Approximation Algorithm for Extended Maximum Concurrent Flow Problem with Saturated Capacity

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Abstract—The Maximum Concurrent Flow is a classical multicommodity flow problem and has a extensive applications in the practice such as transportations, communications and network designs. It has important sense not only in theory, but also in practice to develop this kind of multicommodity flow problem. This work studied a kind of extensive maximum concurrent flow problem (EMCFPSC). The major contributions are as follows: (A) Propose the definition of the problem EMCFPSC and prove its solutions exist. (B) Design an approximation algorithm to solve EMCFPSC through constructing auxiliary network and implementing the binary search by a full polynomial time approximation scheme (FPTAS) , presented by Korte and Vygen in 2000 from the framework of Garg and Könemann , to find approximate solutions of Maximum Multicommodity Flow Problem. (C) Propose and analyse the complexity and the approximation measure of the designed algorithm. (D) Shown that the designed approximation algorithm is a FPTAS to solve the problem EMCFPSC.

Keywords- network; multicommodity flow; concurrent flow; approximation algorithm; approximation measure

I. INTRODUCTION

The multicommodity flow problem is an important compoent of the network flow, which can be used to deal with many practical problems, see, e.g., the references listed afterwards. The Maximum Concurrent Flow Problem (MCFP) is a main kind of multicommodity flow problems, which was introduced by Matula in 1985, see Shahrokhi and Matula [1], and have been severely studied for more than two decades, see, e.g., the listed references [1-6]. Motivated by the published literatures of this research area, there are a kind of specific Maximum Concurrent Flow Problem which were discussed in the present work, termed as Extended Maximum Concurrent Flow Problem with Saturated Capacity (EMCFPSC).

The rest of this article is organized as follows. Some preliminaries are presented in Section 2. Section 3 formulates the problem EMCFPSC and proves the existence of its solutions. Section 4 is specially devoted to designing an algorithm to approximately solve the problem. Section 5 investigates the complexity and precision of the algorithm. Finally, the paper is concluded with Section 6.

II. PRELIMINARIES

This section provides some preliminaries for our sequel research.

A graph $G=(V,E)$ is called as a hybrid graph if its edge can be either directed edge or undirected edge, of which the directed graph and undirected graph are specific cases. Here V and E represent all the nodes (or vertices) and edges (or arcs) of G respectively.

Given graph $G=(V,E)$ with weight (or capacity) c and $H=\{[s_i,t_i]:s_i,t_i\in V; i=1,2,\cdots,k\}$, where $c:E\to R_+$ and $R_+=[0,\infty)$, and s_i and t_i express the source and the terminal of commodity i respectively, we call the triad (G,c,H) as a multicommodity network. Let $[s_i,t_i]\in H$; $s_i,v_1,v_2,\cdots,v_l,t_i$ be some nodes of G and be different from each other except for $s_i=t_i$; e_j be the edge of G with endpoint v_j resp. v_{j+1} for $0\le j\le l$, where $v_0=s_i$ and $v_{l+1}=t_i$. (When e_j is directed, the edges v_j and v_{j+1} must be the original endpoint and the terminal endpoint, respectively.) Then it is called

$$P=\left[(s_i,e_0,v_1),(v_1,e_1,v_2),\cdots,(v_l,e_l,t_i)\right]$$

as an s_i-t_i-path of (G,c,H). (To simplify in notation afterwards, we denote (v_j,e_j,v_{j+1}) as (v_j,v_{j+1}) when e_j needn't be indicated; and denote (v_j,e_j,v_{j+1}) as e_j when v_j and v_{j+1} needn't be indicated.) Let Γ_j be a set of s_i-t_i-paths such that $c(e)>0$ for all $e\in E(P)$ and $P\in\Gamma_i, i=1,2,\cdots,k$, where $E(P)$ denotes the set of all the edges of P. It is called $\Gamma = \bigcup_{i=1}^{k} \Gamma_i$ as a (positive) path system on (G, c, H), which is denoted by $\Gamma|_{(G,c,H)}$, or Γ for conciseness.

