

# Crack Tip Fields for Anti-plane Interfacial Crack between Functionally Graded and Homogeneous Piezoelectric Materials

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**Abstract:** The problem of an anti-plane crack situated in the interface of functionally graded piezoelectric materials (FGPMs) and homogeneous piezoelectric materials (HPMs) is considered under the impermeable assumption of crack surfaces. The mechanical and electrical properties of the FGPMs are assumed to be power functions of  $y$  perpendicular to the crack. The higher order crack tip stress and electric displacement fields for FGPMs and HPMs are obtained by the eigen-expansion method. The stress intensity factor and electric displacement intensity factor are obtained explicitly.

## Introduction

Due to their brittleness, the fracture problems of a cracked body in piezoelectric materials have been investigated in recent years. In particular, they have been attracted extensive attention in order to meet the demand of high strength and high temperature applications. However, for the interfacial crack problem of piezoelectric materials, some results have been reported. Li and Lee [1] solved anti-plane fracture for the weak-discontinuous interface in a non-homogeneous piezoelectric bi-material structure. Li et al. [2] investigated the problem of a penny-shaped interfacial crack between a functionally graded piezoelectric layer and a homogeneous piezoelectric layer. Furthermore, Zhang et al. [3] studied the dynamic interaction of two collinear interfacial cracks between two dissimilar functionally graded piezoelectric/piezomagnetic material strips subjected to the anti-plane shear harmonic stress waves. Ing and Chen [4] investigated the transient response of an interfacial crack between two functionally graded piezoelectric strips. It is worth noting that these literatures on interfacial crack of piezoelectric materials are focused on the singular part of crack tip fields. The fracture behavior of an interfacial crack between FGPMs and HPMs has not been studied by using the eigen-expansion method. The purpose of this paper is to present the higher order fields and explicit expression of intensity factors.

## The higher order crack-tip field

Consider an impermeable crack of length  $2L$  located at the interface subjected to anti-plane shear loads and the in-plane electric displacements. The material properties of the FGPM are assumed as the following power forms

$$c_{44} = c_{440} f(r, q), \quad e_{15} = e_{150} g(r, q), \quad e_{11} = e_{110} j(r, q) \quad (1)$$

$$f(r, q) = (1 + b_1 r \sin q)^{k_1}, \quad g(r, q) = (1 + b_2 r \sin q)^{k_2}, \quad j(r, q) = (1 + b_3 r \sin q)^{k_3} \quad (2)$$

where  $c_{44}^{(2)}$  is the shear modulus,  $e_{15}^{(2)}$  is the piezoelectric coefficient,  $e_{11}^{(2)}$  is the dielectric parameter of the HPM.

The governing equations can be written as

$$\begin{cases} c_{44}\nabla^2 w + \frac{\partial c_{44}}{\partial r} \frac{\partial w}{\partial r} + \frac{\partial c_{44}}{r^2 \partial q} \frac{\partial w}{\partial q} + e_{15}\nabla^2 f + \frac{\partial e_{15}}{\partial r} \frac{\partial f}{\partial r} + \frac{\partial e_{15}}{r^2 \partial q} \frac{\partial f}{\partial q} = 0 \\ e_{15}\nabla^2 w + \frac{\partial e_{15}}{\partial r} \frac{\partial w}{\partial r} + \frac{\partial e_{15}}{r^2 \partial q} \frac{\partial w}{\partial q} - e_{11}\nabla^2 f - \frac{\partial e_{11}}{\partial r} \frac{\partial f}{\partial r} - \frac{\partial e_{11}}{r^2 \partial q} \frac{\partial f}{\partial q} = 0 \end{cases} \quad (3)$$

where  $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial q^2}$  is the two-dimensional Laplace operator.

The displacement component  $w$  and the electric potential  $f$  can be expanded as follows

$$w = \sum_{i=1}^{\infty} r^{\frac{i}{2}} w_i(q), \quad f = \sum_{i=1}^{\infty} r^{\frac{i}{2}} f_i(q) \quad (4)$$

where,  $w_i(q)$  and  $f_i(q)$  are eigen-functions.

Substitute Eq. (4) into Eq. (3). According to the linear independence of  $r^{-3/2}$ ,  $r^{-1}$ ,  $r^{-1/2}$ , ...,  $r^{i/2-2}$ , ..., the system of ordinary differential equations are obtained.

