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Connections between Quartic Bézier Curves with Shape Parameters and Cubic H- Bézier Curves

Ping Xu[a], Desheng Yu[b*]

School of Mathematics and Information, Nanchang Hangkong University, Nanchang 330063, China

[a]1402326305@qq.com, [b]yuds17@163.com

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Abstract. The technique of smooth connections between quartic Bézier curves with shape parameters and cubic H- Bézier curves is investigated. Based on a study of their basis functions, endpoint properties and shape analysis, the G^1, G^2 continuous connection conditions between quartic Bézier curves with shape parameters and cubic H- Bézier curves are presented. Experimental examples show that using the technique of smooth connections between these curves, we can take the advantages of these curves sufficiently and solve the problems of complex curves and surfaces modeling better.

Introduction

The design of complex free-form curves and surfaces always need to use piecewise technique, so the splicing of curves and surfaces has been generated. In the initial stage, the curves and surfaces with the same types and degrees were widely discussed. Zheng studied the curvature continuous connection between rational patches on the triangular domain [1]; Hu et.al. put forward the smooth connection conditions between quartic Bézier curves with shape parameters [2]; Sun et.al. investigated the connections between TC- Bézier curves and surfaces [3], and so on. Although these kinds of connections could solve a lot of problems of curves and surfaces modeling, they did not give full play to the advantages of the curves. Then people researched the connections between different kinds of curves and surfaces with the same degrees. Liu et.al. given G^1 connection conditions between C-B spline and T-B spline, and solved the representing problems of semicircle and semi elliptic arc in C-B spline surfaces modeling [4]. To deal with the representing problems of transcendental curves in CE-Bézier curves modeling, Qin et.al. established the necessary and sufficient conditions for the splicing of CE-Bézier curves and H-Bézier curves [5].As a result, people would ask naturally, Should we solve some practical problems better by using the connections between the curves and surfaces with different types and different degrees? In this respect, Zhang et.al. made a preliminary attempt and discussed the G^0, G^1, G^2 continuous connection conditions between cubic Bézier curves and quardratic uniform B-spline curves, but they did not give concrete modeling instances, and the degrees of the curves were much lower and shape adjustments were limited [6].

On taking all the advantages of Bézier curves, quartic Bézier curves with shape parameters have not only excellent shape adjustments, but also much lower computing complexity than some algebraic polynomial curves [2]. H-Bézier curves retain a lot of favorable characteristics of Bézier curves and can better retain the shape of the curves than Bézier curves in a given control polygon [7-10]. Moreover, H-Bézier curves can accurately present transcendental curves, such as exponential curves, catenary's curves, and so on. So in this paper, the connections between quartic Bézier curves with shape parameters and cubic H-Bézier curves are studied, and the G^1, G^2 continuous connection conditions and algorithms are given, thus the problems of lower degrees and poor shape adjustments are overcome. Numerical examples show that this method can give full play to the advantages of these two kinds of curves and settle the problems of complex geometrical modeling better.

The remainder of the paper is organized as follows. Section 2 introduces the definitions and properties of quartic Bézier curves with shape parameters and cubic H-Bézier curves. Section 3 details the connections between quartic Bézier curves with shape parameters and cubic H-Bézier curves.

Section 4 presents steps and examples of connections between the curves. Section 5 concludes the paper.

The definitions and properties of the curves

The definitions and properties of quartic Bézier curves with shape parameters

Definition 1. For $\forall t \in [0,1], l \in [-3,1]$, the quartic Bézier curves with shape parameters basis functions are defined as follows:

$$\begin{cases} b_{0,3}(t) = (1 - lt)(1-t)^3, \\ b_{1,3}(t) = (3 + l - lt)(1-t)^2 t, \\ b_{2,3}(t) = (3 + lt)(1-t)t^2, \\ b_{3,3}(t) = (1 - l + lt)t^3. \end{cases} \tag{1}$$

Definition 2. Let $\boldsymbol{p}_i \in \boldsymbol{R}^n (n = 2,3, i = 0,1,2,3)$ be control points, then

$$\boldsymbol{P}(t) = \sum_{i=0}^{3} \boldsymbol{p}_i b_{i,3}(t) \tag{2}$$

is called a quartic Bézier curve with shape parameters, where l is called shape control parameter of the curve.