Definition 1. Let Γ be a path system. If mapping $y : \Gamma \to R_+$ satisfies:

$$\sum_{e \in E(P), P \in \Gamma} y(P) \le c(e), \quad \forall e \in E(\Gamma),$$

where $E(\Gamma) = \{e \in E(P) : P \in \Gamma\}$, i.e. the set of all the edges of Γ. Then y is called as a multicommodity flow on Γ defined by the function on the path system, and as a flow for simplicity; and call $V(y) = \sum_{P \in \Gamma} y(P)$ as flow value of y, and $V_i(y) = \sum_{P \in \Gamma_i} y(P)$ as branch flow value of commodity i with y. The set of all the flows on Γ is denoted by $F[\Gamma]$, and by F in brief.

Remark 1. As is known to us, there are two kinds of definitions about the flows, which are the defined by the function on the edge set and by the function on the path system. For the purpose of being simple and clear, the study is only made as far as the flow defined by the function on the path system in the present work.

Generally, the problem to find a multicommodity flow to satisfy certain conditions is called multicommodity flow problems (MFP). A few of usual multicommodity flow problems are as follows. The problem to find a flow y so that $V(y)$ is maximized, i.e.

$$V(y) = OPT[\Gamma] = \hat{V}$$
$$(= \max\{V(y) : y \in F[P]\}),$$

is called the maximum multicommodity flow problem on Γ（MMFP）. A flow y satisfying MMFP is called a solution of MMFP. The problem to find a flow y so that $V_i(y) \le b_i$ for given b_i, $i = 1, 2, \cdots, k$, and $V(y)$ is maximized, i.e. $V(y) = \overline{V}(= \max\{V(y) : V_i(y) \le b_i,\ i = 1, 2, \cdots, k; y \in F\})$, is called the maximum flow problem with bounds (MMFP-B). The problem to find a flow y so that $V(y)$ is maximized under the condition $V_i(y) = \lambda b_i$ for given $b_i, i = 1, 2, \cdots, k$, and for $\lambda \in [0,1]$ is called the Maximum Concurrent Flow Problem (MCFP) (see, e.g., Shahrokhi and Matula [6], or Garg and Könemann [5]). Call the maximized λ as concurrent point of problem MCFP.

III. MODEL FORMULATION

Now we begin to introduce the problem which is attacked in the present work.

Definition 2. Suppose Γ is a path system on the multicommodity network (G, c, H), and

$$\mathbf{b} = (b_1, b_2, \cdots, b_k).$$

Call the problem to find a flow y' so that

$$\min_{1 \le i \le k}\left[\frac{1}{b_i} V_i(y')\right] = \max\left\{\min_{1 \le i \le k}\left[\frac{1}{b_i} V_i(y)\right] : y \in F, V_i(y) \le b_i\right\}$$

as Extended Maximum Concurrent Flow Problem (EMCFP).

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and

$$V(y') = \max\{V(y) : y \text{ is the solution of EMCFP}\}$$

as Extended Maximum Concurrent Flow Problem with Saturated Capacity (EMCFPSC). ($\frac{1}{b_i} V_i(y)$ can be interpreted as the met rate of commodity i with y.)

Remark 2. Let y be a solution of EMCFP and y' be a solution of MCFP. Then $\min_{1 \le i \le k}\left[\frac{1}{b_i} V_i(y)\right] = \lambda$ (concurrent point of the problem MCFP). And $V_i(y) \ge V_i(y')$ for all i.

Theorem 1. The solution of the problem EMCFPSC exists.