Solving the system of ordinary differential equations, we can obtain the results

$$\begin{cases} w_1(q) = B_{11} \sin \frac{q}{2} \\ f_1(q) = B_{12} \sin \frac{q}{2} \end{cases} \quad (5)$$

$$\begin{cases} w_2(q) = B_{21} \cos q \\ f_2(q) = B_{22} \cos q \end{cases} \quad (6)$$

$$\begin{cases} w_3(q) = B_{31} \sin \frac{3q}{2} + \left[ -\frac{e_{150}^2 k_2 b_2 + c_{440} e_{110} k_1 b_1}{12(c_{440} e_{110} + e_{150}^2)} B_{11} + \frac{e_{150} e_{110} (k_3 b_3 - k_2 b_2)}{12(c_{440} e_{110} + e_{150}^2)} B_{12} \right] \cdot \\ \quad \left( 3 \cos \frac{q}{2} + \cos \frac{3q}{2} \right) \\ f_3(q) = B_{32} \sin \frac{3q}{2} + \left[ \frac{c_{440} e_{150} (k_2 b_2 - k_1 b_1)}{12(c_{440} e_{110} + e_{150}^2)} B_{11} - \frac{c_{440} e_{110} k_3 b_3 + e_{150}^2 k_2 b_2}{12(c_{440} e_{110} + e_{150}^2)} B_{12} \right] \cdot \\ \quad \left( 3 \cos \frac{q}{2} + \cos \frac{3q}{2} \right) \end{cases} \quad (7)$$

$$\begin{cases} w_4(q) = B_{41} \cos 2q \\ f_4(q) = B_{42} \cos 2q \end{cases} \quad (8)$$

