An Approximate Semantic Model of Hybrid Systems

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***Abstract*—Hybrid system is a dynamic system. However, it is difficult to formally analyze hybrid systems due to the continuous parts. In this paper, we propose an approximate semantic model of hybrid systems, in order to employ formal analyzing techniques; furthermore, the error could be under control. This method is based on incomplete low-up matrix decomposition, which is used to generate approximating transition between states for the continuous components of hybrid systems. The technique reduces the complexity of analyzing computation. Moreover, the model is also used to approximating control the transition conditions, which simplifies the conditions.**

Keywords-hybrid systems; approximate analyzing model; incomplete decomposition; error analysis

I. INTRODUCTION

Approximation of purely discrete systems has been based on language inclusion and equivalence with notions such as simulation or bisimulation relations [1].The notion of bisimulation has been instrumented in obtaining decidability results for various classes of hybrid systems[2].Notions that are similar to bisimulation have been considered in supervisory control of discrete event systems [3], and hybrid systems [4]. Bisimulations have also been used as a controller synthesis tool for discrete-event systems[5].However, these methods can only ensure the systems approximately satisfy the given rang and not be able to calculate the allowable error of transition between states.

The basic method of incomplete factorizations is used of incomplete LU decomposition. The most well-known is the incomplete triangular decomposition ILU0 [6]. However, the ILU preconditioner does not perform well for some PDE problems [7]. Thus, modified ILU (MILU) preconditioner is proposed [8].

In this work, we review the hybrid systems and its classical semantic model in section 2. In section 3, approximate techniques employed should be demonstrated. Then the approximate semantic model and its error analyzing are considered in detail in section 4. Section 5 draws conclusion and proposes the future works.

II. HYBRID SYSTEMS AND ITS SEMANTIC MODEL

Definition 1 A hybrid system [9]is defined as a tuple

$$H = (L, n, p, E, F, Inv, G, R, Q^0),\ \text{where}$$

- L is a finite set of locations or discrete states. $|L|$ denotes the number of elements of L, $L = \{1, \cdots, |L|\}$.

- n: $L \to \mathrm{N}$, where for every $l \in L$, n_l is the dimension ofthe continuous state space in the location l. The set of states of the hybrid system is $Q = \bigcup_{l \in L} \{l\} \times R^{n_l}$.

- p: $L \to \mathrm{N}$, where for every $l \in L$, p_l is the dimension of the continuous observation of the hybrid system in the location l. The set of observations is $\Pi = \bigcup_{l \in L} \{l\} \times R^{p_l}$.

- $E \subseteq L \times L$ is the set of events or discrete transitions.

- $F = \{F_l | l \in L\}$ defines the continuous dynamics in each location. For each $l \in L$, F_l is a triple (f_l, g_l, U_l) where f_l: $R^{n_l} \times U_l \to R^{n_l}$, g_l: $R^{n_l} \to R^{p_l}$ and $U_l \subseteq R^{m_l}$ is a compact set of internal inputs accounting for disturbances and modeling uncertainties. While the discrete part of the state is l, the continuous part evolves according to

$$\begin{cases} \dot{x}(t) = f_l(x(t), u(t)), u(t) \in U_l \\ y(t) = g_l(x(t)). \end{cases}$$

- $Inv = \{Inv_l | l \in L\}$ defines an invariant set in each location. For each $l \in L$, $Inv_l \subseteq R^{n_l}$ constrains the value of the continuous part of the state while the discrete part is l.

● $G = \{G_e | e \in E\}$ defines the guard for each discrete transition. For each $e = (l.l') \in E$, $G_e \subseteq Inv_l$. The discrete transition e is enabled when the continuous part of the state is in G_e.

● $R = \{R_e | e \in E\}$ defines the reset map for each discrete transition. For each $e = (l.l') \in E$, R_e: $G_e \to 2^{Inv_{l'}}$. When the event e occurs, the continuous part of the state is reset using R_e.

● $Q^0 \subseteq Q$ is the set of initial states: $Q^0 = \bigcup_{l \in L} \{l\} \times I_l^0$, with $I_l^0 \subseteq Inv_l$.

