$\frac{1}{2} < 1 - \frac{1}{\theta+\varepsilon}\frac{\mathrm{d}\beta(x)}{\mathrm{d}\log x} < 2 \text{ for } x > x_0.$$

Let $\lambda_{n_0}$ be a fixed $\lambda_n$ greater than $x_0$ and $\lambda_{n_3}$, then $\varphi'(\lambda_{n_0}) > 0$ for $0 < \sigma < \sigma_1$. Also $\varphi'(\lambda_{n(\sigma)+1}) < 0$ for all $0 < \sigma < \sigma_2$. Now for $0 < \sigma < \sigma_0 = \min(\sigma_1, \sigma_2)$, we denote by $x_*(\sigma)$ the point where $\varphi(x_*(\sigma)) = \max_{\lambda_{n_0} \le x \le \lambda_{n(\sigma)+1}} \varphi(x)$, then $\lambda_{n_0} < x_*(\sigma) < \lambda_{n(\sigma)+1}$ and

$$x_*(\sigma) = \beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{1-\mathrm{d}(\sigma)}),$$

Where $\mathrm{d}(\sigma) = \frac{1}{\theta+\varepsilon}\frac{\mathrm{d}\beta(x)}{\mathrm{d}\log x}\bigg|_{x=x_*(\sigma)}$, and so

$$\max_{n_0 \le n \le n(\sigma)+1}\{\| a_n \| \mathrm{e}^{-\lambda_n\sigma}\} \le \exp\{\varphi(x_*(\sigma))\}$$

$$\le \exp\{\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{1-\mathrm{d}(\sigma)})\frac{\sigma}{1-\mathrm{d}(\sigma)}$$

$$-\sigma\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{1-\mathrm{d}(\sigma)})\}$$

$$\le \exp\{\frac{\sigma\mathrm{d}(\sigma)}{1-\mathrm{d}(\sigma)}\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{1-\mathrm{d}(\sigma)})\frac{\sigma}{1-\mathrm{d}(\sigma)}$$

$$\le \exp\{\sigma\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{2})\} \qquad (11)$$

Now, for $0 < \sigma < \sigma_0$, we have

$$M_u(\sigma, F) \le P(n_0) + \sum_{n=n_0}^{n(\sigma)} \exp\{\lambda_n \exp(-\frac{\beta(\lambda_n)}{\theta+\varepsilon}) - \lambda_n\sigma\}$$

$$+ \sum_{n=n(\sigma)+1}^{\infty} \exp\{\lambda_n \exp(-\frac{\beta(\lambda_n)}{\theta+\varepsilon}) - \lambda_n\sigma\}$$

Where $P(n_0)$, the sum of first $n_0$ terms, is bounded. Using (10), (11) and definition of $n(\sigma)$, the above inequality gives

$$M(\sigma, F)$$

$$\le P(n_0) + n(\sigma)\exp\{\sigma\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{2}) + o(1)$$

$$\leq P(n_0) + \overline{D}\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{2})\bullet$$

$$\exp\{\sigma\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{2})\} + o(1)$$

Or

$$\log^{+} M_u(\sigma, F) \leq \sigma\beta^{-1}(-(\theta+\varepsilon)\log\frac{\sigma}{2})(1+o(1)) .$$

Since $\beta \in \Delta_0$, this easily gives

$$\rho(\beta, F) \leq \theta \qquad (12)$$

The theorem now follows from (8) and (12).

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