Proof. It is obvious that the solution of MCFP is also the feasible solution of EMCFP. Hence,

$$B = \{y \in F : y \text{ is the feasible solution of EMCFP}\} \ne \emptyset$$

from the well known fact the solution of the problem MCFP exists. When the member of B is finite, the conclusion of Theorem 1 is trivial. Otherwise,

$$\sup\{V(y) : y \in B\}$$

exists and is finite. Let it be b. Then there is at least a sequence $\{y_j\}$ in B such that $\lim_{j \to \infty} V(y_j) = b$, and $\lim_{j \to \infty} y_j(P)$ exists for all $P \in \Gamma$. Now let

$$y(P) = \lim_{j \to \infty} y_j(P)$$

for all $P \in \Gamma$. Then $y \in B$, and $V(y) = b$. That is, y is the solution of EMCFPSC. □

Example. During the winter between the year 2009 and 2010, in major regions of the world, the amount of energy consumed for heating is far more than the ordinary winter due to the abnormal lower air temperatures. With this circumstances, the electrical networks emerge to congestion for a lot of districts in the world. Assume the purpose of supplying power is firstly to maximize the minimum of the met rates of cities, and secondly to maximize the amount of the power supplement. Then the optimal problem with the power supplement can be largely regarded as the mould EMCFPSC. One can believe the mean of applications of the mould EMCFPSC from this instance.

IV. ALGORITHM

In this section, we are devoted to designing an approximation algorithm for the problem EMCFPSC.

Algorithm (Approximation algorithm for EMCFPSC):

Input Multicommodity network (G,c,H), path system Γ, vector $\mathbf{b}$ and error parameter η.

($\min b_i > 0, V(y) > 0, \lambda > 0,$ where λ is the concurrent point of the problem MCFP.)

Output Flow y^* on Γ, which has the characters specified by the Theorem 2 in Section 5.

1. Put

$$\varepsilon = \min\left\{\frac{\eta}{\sum_{i=1}^{k} b_i}, \frac{1}{7}\right\}; b_i^0 = b_i, i = 1,2,\cdots,k; l = 0, h = 0.$$

2. Set $l := l+1$. If $l\eta < 1$, implement the next Step. Otherwise, put $l^* = l$ and go to the Step 4.

3. Set $b_i := l\eta b_i^0, \mathbf{b} := (b_1, b_2, \cdots, b_n)$. For Γ, $\mathbf{b}$ and ε, obtain an approximation solution y_1 of MMFP-B by Algorithm 2 of Cheng and Li [7]. If

$$V(y_1) < \frac{1}{1+\varepsilon}\left(\sum_{i=1}^{k} b_i\right),$$

put $l^* = l$ and implement the next step. Otherwise, return to (2).

4. Construct the auxiliary network and demanding vector $\mathbf{b}$ as follows. Set

$$V' = V \cup \{t_0', t_i' : i = 1,2,\cdots,k\}\ (t_0', t_i' \overline{\in} V),$$

$$E' = E \cup \{(t_i, t_0'), (t_i, t_i') : i = 1,2,\cdots,k\},$$

$$H = \{[s_i, t'], [s_i, t_0'] : i = 1,2,\cdots,k\},$$

$$G' = (V', E'),$$

$$\Gamma_i' = \{P + (t_i, t_i') : P \in \Gamma_i\},$$

$$\Gamma_0'^i = \{P + (t_i, t_0') : P \in \Gamma_i\}, i = 1,2,\cdots,k;$$

$$\Gamma' = \left(\bigcup_{i=1}^{k} \Gamma'\right) \cup \left(\bigcup_{i=1}^{k} \Gamma_0'^i\right);$$

$$b_i' = (l^* - l)\eta b_i^0, b_0' = b, \mathbf{b}' = (b_1', b_2', \cdots, b_k', b_0'),$$

and

$$c'(e) = \begin{cases} c(e) & e \in E(G) \\ b_i' & e = (t_i, t_i') \\ b_i - b_i' & e = (t_i, t_i^0). \end{cases}$$

Put $y' = (0)$.