$$\begin{aligned}
w_5(q) = & B_{51} \sin \frac{5q}{2} + \frac{1}{48(e_{150}^2 + e_{110}c_{440})^2} \{ [(k_2^2 e_{150}^4 - 5c_{440} e_{150}^2 e_{110} k_2^2 + 2k_2 e_{150}^4 + \\
& 2c_{440} e_{150}^2 e_{110} k_2) b_2^2 + (c_{440}^2 e_{110}^2 k_1^2 + 2c_{440}^2 e_{110}^2 k_1 - 2c_{440} e_{150}^2 e_{110} k_1^2 + 2c_{440} \cdot \\
& e_{150}^2 e_{110} k_1) b_1^2 + 3c_{440} e_{150}^2 e_{110} (3k_1 b_1 k_2 b_2 + k_2 b_2 k_3 b_3 - k_1 b_1 k_3 b_3)] B_{11} + \\
& [2(-c_{440} e_{150}^2 e_{110} k_2^2 + c_{440} e_{150}^2 e_{110} k_2 + e_{150}^3 e_{110} k_2 + 2e_{150}^3 e_{110} k_2^2) b_2^2 + (-c_{440} \cdot \\
& e_{150} e_{110}^2 k_3^2 - 2e_{150}^3 e_{110} k_3 + 2e_{150}^3 e_{110} k_3^2 - 2c_{440} e_{150} e_{110}^2 k_3) b_3^2 - 6e_{150}^3 e_{110} k_2 \cdot \\
& b_2 k_3 b_3 + 3c_{440} e_{150}^2 e_{110} (k_1 b_1 k_2 b_2 + k_2 b_2 k_3 b_3 - k_1 b_1 k_3 b_3)] B_{12} \} \sin \frac{q}{2} + \\
& \frac{1}{20(e_{150}^2 + e_{110}c_{440})^2} (\cos \frac{5q}{2} - 5 \cos \frac{q}{2}) \{ [(c_{440} e_{150}^2 e_{110} + c_{440}^2 e_{110}^2) k_1 b_1 + \\
& (c_{440} e_{150}^2 e_{110} + e_{150}^4) k_2 b_2] B_{31} + (e_{150}^3 e_{110} + c_{440} e_{150} e_{110}^2) (k_2 b_2 - k_3 b_3) B_{32} \} + \\
& \frac{1}{32(e_{150}^2 + e_{110}c_{440})^2} \sin \frac{3q}{2} \{ [(c_{440}^2 e_{110}^2 k_1^2 + 2c_{440}^2 e_{110}^2 k_1 - 2c_{440} e_{150}^2 e_{110} k_1^2 + \\
& 2c_{440} e_{150}^2 e_{110} k_1) b_1^2 + (k_2^2 e_{150}^4 - 5c_{440} e_{150}^2 e_{110} k_2^2 + 2k_2 e_{150}^4 + 2c_{440} e_{150}^2 e_{110} \cdot \\
& k_2) b_2^2 + 3c_{440} e_{150}^2 e_{110} (3k_1 b_1 k_2 b_2 + k_2 b_2 k_3 b_3 - k_1 b_1 k_3 b_3)] B_{11} + [(-2c_{440} \cdot \\
& e_{150} e_{110}^2 k_2^2 + 2c_{440} e_{150} e_{110}^2 k_2 + 2e_{150}^3 e_{110} k_2 + 4e_{150}^3 e_{110} k_2^2) b_2^2 + (-c_{440} e_{150} e_{110}^2 \cdot \\
& k_3^2 - 2e_{150}^3 e_{110} k_3 + 2e_{150}^3 e_{110} k_3^2 - 2c_{440} e_{150} e_{110}^2 k_3) b_3^2 - 6e_{150}^3 e_{110} k_2 b_2 k_3 b_3 + \\
& 3c_{440} e_{150}^2 e_{110} (k_1 b_1 k_2 b_2 + k_2 b_2 k_3 b_3 - k_1 b_1 k_3 b_3)] B_{12} \} \\
f_5(q) = & B_{52} \sin \frac{5q}{2} + \frac{1}{48(e_{150}^2 + e_{110}c_{440})^2} \{ [2(-2c_{440} e_{150}^3 k_2^2 - c_{440}^2 e_{150} e_{110} k_2 - c_{440} \cdot \\
& e_{150}^3 k_2 + c_{440}^2 e_{150} e_{110} k_2^2) b_2^2 + (c_{440}^2 e_{150} e_{110} k_1^2 + 2c_{440}^2 e_{150} e_{110} k_1 + 2c_{440} e_{150}^3 k_1 - \\
& 2c_{440} e_{150}^3 k_1^2) b_1^2 + 3c_{440}^2 e_{150} e_{110} (k_1 b_1 k_3 b_3 - k_2 b_2 k_3 b_3 - k_1 b_1 k_2 b_2) + 6c_{440} e_{150}^3 \cdot \\
& k_1 b_1 k_2 b_2] B_{11} + [(c_{440}^2 e_{110}^2 k_3^2 + 2c_{440} e_{150}^2 e_{110} k_3 + 2c_{440}^2 e_{110}^2 k_3 - 2c_{440} e_{150}^2 e_{110} k_3^2) \cdot \\
& b_3^2 + (-5c_{440} e_{150}^2 e_{110} k_2^2 + e_{150}^4 k_2^2 + 2c_{440} e_{150}^2 e_{110} k_2 + 2e_{150}^4 k_2) b_2^2 + 3c_{440} e_{150}^2 \cdot \\
& e_{110} (k_1 b_1 k_2 b_2 + 3k_2 b_2 k_3 b_3 - 3k_1 b_1 k_3 b_3)] B_{12} \} \sin \frac{q}{2} + \frac{1}{20(e_{150}^2 + e_{110}c_{440})^2} (\cos \frac{5q}{2} - \\
& 5 \cos \frac{q}{2}) \{ (c_{440}^2 e_{150} e_{110} + c_{440} e_{150}^3) (k_2 b_2 - k_1 b_1) B_{31} + [(c_{440} e_{150}^2 e_{110} - c_{440}^2 e_{110}^2) k_3 b_3 - \\
& (c_{440} e_{150}^2 e_{110} + e_{150}^4) k_2 b_2] B_{32} \} + \frac{1}{32(e_{150}^2 + e_{110}c_{440})^2} \sin \frac{3q}{2} \{ [(c_{440}^2 e_{150} e_{110} k^2 + \\
& 2c_{440}^2 e_{150} e_{110} k_1 + 2c_{440} e_{150}^3 k_1 - 2c_{440} e_{150}^3 k_1^2) b_1^2 + 2(-2c_{440} e_{150}^3 k_2^2 - c_{440}^2 e_{150} e_{110} k_2 - \\
& c_{440} e_{150}^3 k_2 + c_{440}^2 e_{150} e_{110} k_2^2) b_2^2 + 6c_{440} e_{150}^3 k_1 b_1 k_2 b_2 + 3c_{440}^2 e_{150} e_{110} (k_1 b_1 k_3 b_3 - \\
& k_2 b_2 k_3 b_3 - k_1 b_1 k_2 b_2)] B_{11} + [(c_{440}^2 e_{110}^2 k_3^2 + 2c_{440} e_{150}^2 e_{110} k_3 + 2c_{440}^2 e_{110}^2 k_3 - 2c_{440} e_{150}^2 e_{110} k_3^2) \cdot \\
& e_{110} k_3^2) b_3^2 + (e_{150}^4 k_2^2 - 5c_{440} e_{150}^2 e_{110} k_2^2 + 2c_{440} e_{150}^2 e_{110} k_2 + 2e_{150}^4 k_2) b_2^2 + 3c_{440} e_{150}^2 \cdot \\
& e_{110} (k_1 b_1 k_2 b_2 + 3k_2 b_2 k_3 b_3 - k_1 b_1 k_3 b_3)] B_{12} \} \}
\end{aligned} \tag{9}$$

