The Existence and Simulations of Periodic Solution of Predator-prey Models with Impulsive Perturbations and Holling Type III Functional Responses

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***Abstract*—In this paper, a criterion for the existence of periodic solutions of predator-prey models with impulsive perturbations and Holling type III functional responses is established using the continuation theorem of coincidence degree theory and analysis techniques. Further, some numerical simulations show that our models can occur in many forms of complexities including periodic oscillation and Gui strange attractors.**

Keywords-periodic solution; predator-prey model; coincidence degree theory; impulses

I. INTRODUCTION

The prey–predator systems of Lotka–Volterra type have been discussed widely in the last century. In the literature many studies considered the predator species as density independence. However, considerable evidence show that some predator species may be density dependence because of the environmental factors [1,2,3]. It is more appropriate to add the density-dependent term to these models in such a circumstance. On the other hand, many systems arising in physical, chemical and biological phenomena exhibit impulsive dynamical behaviours due to the abrupt jumps during the evolution processes, which can be modelled by impulsive differential equations [4-9]. In this paper, we shall explore the dynamics of the nonautonomous predator–prey system with impulsive perturbations and Holling type III functional responses in a more general form.

$$\begin{cases} \left.\begin{aligned} &\frac{dy_1(t)}{dt} = y_1(t)[r(t) - a(t)y_1(t)] - \frac{b(t)y_1^2(t)y_2(t)}{\beta^2(t)+y_1^2(t)}, \\ &\frac{dy_2(t)}{dt} = -d(t)y_2(t) + \frac{c(t)y_1^2(t)y_2(t)}{\beta^2(t)+y_1^2(t)} - e(t)y_2^2(t), \end{aligned}\right\} t \neq t_k, \\ \left.\begin{aligned} &\Delta y_1(t_k) = c_k y_1(t_k), \\ &\Delta y_2(t_k) = d_k y_2(t_k), \end{aligned}\right\} t = t_k,\ k = 1,2,\ldots \end{cases} \tag{1}$$

where $y_1(t)$ and $y_2(t)$ represent densities of prey and predator at time t respectively. $r(t)$, $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, $\beta(t)$ are positive periodic continuous functions with period ω, $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{t\to+\infty} t_k = +\infty$. Assume that $c_k, d_k (k \in \mathbb{Z}_+)$ are constants and there exists a positive integer q such that $t_{k+q} = t_k + \omega$, $c_{k+q} = c_k$, $d_{k+q} = d_k$ and $0 < t_{k+1} - t_k < \omega$.

Let $g(t)$ be a bounded continuous function on $\mathbb{R}$. Define $g^l = \inf_{t\in R} g(t)$, $g^u = \sup_{t\in R} g(t)$. Particularly, if $g(t)$ are ω-periodic function with respect to t, then

$$\bar{g} = \frac{1}{\omega}\int_0^\omega g(t)dt.$$

II. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

To prove our results, we need the notion of the Mawhin's continuation theorem formulated in [10].

Lemma 1. Let X and Y be two Bannach space. Consider an operator equation $Lx = \lambda Nx$, where $L: \mathrm{Dom}\, L \cap X \to Y$ is a Fredholm operator of index zero and $\lambda \in [0,1]$ is a parameter. Then there exist two projectors $P: X \to \mathrm{Ker}\, L$ and $Q: Y \to Y/\mathrm{Im}\, L$. Assume that $N: \bar{\Omega} \to Y$ is L-compact on $\bar{\Omega}$, where Ω is open bounded in X. Furthermore, assume that:

(a) for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \mathrm{Dom}\, L$, $Lx \neq \lambda Nx$.

(b) for each $x \in \partial\Omega \cap \mathrm{Ker}\, L$, $QNx \neq 0$,

(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}\, L, 0\} \neq 0$, where $J: \mathrm{Im}\, Q \to \mathrm{Ker}\, L$ is an isomorphism and $\deg\{*\}$ represents the Brouwer degree.

Then equation $Lx = Nx$ has a solution on $\bar{\Omega} \cap \mathrm{Dom}\, L$.

Now we are ready to state and prove the main results of the present paper.