Here, we consider that hybrid systems can be formulated as transition systems [10]. The results can be reviewed in much detail in [11].

Definition 2 A labeled transition system with observations is a tuple $T = (Q, \Sigma, \to, Q^0, \Pi, \langle\langle . \rangle\rangle)$ that consists of: a set Q of states, a set Σ of labels, a transition relation $\to \subseteq Q \times \Sigma \times Q$, a set $Q^0 \subseteq Q$ of initial states, a set Π of observations, and an observation map $\langle\langle . \rangle\rangle : Q \to \Pi$.

A state trajectory of T is a sequence of transitions,

$$q^0 \xrightarrow{\sigma^0} q^1 \xrightarrow{\sigma^1} q^2 \xrightarrow{\sigma^2} \cdots, \qquad \text{where } q^0 \in Q^0.$$

Let $T_1 = (Q_1, \Sigma_1, \to_1, Q_1^0, \Pi_1, \langle\langle . \rangle\rangle_1)$ and $T_2 = (Q_2, \Sigma_2, \to_2, Q_2^0, \Pi_2, \langle\langle . \rangle\rangle_2)$ be two labeled transition systems with the same set of labels $(\Sigma_1 = \Sigma_2 = \Sigma)$ and the same set of observations $(\Pi_1 = \Pi_2 = \Pi)$ (i.e. T_1 and T_2 are elements of $T(\Sigma, \Pi)$). Let us assume that the set of observation Π is a metric space; d_Π denotes the metric of Π.

Definition 3 A relation $S_\delta \subseteq Q_1 \times Q_2$ is a $\delta -$approximate simulation relation of T_1 by T_2 if for all $(q_1, q_2) \in S_\delta$:

(1) $d_\Pi(\langle\langle q_1 \rangle\rangle_1, \langle\langle q_2 \rangle\rangle_2) \le \delta$,

(2) $\forall q_1 \xrightarrow{\sigma}_1 q_1', \exists q_2 \xrightarrow{\sigma}_2 q_2'$ such that $(q_1', q_2') \in S_\delta$.

Definition 4 T_2 approximately simulates of T_1 with the precision δ (noted $T_1 \preceq_\delta T_2$), if there exists S_δ, a $\delta -$approximate simulation relation of T_1 by T_2 such that for all $q_1 \in Q_1^0$, there exists $q_2 \in Q_2^0$ such that $(q_1, q_2) \in S_\delta$.

If T_2 approximately simulates T_1 with the precision δ then the language of T_1 is approximated with precision δ by the language of T_2.

Theorem 5 If $T_1 \preceq_\delta T_2$, then for all external trajectories of T_1,

$$\pi_1^0 \xrightarrow{\sigma^0} \pi_1^1 \xrightarrow{\sigma^1} \pi_1^2 \xrightarrow{\sigma^2} \cdots,$$

there exists an external trajectory of T_2 with the same sequence of labels

$$\pi_2^0 \xrightarrow{\sigma^0} \pi_2^1 \xrightarrow{\sigma^1} \pi_3^2 \xrightarrow{\sigma^2} \cdots,$$

such that for all $i \in N, d_\Pi(\pi_1^i, \pi_2^i) \le \delta$ [11]. (Proof omitted.)

III. Approximate and Incomplete Factorizations

The approximate solution for a large sparse linear system $Ax = b$ is to find a matrix M (the preconditioner) such that the original linear system is transformed into an equivalent linear system $M^{-1}Ax = M^{-1}b$. Here, we focus on incomplete factorization preconditioners which are ILU and MILU. The basic idea of ILU preconditioner is to modify Gaussian elimination to allow fill-ins at only a restricted set of positions in the LU factors.

Let the allowable fill-in positions be given by the index set S, i.e.

(1) $l_{i,j} = 0$ if $j > i$ or $(i, j) \notin S$; $u_{i,j} = 0$ if $i > j$ or $(i, j) \notin S$.

A commonly used strategy is to define S by:

(2) $S = \{(i, j) | a_{i,j} \neq 0\}$.