5. Put $h := h+1, b := h\eta$. If $b > \sum_{i=1}^{k} b_i^0$, put $h^* = h$ and implement the next step. Otherwise for Γ', b' and ε, obtain an approximation solution y_2 by Algorithm 2 of [7]. If $V(y_2) < \frac{1}{1+\varepsilon}\left[\left(\sum_{i=1}^{k} b_i'\right) + b\right]$, put $h^* = h$ and implement the next step. Otherwise, put $y' := y_2$, and then return to 5.

6. Put

$$y^*(P) = y'(P + (t_i, t_i')) + y'(P + (t_i, t_0)), \forall P \in \Gamma_i,$$

$i = 1,2,\cdots,k.$ Stop.

V. ALGORITHM ANALYSIS

This section is specially devoted to discussing the correctness approximate precision and complexity of the algorithm which have been presented above.

Theorem 2. The complexity of the Algorithm is

$$O\left(\frac{1}{\eta^3} km^2 \log n\right),$$

where $k = |H|, n = |V(G)|, m = |E(G)|$. Let y be a solution of the EMCFPSC and y^* be the output of the Algorithm. Then y^* is a flow on Γ,

$$V_i(y^*) \le b_i, i = 1,2,\cdots,k, V(y) \le \left[l^*\left(\sum_{i=1}^{k} b_i\right) + h^*\right]\eta,$$

and

$$\left[(l^* - l)\left(\sum_{i=1}^{k} b_i\right) + h^*\right]\eta - 3\eta \le V(y^*) \le \left[(l^* - l)\left(\sum_{i=1}^{k} b_i\right) + h^*\right]\eta, \quad (1)$$

$$(l^* - 1)\eta - \frac{2\eta}{\min b_i} \le \min \frac{V_i(y^*)}{b_i} \le \min \frac{V_i(y)}{b_i} = \lambda \le l^*\eta. \quad (2)$$

Here λ is the concurrent point of the corresponding problem MCFP. And moreover, we have

$$V(y)(1-\delta) \le V(y^*), \quad (3)$$

$$\lambda(1-\delta) \le \min \frac{V_i(y^*)}{b_i} \le \lambda, \quad (4)$$

where

$$\delta = \eta \max\left\{\frac{1}{V(y)}\left[\left(\sum_{i=1}^{k} b_i + 3\right), \frac{1}{\lambda}\left(1 + \frac{2}{\min b_i}\right), \frac{km}{V(y)}\right]\right\}.$$

Proof. Firstly, it is obvious that the complexity of the Algorithm depends on the complexity of the subroutine Algorithm 2 of [7] and the times of the iteration that is operated on the subroutine. By simply analyzing the

Algorithm, it can be known that the times is no more than $\max b_i\left(\frac{k}{\eta}+\frac{1}{\eta}\right)$, which can be denoted as $O\left(\frac{1}{\eta}\right)$. On the other hand, the complexity of the subroutine is

$$O\left(\frac{1}{\varepsilon^2}km^2\log n\right)=O\left(\frac{1}{\eta^2}km^2\log n\right),$$

see [7]. Hence the complexity of Algorithm is

$$O\left(\frac{1}{\eta^3}km^2\log n\right).$$

By further analyzing the Algorithm, it can be easily known that y^* is a flow on Γ,

$$V\left(y^*\right)\le\left[\left(l^*-1\right)\left(\sum_{i=1}^{k}b_i\right)+h^*\right]\eta,$$

and $\min\frac{V_i\left(y^*\right)}{b_i}\le\min\frac{V_i\left(y\right)}{b_i}\le l^*\eta$. Hence we only need to specify $\left[\left(l^*-1\right)\left(\sum_{i=1}^{k}b_i\right)+h^*\right]\eta-3\eta\le V\left(y^*\right)$ and

$$\min\frac{V_i\left(y^*\right)}{b_i}\ge\left(l^*-1\right)\eta-\frac{2\eta}{\min b_i}.$$