Theorem 1 Assume that the following conditions hold:

$$C_\pm := 2\bar{r}\omega \pm \ln\prod_{k=1}^{q}(1+c_k) > 0,$$

$$D_\pm := 2\bar{d}\omega \pm \ln\prod_{k=1}^{q}(1+d_k) > 0,$$

$$m_1 := \frac{1}{2}\ln\frac{\beta^{2l}(D_- - \bar{d}\omega)}{\bar{c}\omega} - C_+ > 0,$$

$$M_1 := \ln\frac{C_+ - \bar{r}\omega}{\bar{a}\omega} + C_+ > 0,$$

$$m_2 := \ln\frac{1}{\bar{e}}\left(\frac{c^l\omega e^{2m_1}}{\omega^{2u} + e^{2m_1}} - D_- + \bar{d}\omega\right) - D_+ > 0,$$

$$M_2 := \ln\frac{\bar{c}\omega - D_-}{\bar{e}\omega} + D_+ > 0$$

then system (1) has at least one ω-periodic solution.

Proof. Make the change of variables $y_1(t) = \exp\{x_1(t)\}$, $y_2(t) = \exp\{x_2(t)\}$, then system (1) can be reformulated as

$$\begin{cases} \left.\begin{aligned} \frac{dx_1(t)}{dt} &= r(t) - a(t)e^{x_1(t)} - \frac{b(t)e^{x_1(t)+x_2(t)}}{\beta^2(t)+e^{2x_1(t)}}, \\ \frac{dx_2(t)}{dt} &= -d(t) + \frac{c(t)e^{2x_1(t)}}{\beta^2(t)+e^{2x_1(t)}} - e(t)e^{x_2(t)}, \end{aligned}\right\} t \neq t_k, \\ \left.\begin{aligned} \Delta x_1(t_k) &= \ln(1+c_k), \\ \Delta x_2(t_k) &= \ln(1+d_k), \end{aligned}\right\} t = t_k,\ k = 1, 2, \ldots \end{cases} \tag{2}$$

Let

$$PC(J,\mathbb{R}) = \left\{ \begin{array}{l} x : J \to \mathbb{R} \,|\, x(t) \text{ is continuous with respect to} \\ t \neq t_1, \cdots, t_q; x(t^+) \text{ and } x(t^-) \text{ exist at } t_1, \cdots, t_q; \\ \text{and } x(t_k) = x(t_k^-),\ k = 1, 2, \cdots, q \end{array} \right\}$$

To complete the proof, we only need to search for an appropriate open bounded subset $\Omega \subset X$ verifying all the requirements in Lemma 1.

Note $x = (x_1, x_2)^T$, define

$$X = \left\{ x \in PC(\mathbb{R}, \mathbb{R}^2) : x(t+\omega) = x(t) \right\},$$

$Y = X \times \mathbb{R}^{2q}$, then it is standard to show that both X and Y are Banach space when they are endowed with the norms $\| x \|_c = \sup_{t\in[0,\omega]} | x(t) |$ and

$$\| (x, c_1, \cdots, c_q) \| = (\| x \|_c^2 + | c_1 |^2 + \cdots + | c_q |^2)^{1/2}.$$

Let

$$\mathrm{Dom}\, L = \{ x \in C^1[0,\omega; t_1, \cdots, t_q] \,|\, x(0) = x(\omega) \},$$

$$L : \mathrm{Dom}\, L \to Y,\ Lx = (x', \Delta x(t_1), \cdots, \Delta x(t_q)),\ \ N : X \to Y,$$

$$Nx = N\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \left(\begin{pmatrix} r(t) - a(t)e^{x_1(t)} - \dfrac{b(t)e^{x_1(t)+x_2(t)}}{\beta^2(t)+e^{2x_1(t)}} \\ -d(t) + \dfrac{c(t)e^{2x_1(t)}}{\beta^2(t)+e^{2x_1(t)}} - e(t)e^{x_2(t)} \end{pmatrix}, \begin{pmatrix} \ln(1+c_k) \\ \ln(1+d_k) \end{pmatrix}_{k=1}^{q} \right).$$

It is easy to prove that L is a Fredholm operator of index zero. Consider the operator equation

$$Lx = \lambda Nx,\ \ \lambda \in (0,1). \tag{3}$$

Integratinrg (3) over the interval $[0,\omega]$, we obtain

$$\begin{cases} \displaystyle\int_0^\omega \left[r(t) - a(t)e^{x_1(t)} - \frac{b(t)e^{x_1(t)+x_2(t)}}{\beta^2(t)+e^{2x_1(t)}} \right] dt \\ \displaystyle\qquad = -\sum_{k=1}^{q} \ln(1+c_k) = \ln\prod_{k=1}^{q} \frac{1}{1+c_k}, \\ \displaystyle\int_0^\omega \left[-d(t) + \frac{c(t)e^{2x_1(t)}}{\beta^2(t)+e^{2x_1(t)}} - e(t)e^{x_2(t)} \right] dt \\ \displaystyle\qquad = -\sum_{k=1}^{q} \ln(1+d_k) = \ln\prod_{k=1}^{q} \frac{1}{1+d_k}. \end{cases}$$