Let the preconditioner M be defined by the product of the resulting LU factors, i.e. $M = LU$.

(3) $m_{i,j} = a_{i,j}$ if $(i, j) \in S$.

The basic idea of MILU preconditioner is: in the condition (3) for ILU, the condition $m_{i,j} = a_{i,i}$ is removed and a new row sum condition is added. That is, (3) is replaced by:

(4) $\sum_{j=1}^{n} m_{i,j} = \sum_{j=1}^{n} a_{i,j}$ $\forall i$ and $m_{i,j} = a_{i,j}$ if $i \neq j$ and $(i, j) \in S$.

Here, accuracy refers to the degree of preconditioner M and matrix A, can be measured by the size of the $\|M - A\|_F$.

IV. Approximate Semantic Model

Let $H = (L, n, p, E, F, Inv, G, R, Q^0)$ be a hybrid system and $T = (Q, \Sigma, \to, Q^0, \Pi, \langle\langle . \rangle\rangle)$ be the associated transition system. Here, we only consider the continuous state space in the location $l \in L$ and the relation of states transition is $\dot{x} = Ax$. For, the continuous dynamic behavior $q \in Q$ is abstracted into a plurality of continuous dynamic behaviors $\cdots q_1 q_2 q_3 \cdots$, and every $q_i (i \in N)$, where N is the set of positive integers, has the form of changing rules $\dot{x} = Ax$. For $t \in R^+$, where R^+ is the set of positive real numbers, the continuous dynamic behavior trajectory is abstracted into:

$$\cdots q_1 \xrightarrow{\sigma^1} q_2 \xrightarrow{\sigma^2} q_3 \xrightarrow{\sigma^3} \cdots,$$

and the mathematical relationship of $q_1 \xrightarrow{\sigma^1} q_2$, $q_2 \xrightarrow{\sigma^2} q_3$ is respective $\dot{x} = A_1 x$, $\dot{y} = A_2 y$. Thus, a transition from behavior q_1 to behavior q_3 can be expressed as $q_1 \xrightarrow{\sigma^2 \sigma^1} q_3$.

Consider the continuous dynamic behavior $q_1 \xrightarrow{\sigma^1} q_2$ is following as:

$$\begin{cases} \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ \cdots\cdots \\ \dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n \end{cases}, t \in [t_1, t_2].$$

And the initial behavior is $x(t_1) = (b_1, b_2, \cdots, b_n)$.

It can be expressed as: $\dot{x} = A(t)x$, $t \in [t_1, t_2]$,

where
$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}.$$

The original system $\dot{x} = A(t)x$ is transformed into an equivalent system $M^{-1}Ax = M^{-1}\dot{x}$ by the mean of ILU, where $M = LU$ is the preconditioner, L is a triangular matrix, U is an upper triangular matrix. Because the system $\dot{x} = A(t)x$ is a portrait of the transition process from q_1 to q_2. $A - M$, an error matrix, has the meaning that q_1 can be translated into q_2 in the error, and its degree of approximation is $\|M - A\|_F$.

Definition 6 The approximate distance between q_1 and q_2 as $d(q_1, q_2) = \|M - A\|_F$.

We can call $d(q_1, q_2) = \|M - A\|_F$ as the degree of approximation of q_1 translated into q_2. Because $x(t_1) = q_1, x(t_2) = q_2$, the solution y of the equivalent system $M^{-1}Ax = M^{-1}\dot{x}$ is the approximate solution of the original system $\dot{x} = A(t)x$, where $y(t_1) = q_1, y(t_2) = q_2'$. The mathematical relationship of $q_1 \xrightarrow{\sigma} q_2'$ is: $\dot{y} = A(t)y$, $t \in [t_1, t_2]$,

where $y = (y_1 \quad y_2 \quad \cdots \quad y_n)^T$, $\dot{y} = (\dot{y}_1 \quad \dot{y}_2 \quad \cdots \quad \dot{y}_n)^T$.

And the initial state is $y(t_1) = (b_1, b_2, \cdots, b_n)$.

Therefore, the accuracy between exact solution x and approximate solution y is $\delta = \|M - A\|_F$.