$$\left\{ \begin{aligned} w_6(q) = & B_{61} \cos 3q + \frac{3 \sin q - \sin 3q}{12(e_{150}^2 + c_{440} e_{110})} [(c_{440} e_{110} k_1 b_1 + e_{150}^2 k_2 b_2) B_{41} + \\ & e_{150} e_{110} (k_2 b_2 - k_3 b_3) B_{42}] \\ f_6(q) = & B_{62} \cos 3q + \frac{3 \sin q - \sin 3q}{12(e_{150}^2 + c_{440} e_{110})} [c_{440} e_{150} (k_1 b_1 - k_2 b_2) B_{41} + \\ & (c_{440} e_{110} k_3 b_3 + e_{150}^2 k_2 b_2) B_{42}] \end{aligned} \right. \tag{10}$$

$$\left\{ \begin{aligned} w = & \sum_{i=1}^{+\infty} [r^{i-1/2} A_{(2i-1)1} \sin(i - \frac{1}{2})q + r^i A_{(2i)1} \cos iq] \\ f = & \sum_{i=1}^{+\infty} [r^{i-1/2} A_{(2i-1)2} \sin(i - \frac{1}{2})q + r^i A_{(2i)2} \cos iq] \end{aligned} \right. \tag{11}$$

where  $\mathbf{A}_{ij} = \begin{bmatrix} A_{ij} \\ B_{ij} \end{bmatrix}$ ,  $\mathbf{C}_{ij} = \begin{bmatrix} C_{ij} \\ D_{ij} \end{bmatrix}$  are the undetermined coefficients.

Substituting Eq. (5)-(10) and (11) into Eq.(4), the displacement component and the electric potential for the FGPM and HPM regions are obtained. Then the stress and electric displacement fields will be obtained based on constitutive relations as

$$\begin{aligned} t_{xz}^{(k)} &= c_{44}^{(k)} w_{,x}^{(k)} + e_{15}^{(k)} f_{,x}^{(k)}, \quad t_{yz}^{(k)} = c_{44}^{(k)} w_{,y}^{(k)} + e_{15}^{(k)} f_{,y}^{(k)} \\ D_x^{(k)} &= e_{15}^{(k)} w_{,x}^{(k)} - e_{11}^{(k)} f_{,x}^{(k)}, \quad D_y^{(k)} = e_{15}^{(k)} w_{,y}^{(k)} - e_{11}^{(k)} f_{,y}^{(k)} \end{aligned} \quad (12)$$

The mechanical and electric conditions must satisfy the following continuity conditions along the interface

$$t_{qz}^{(I)} = t_{qz}^{(II)}, \quad D_q^{(I)} = D_q^{(II)} \quad q = 0 \quad (13)$$

Substituting Eq.(12) into Eq.(13), the relations between  $A_{ij}$  and  $C_{ij}$  can be obtained. The higher order stress and electric displacement fields will be given based on Eq. (12) finally. Then we obtain the stress intensity factor (SIF) and electric displacement intensity factor (EDIF) as

$$\begin{cases} K^T = \lim_{r \rightarrow 0} \sqrt{2\pi r} S_{yz}^{(I)}(r, 0) = \frac{\sqrt{2\pi}}{2} (c_{44}^{(2)} A_{11} - e_{15}^{(2)} B_{11}) \\ K^D = \lim_{r \rightarrow 0} \sqrt{2\pi r} D_y^{(I)}(r, 0) = \frac{\sqrt{2\pi}}{2} (e_{15}^{(2)} A_{11} + e_{11}^{(2)} B_{11}) \end{cases} \quad (14)$$

## Conclusions

The higher order fields of an interfacial crack between FGPMs and HPMs under the impermeable electrical boundary condition of crack surfaces are presented. The first order displacement and electric potential fields of FGPMs are identical to the ones of HPMs. The second order displacement and electric potential fields of them have the same mathematical form; however, the magnitudes of them depend on the undetermined coefficients. The fields are the theoretical basis for determining stress intensity factor and carrying out the experimental analysis. Therefore, the study is of obvious universal theoretical significance and engineering value to various physical weak-discontinuous problems of piezoelectric materials.

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## References

- [1] Y.D. Li & K.Y. Lee. Anti-plane fracture analysis for the weak-discontinuous interface in a non-homogeneous piezoelectric bi-material structure. *European Journal of Mechanics - A/Solids*. 28, pp. 241-247, 2009.
- [2] Y.S. Li, W.J. Feng & Z.H. Xu. A penny-shaped interface crack between a functionally graded piezoelectric layer and a homogeneous piezoelectric layer. *Meccanica*. 44, pp. 377-387, 2009.
- [3] P.W. Zhang, Z.G. Zhou & B. Wang. Dynamic behavior of two collinear interface cracks between two dissimilar functionally graded piezoelectric/piezomagnetic material strips. *Applied Mathematics and Mechanics-English Edition*. 28, pp. 615-625, 2007.
- [4] Y.S. Ing & J.H. Chen. Dynamic fracture analysis of an interfacial crack in a two-layered functionally graded piezoelectric strip. *Theoretical and Applied Fracture Mechanics*. 63, pp. 40-49, 2013.