Obviously, when the shape parameter l is zero, the curve (2) degenerates into a conventional cubic Bézier curve. From equation (1) and (2), we know that quartic Bézier curves with shape parameters have many properties in common with Bernstein basis functions, including symmetry, convex-preservation, diminishing variation, invariant geometrical properties, and so on. And the endpoints of the curve have the following properties:

$$\begin{cases} \boldsymbol{P}(0) = \boldsymbol{p}_0, \\ \boldsymbol{P}(1) = \boldsymbol{p}_3, \\ \boldsymbol{P}'(0) = (3 + l)(\boldsymbol{p}_1 - \boldsymbol{p}_0), \\ \boldsymbol{P}'(1) = (3 + l)(\boldsymbol{p}_3 - \boldsymbol{p}_2), \\ \boldsymbol{P}''(0) = 6(\boldsymbol{p}_0 - 2\boldsymbol{p}_1 + \boldsymbol{p}_2) + 6l(\boldsymbol{p}_0 - \boldsymbol{p}_1), \\ \boldsymbol{P}''(1) = 6(\boldsymbol{p}_3 - 2\boldsymbol{p}_2 + \boldsymbol{p}_1) + 6l(\boldsymbol{p}_3 - \boldsymbol{p}_2). \end{cases}$$

The definitions and properties of H-Bézier curves

Definition 3. For $\forall t \in [0,q], q \in R^+$, the cubic H-Bézier basis functions are defined as follows:

$$\begin{cases} z_3(t) = \dfrac{t - \sinh t}{q - \sinh q}, \\ z_2(t) = M\left[\dfrac{\sinh(q-t) + t\cosh q - \sinh q}{q\cosh q - \sinh q} - z_3(t)\right], \\ z_1(t) = z_2(q-t), \\ z_0(t) = z_3(q-t), \end{cases}$$

where $M = \dfrac{q\cosh q - \sinh q}{q\cosh q - 2\sinh q + q}, q > 0$.

Definition 4. Let $\boldsymbol{q}_i \in \boldsymbol{R}^n (n = 2,3; i = 0,1,2,3)$, be control points, then

$$\boldsymbol{Q}(t) = \sum_{i=0}^{3} \boldsymbol{q}_i z_{i,3}(t)$$

is called a cubic H-Bézier curve. And the endpoints of the curve have the following properties:

$$\begin{cases} \boldsymbol{Q}(0)=\boldsymbol{q}_0, \boldsymbol{Q}(q)=\boldsymbol{q}_3, \\ \boldsymbol{Q}'(0)=m(\boldsymbol{q}_1-\boldsymbol{q}_0), \\ \boldsymbol{Q}'(q)=m(\boldsymbol{q}_3-\boldsymbol{q}_2), \\ \boldsymbol{Q}''(0)=n(\boldsymbol{q}_0-\boldsymbol{q}_1)+l(\boldsymbol{q}_2-\boldsymbol{q}_1), \\ \boldsymbol{Q}''(q)=n(\boldsymbol{q}_3-\boldsymbol{q}_2)+l(\boldsymbol{q}_1-\boldsymbol{q}_2), \end{cases}$$

where $m=\dfrac{\cosh q-1}{\sinh q-q}, n=\dfrac{\sinh q}{\sinh q-q}, l=\dfrac{\sinh q}{q\cosh q-2\sinh q+q}$.