In terms of the Algorithm,

$$\varepsilon\le\frac{\eta}{\sum_{i=1}^{k}b_i},\left(l^*-1\right)\eta\le 1,\left(h^*-1\right)\eta\le\sum_{i=1}^{k}b_i,$$

and

$$V\left(y^*\right)\ge\frac{1}{1+\varepsilon}\left[\left(\sum_{i=1}^{k}\left(l^*-1\right)\eta b_i\right)+\left(h^*-1\right)\eta\right].$$

Therefore

$$\begin{aligned}V\left(y^*\right)&\ge\frac{1}{1+\varepsilon}\left[\left(\sum_{i=1}^{k}\left(l^*-1\right)\eta b_i\right)+\left(h^*-1\right)\eta\right]\\&=\left[\left(l^*-1\right)\left(\sum_{i=1}^{k}b_i\right)+\left(h^*-1\right)\right]\eta-\\&\quad\frac{\varepsilon}{1+\varepsilon}\left[\left(l^*-1\right)\eta\left(\sum_{i=1}^{k}b_i\right)+\left(h^*-1\right)\eta\right]\\&\ge\left[\left(l^*-1\right)\left(\sum_{i=1}^{k}b_i\right)+\left(h^*-1\right)\right]\eta\\&\quad-\varepsilon\left[\left(\sum_{i=1}^{k}b_i\right)+\left(\sum_{i=1}^{k}b_i\right)\right]\\&\ge\left[\left(l^*-1\right)\left(\sum_{i=1}^{k}b_i\right)+\left(h^*-1\right)\right]\eta-2\eta.\end{aligned}\tag{5}$$

Let $V_i'=\sum_{P\in\Gamma_i'}y_i'(P),i=1,2,\cdots,k$ and $V_i'^0=\sum_{P\in\Gamma_0'^i}y_i'^0(P)$. Then $V_i\left(y^*\right)=V_i'+V_i'^0$ since $y_i=y_i'+y_i'^0$. For $b_i'=\left(l^*-1\right)\eta b_i$, we have $V_i'\le\left(l^*-1\right)\eta b_i$. Suppose $V_j\left(y^*\right)<\left(l^*-1\right)\eta b_j-2\eta$ for some j. Then

$$\begin{aligned}V\left(y^*\right)&=\sum V_i=\left[\sum_{i\ne j}\left(V_i'+V_i'^0\right)\right]+V_j\left(y^*\right)\\&<\sum_{i\ne j}V_i'+\sum_{i\ne j}V'^0+\left(l^*-1\right)\eta b_j-2\eta.\end{aligned}\tag{6}$$

On the other hand, $\sum_{i\ne j}V_i'^0\le\sum V_i'^0\le\left(h^*-1\right)\eta$. Hence, (6) implies

$$\begin{aligned}V\left(y^*\right)&<\left[\left(\sum_{i\ne j}\left(l^*-1\right)\eta b_i\right)+\left(h^*-1\right)\eta\right]\\&\quad+\left(h^*-1\right)\eta b_j-2\eta\\&=\left[\left(\sum\left(l^*-1\right)\eta b_i\right)+h^*\eta\right]-3\eta.\end{aligned}\tag{7}$$

It is clearly that (7) is in contradiction with (5). So, $V\left(y^*\right)\ge\left(l^*-1\right)\eta b_i-2\eta$ for any i. This implies

$$\min\frac{V_i\left(y^*\right)}{b_i}\ge\left(l^*-1\right)\eta-\frac{2\eta}{\min b_i}.$$