Theorem 7Let y be the approximate solution of $\dot{x} = A(t)x$, then $\dot{y} = A(t)y$ simulates approximately to $\dot{x} = A(t)x$ with precision $\lim_{t\to t_2}\|A(t)\|_F\|M - A\|_F$.

Proof: $$\lim_{t\to t_2}\left\|\dot{x}-\dot{y}\right\|_F = \lim_{t\to t_2}\|A(t)x - A(t)y\|_F = \lim_{t\to t_2}\|A(t)(x-y)\|_F$$
$$\le \lim_{t\to t_2}\|A(t)\|_F\|x-y\|_F \le \lim_{t\to t_2}\|A(t)\|_F \lim_{t\to t_2}\|x-y\|_F$$
$$= \lim_{t\to t_2}\|A(t)\|_F\|M - A\|_F$$

Hence, we can say the continuous dynamic behavior q_2 can be approximately simulated by q_2'.

For location $l \in L$, the continuous dynamic behavior q is abstracted into a plurality of continuous dynamic behaviors $\cdots q_1q_2q_3\cdots$. q is approximately simulated into q', that is $\cdots q_1'q_2'q_3'\cdots$. This means the set of continuous dynamic behavior $\cdots q_1'q_2'q_3'\cdots$ can be abstracted into a continuous behavior q'.

Here, we mainly focus on the discrete switching between different discrete states. The continuous dynamic state q in the location $l \in L$ is abstracted into a set of continuous dynamic states $\cdots q_1q_2q_3\cdots$, then the set of continuous dynamic states $\cdots q_1q_2q_3\cdots$ are approximately simulated to $\cdots q_1'q_2'q_3'$ via the ILU method. That is, $l \xrightarrow{\tau} l'$ (iff$(l, l') = e \in E$) becomes $\cdots q_1'q_2'q_3'\cdots \xrightarrow{\tau} l'$, here, τ is a discrete transition.

We discuss the discrete transition τ (condition). Its form likes this: $g_i(x_1, x_2, \cdots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$ $(i = 1, 2, \cdots, n)$.

We only study the form of $g_i(x_1, x_2, \cdots, x_n)$. Sometimes, all of variables are difficult to satisfy the scope of $g_i(x_1, x_2, \cdots, x_n)$. Thus we can simplify the condition. We put all of the condition considered. It likes this:

$$\begin{cases} g_1(x_1, x_2, \cdots, x_n) = a_{11}x_1 + a_{12}x_2 + \cdots + a_{nn}x_n \\ g_2(x_1, x_2, \cdots, x_n) = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \\ g_n(x_1, x_2, \cdots, x_n) = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases} \quad (1)$$

Here, let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, x' = \begin{bmatrix} g_1(x_1, x_2, \cdots, x_n) \\ g_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ g_n(x_1, x_2, \cdots, x_n) \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Thus (1) is equivalent to $x' = Ax$. We can obtain the preconditioner matrix M of matrix A by the mean of ILU. We assume that the form of M likes this:

$$M = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

Then M replaces A in the equation $x' = Ax$, thereby we can obtain equations like this:

$$\begin{cases} g_1(x_1, x_2, \cdots, x_n) = b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n \\ g_2(x_1, x_2, \cdots, x_n) = b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n \\ \vdots \\ g_n(x_1, x_2, \cdots, x_n) = b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n \end{cases} \quad (2)$$

The values of variables in (1) are different from that of (2) due to the preconditioner M. It's difficult to control the precise conditions for states transitions. So we only find the approximating conditions for states transitions.

Error of exact solution for (1) is $\|A-M\|_F$. We assume that the precise conditions is η, so the range of variables only belongs to $[-\|A-M\|_F+\eta, \|A-M\|_F+\eta]$,there states transition can come true. Here, states transition only is approximation.

V. Conclusion

A new modeling of hybrid systems is proposed by means of incomplete low-up matrix decomposition. Extending hybrid systems' semantic models to approximation semantic models enables the formal techniques to analyze hybrid systems. In addition, we analyze the error in detail, which could be under control according to proper decomposition condition.

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