Connections between Quartic Bézier Curves with Shape Parameters and Cubic H-Bézier Curves

G^1 smooth connection

Theorem 1. The necessary and sufficient conditions of G^1 connections between the curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ are

$$\boldsymbol{q}_0=\boldsymbol{p}_3=\boldsymbol{p}_2+\frac{m}{(3+l)d}(\boldsymbol{q}_1-\boldsymbol{q}_0). \tag{3}$$

Proof. First, the endpoint of $\boldsymbol{P}(t)$ and the initial point of $\boldsymbol{Q}(t)$ satisfy G^0 continuity, i.e.

$$\boldsymbol{q}_0=\boldsymbol{p}_3; \tag{4}$$

Then the curves have the same tangent vector direction at the joining point, i.e.

$$\boldsymbol{Q}'(0)=d\boldsymbol{P}'(1)\ (d>0). \tag{5}$$

And according to the endpoint properties of the curves, we have following:

$$\begin{cases} \boldsymbol{Q}'(0)=m(\boldsymbol{q}_1-\boldsymbol{q}_0), \\ \boldsymbol{P}'(1)=(3+l)(\boldsymbol{p}_3-\boldsymbol{p}_2). \end{cases} \tag{6}$$

Then according to (4), (3) is obtained by substituting (6) into (5).

The geometric meaning of G^1 smooth connections between these curves at a joining point is: the control points $\boldsymbol{p}_2, \boldsymbol{p}_3(\boldsymbol{q}_0)$ and $\boldsymbol{q}_1$ must be collinear ordered arrangement when splicing. This line is the common tangent at the joining point.

G^2 smooth connection

The necessary and sufficient conditions of G^2 connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ at a joining point are: (a) satisfying G^1 smooth connection conditions; (b) having the same curvature; (c) having the same bi-normal vector direction.

Theorem 2. The necessary and sufficient conditions of G^2 connections between the curves $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ are

$$\begin{cases} \boldsymbol{q}_0=\boldsymbol{p}_3=\boldsymbol{p}_2+\dfrac{m}{(3+l)d}(\boldsymbol{q}_1-\boldsymbol{q}_0), \\ 6\boldsymbol{p}_1-6(2+l)\boldsymbol{p}_2=[k-6(1+l)]\boldsymbol{q}_0-(k+ld^{-2})\boldsymbol{q}_1+ld^{-2}\boldsymbol{q}_2, \end{cases} \tag{7}$$

where $k=nd^{-2}-md, d>0, d\in R$.

Proof. The first equation of (7) is obtained by theorem 1. Moreover, the bi-normal vectors of $\boldsymbol{P}(t)$ at $t=1$ and $\boldsymbol{Q}(t)$ at $t=0$ respectively are:

$$\boldsymbol{D}_1=\boldsymbol{P}'(1)\times\boldsymbol{P}''(1), \boldsymbol{D}_2=\boldsymbol{Q}'(0)\times\boldsymbol{Q}''(0).$$

According to (c) and equation (4) and (5), we obtain that $\boldsymbol{P}'(1), \boldsymbol{P}''(1), \boldsymbol{Q}'(0), \boldsymbol{Q}''(0)$ are coplanar, and

$$\boldsymbol{P}''(1)=c\boldsymbol{Q}''(0)+d\boldsymbol{Q}'(0), c>0.$$

Then the following equation is obtained according to (b):

$$\frac{|\boldsymbol{Q}'(0)\times\boldsymbol{Q}''(0)|}{|\boldsymbol{Q}'(0)|^3}=\frac{d\,|\boldsymbol{P}'(1)\times\boldsymbol{P}''(1)|}{cd^3\,|\boldsymbol{P}'(1)|^3}=\frac{|\boldsymbol{P}'(1)\times\boldsymbol{P}''(1)|}{|\boldsymbol{P}'(1)|^3},$$

so $c=d^{-2}$ and $\boldsymbol{P}''(1)=d^{-2}\boldsymbol{Q}''(0)+d\boldsymbol{Q}'(0)$.