Finally, in terms of

$$\eta=\frac{\varepsilon}{\max\left\{\frac{1}{V\left(y\right)}\left[\left(\sum_{i=1}^{k}b_i\right)+3\right],\frac{1}{\lambda}\left(1+\frac{2}{\min b_i}\right),\frac{km}{V\left(y\right)}\right\}},$$

$$V\left(y\right)\le\left[l^*\left(\sum_{i=1}^{k}b_i\right)+h^*\right]\eta,$$

and (1), then

$$\begin{aligned}&V\left(y\right)\left(1-\varepsilon\right)=V\left(y\right)-V\left(y\right)\cdot\\&\left[\max\left\{\frac{1}{V\left(y\right)}\left[\left(\sum_{i=1}^{k}b_i\right)+3\right],\frac{1}{\lambda}\left(1+\frac{2}{\min b_i}\right),\frac{km}{V\left(y\right)}\right\}\eta\right]\\&\le\left[l^*\left(\sum_{i=1}^{k}b_i\right)+h^*\right]-\left[\left(\sum_{i=1}^{k}b_i\right)+3\right]\eta\le V\left(y^*\right).\end{aligned}$$

That is, (3) holds. And in terms of

$$\varepsilon=\max\left\{\frac{1}{V\left(y\right)}\left[\left(\sum_{i=1}^{k}b_i\right)+3\right],\frac{1}{\lambda}\left(1+\frac{2}{\min b_i}\right),\frac{km}{V\left(y\right)}\right\}\eta$$

and (2), then

$$\begin{aligned}&\lambda\left(1-\varepsilon\right)=\\&\lambda-\lambda\left[\max\left\{\frac{1}{V\left(y\right)}\left[\left(\sum_{i=1}^{k}b_i\right)+3\right],\frac{1}{\lambda}\left(1+\frac{2}{\min b_i}\right),\frac{km}{V\left(y\right)}\right\}\eta\right]\\&\le l^*\eta-\left(1+\frac{2}{\min b_i}\right)\eta\le\min\frac{V_i\left(y^*\right)}{b_i}\le\lambda.\end{aligned}$$

That is, (4) holds . This completes the proof. □

Remark 3. (i) Except for (1) and (2), we guess

$$V\left(y^*\right)\le V\left(y\right)+km\eta,\tag{8}$$

while η is small enough. This leads to

$$V\left(y^*\right)\le V\left(y\right)\left(1+\varepsilon\right).\tag{9}$$

Note that the complexity of the Algorithm is

$$O\left(\frac{1}{\eta^3}km^2\log n\right).$$

Combining (3), (4) and (9), it can be believed that the Algorithm is a FPTAS (fully polynomial time approximation scheme) to solve EMCFPSC, to a great extent. For the concept of FPTAS, please see Du et al [8] or Korte and Vygen [9]. (ii) Recently, Büsing and Stiller [10] considered a kind of network flow problem arising in line planning. It may be an interesting topic to explore the application of the present work in line planning. Moreover, Soleimani-damaneh[11] investigated the determination of maximal flow in a fuzzy dynamic network with multiple sinks. Mehri [11] studied the inverse maximum dynamic flow problem. It is believed to extend the present work in a fuzzy dynamic network and to address the inverse problem EMCFPSC may also be interesting topics. For the recent work, one can also see, e.g. , [13-14].

VI. CONCLUSIONS

In this paper, the problem Extended Maximum Concurrent Flow Problem with Saturated Capacity is introduced. The existence of its solutions is proved. A approximation algorithm to solve the problem is designed, and the effectiveness of the algorithm, including the complexity and approximation measures, is discussed. The approach designing the algorithm is the specific contribution of the present work, whose main characters are to construct auxiliary network and to implement iterating search through a subroutine (a known algorithm). To solve other network problems with this approach is an interesting topic for further researches in the future.

ACKNOWLEDGMENT

The authors cordially thank the anonymous referees for their valuable comments which lead to the improvement of this paper. This research is Supported by the Education Department Foundations of Liaoning Province L2010514 and L2013420, and the Science and Technology Department Natural Fund of Liaoning Province 2014020120.

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