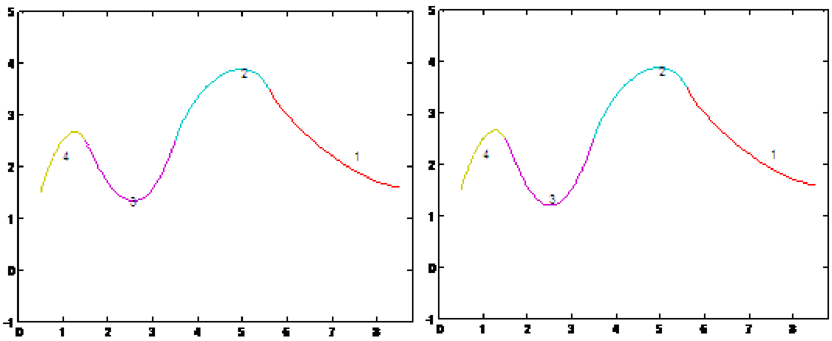
Then, theorem 2 is proved by substituting $\boldsymbol{P}''(1),\boldsymbol{Q}''(0),\boldsymbol{Q}'(0)$, into this equation.

The steps and examples of connections between the curves

According to theorem 1, the steps of G^1 connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ are : (i) Giving control points $\boldsymbol{p}_i\in \boldsymbol{R}^2, i=0,1,2,3)$, $l\in[-3,1]$**;** (ii) Letting $\boldsymbol{q}_0=\boldsymbol{p}_3$, to achieve G^0 continuity between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$; (iii) Giving parameters l,d freely, then computing control point $\boldsymbol{q}_1$ of $\boldsymbol{Q}(t)$ by (3); (iv) Giving remaining control points $\boldsymbol{q}_2,\boldsymbol{q}_3$ of $\boldsymbol{Q}(t)$ freely, to achieve G^1 smooth connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$.

And according to theorem 2, the steps of G^2 connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ are: Processing steps (i), (ii), and (iii) of G^1 continuity of $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ firstly, and then going on with steps: (iv) Giving parameter d freely and computing control point $\boldsymbol{q}_2$ by the second equation of (7); (v) Giving remaining control point $\boldsymbol{q}_3$ freely, achieving G^2 smooth connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$.

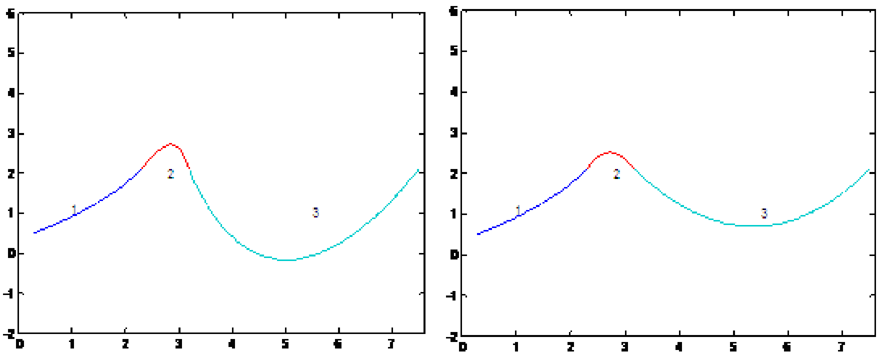
Example 1. A part of the wire in Fig. 1 is constructed by the G^1 connections between quartic Bézier curves with shape parameters and cubic H-Bézier. Curves 1 and 3 are H-Bézier, because H-Bézier curves can represent catenaries, such as stone arch bridge and high voltage wire. Change the values of the degree of freedom d and shape parameter l to adjust the local shape of wire in Fig. (b).



(a) (b)

Figure 1: The G^1 connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$.

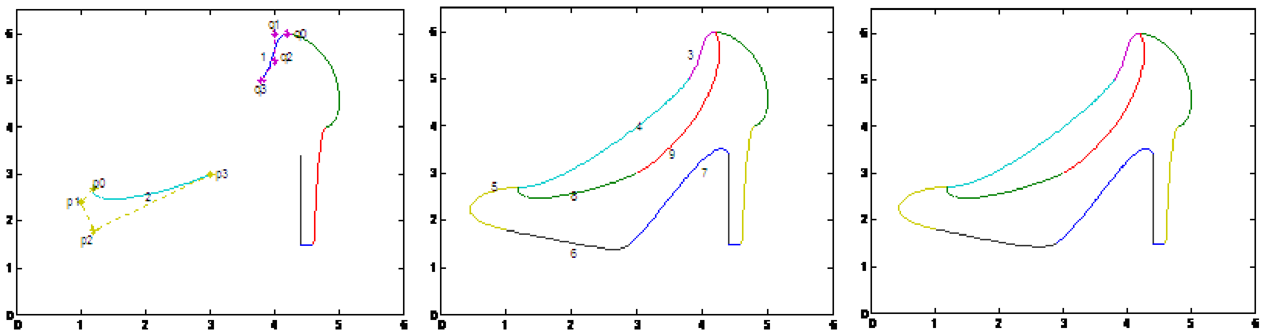
Example 2. A chain in Fig. 2 is constructed by the G^2 connections between quartic Bézier curves with shape parameters and cubic H-Bézier curves, and curves 1 and 3 are H-Bézier. Change the values of the degree of freedom d,d and shape parameter l to adjust the local shape of chain in Fig. (b).



(a) (b)

Figure 2: The G^2 connections between $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$.

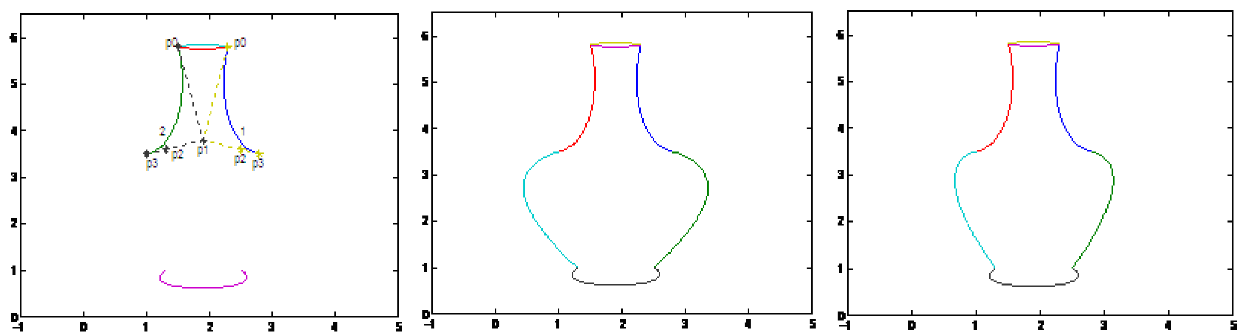
Example 3. A high-heeled shoes model is constructed in Fig. 3. High heels needed splicing is given in Fig. (a). Curve 1 is represented by H-Bézier curves, and their control points are $\boldsymbol{q}_0(4.2,6), \boldsymbol{q}_1(4,6), \boldsymbol{q}_2(4,5.4), \boldsymbol{q}_3(3,3)$ respectively; Curve 2 is represented by Quartic Bézier curves with shape parameters, and the control points are $\boldsymbol{p}_0(1.2,2.7), \boldsymbol{p}_1(1,2.4), \boldsymbol{p}_2(1.2,1.8), \boldsymbol{p}_3(3,3)$ respectively. In Fig. (b), the curves 3, 4, 5, 6, 7 and curves 8, 9 are all constructed by the G^1 connections between quartic Bézier curves with shape parameters and cubic H-Bézier curves; And Fig. (c) is constructed by modifying parameters.



(a) High heels needed splicing. (b) Connection design. (c) After changing parameters.

Figure 3: The design of high heels.

Example 4. The vase model is constructed in Fig. 4. The vase needed splicing is given in Fig. (a), and its two sides are quartic Bézier curves with shape parameters. The control points of curve 1 and curve 2 are $\boldsymbol{p}_0(2.3,5.8), \boldsymbol{p}_1(1.9,3.8), \boldsymbol{p}_2(2.5,3.6), \boldsymbol{p}_3(2.8,3.5)$ and $\boldsymbol{p}_0(1.5,5.8), \boldsymbol{p}_1(1.9,3.8), \boldsymbol{p}_2(1.3,3.6)$ $\boldsymbol{p}_3(1,3.5)$, respectively. The curves on both sides are formed by the G^2 connections between quartic Bézier curves with shape parameters and cubic H-Bézier curves in Fig. (b). To ensure the symmetry of the vase, the selected parameters on both sides are equal ($l=-1.5, d=0.6, d=6$). According to the user's requirement of the capacity and shape of the vase, we need to adjust vase locally. The local shapes of vase are changed by modifying degrees of freedom d, d to $0.75, 5$ in Fig. (c).



(a) The vase needed splicing. (b) Connection design. (c) After changing parameters.

Figure 4: The design of vase.

Example 5. In Fig. 5, a camel is constructed by the G^1 and G^2 hybrid connections between quartic Bézier curves with shape parameters and cubic H-Bézier curves, and the graph is adjusted locally via changing shape parameters.

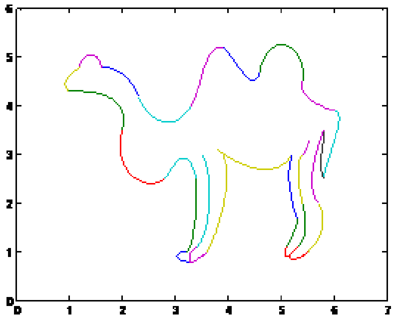
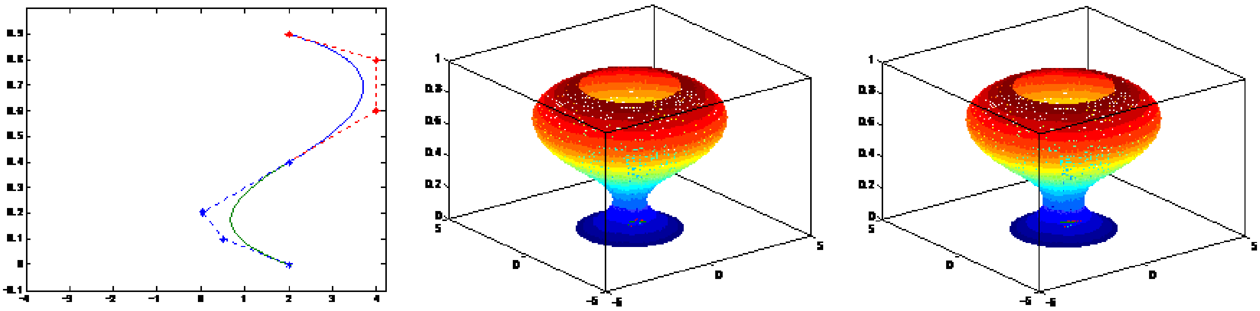


Figure 5: The design of camel.

Example 6. The splicing between these two kinds of curves can also be applied to the modeling of rotating surfaces. In Fig. 6, the rotating surfaces of a chimney are given.



(a)The generatrix of rotating chimney . (b) Connection design. (c) After changing parameters.

Figure 6: The rotating surface of a chimney.

Summary

The G^1, G^2 smooth connection conditions between quartic Bézier curves with shape parameters and H-Bézier curves have been established. The connection conditions are simple and intuitive via theoretical analysis and illustrative examples.

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