It follows that

$$\bar{r}\omega - \ln\prod_{k=1}^{q}\frac{1}{1+c_k} = \int_0^\omega a(t)e^{x_1(t)}dt + \int_0^\omega \frac{b(t)e^{x_1(t)+x_2(t)}}{\beta^2(t)+e^{2x_1(t)}}dt \tag{4}$$

$$\bar{d}\omega + \ln\prod_{k=1}^{q}\frac{1}{1+d_k} = \int_0^\omega \frac{c(t)e^{2x_1(t)}}{\beta^2(t)+e^{2x_1(t)}} - \int_0^\omega e(t)e^{x_2(t)}dt. \tag{5}$$

We can derive

$$\int_0^\omega |x_1'(t)| dt \leq 2\bar{r}\omega + \ln\prod_{k=1}^{q}(1+c_k) = C_+, \tag{6}$$

$$\int_0^\omega |x_2'(t)| dt \leq 2\bar{d}\omega + \ln\prod_{k=1}^{q}(1+d_k) = D_+. \tag{7}$$

Since $x_i(t) \in PC([0,\omega], \mathbb{R})$, there exist $\xi_i, \eta_i \in [0,\omega] \cup \{t_1^+, t_2^+, \ldots, t_p^+\}$, such that

$$x_i(\xi_i) = \inf_{0\leq t\leq\omega} |x_i(t)|,\ \ x_i(\eta_i) = \sup_{0\leq t\leq\omega} |x_i(t)|,\ \ i = 1, 2.$$

From (4) and (5), we can see

$$\int_0^\omega a(t)e^{x_1(\xi_1)}dt \le \int_0^\omega a(t)e^{x_1(t)}dt$$
$$\le \bar{r}\omega - \ln\prod_{k=1}^{q}\frac{1}{1+c_k}$$
$$= \bar{r}\omega + \ln\prod_{k=1}^{q}(1+c_k),$$

$$\int_0^\omega e(t)e^{x_2(\xi_2)}dt \le \int_0^\omega e(t)e^{x_2(t)}dt$$
$$\le \int_0^\omega c(t)dt - \bar{d}\omega - \ln\prod_{k=1}^{q}\frac{1}{1+d_k}$$
$$= (\bar{c}-\bar{d})\omega + \ln\prod_{k=1}^{q}(1+d_k),$$

It follows that

$$x_1(\xi_1) \le \ln\frac{\bar{r}\omega + \ln\prod_{k=1}^{q}(1+c_k)}{\bar{a}\omega} \tag{8}$$

$$x_2(\xi_2) \le \ln\frac{(\bar{c}-\bar{d})\omega + \ln\prod_{k=1}^{q}(1+d_k)}{\bar{e}\omega} \tag{9}$$

On the other hand, by (5), we also have

$$\bar{d}\omega + \ln\prod_{k=1}^{q}\frac{1}{1+d_k} \le \int_0^\omega \frac{c(t)e^{2x_1(t)}}{\beta^2(t)+e^{2x_1(t)}}dt$$
$$\le \int_0^\omega \frac{c(t)e^{2x_1(t)}}{\beta^{2l}}dt$$
$$\le \frac{\bar{c}\omega}{\beta^{2l}}e^{2x_1(\eta_1)}$$

It follows that

$$x_1(\eta_1) \ge \frac{1}{2}\ln\frac{\beta^{2l}}{\bar{c}\omega}\left(\bar{d}\omega - \ln\prod_{k=1}^{q}(1+d_k)\right). \tag{10}$$

Then for $\forall t\in[0,\omega]$, by (6) , (8) and (10), we have

$$x_1(t) \le x_1(\xi_1) + \int_0^\omega |x_1'(t)|dt$$
$$\le \ln\frac{\bar{r}\omega + \ln\prod_{k=1}^{q}(1+c_k)}{\bar{a}\omega} + 2\bar{r}\omega + \ln\prod_{k=1}^{q}(1+c_k)$$
$$:= M_1,$$

$$x_1(t) \ge x_1(\eta_1) - \int_0^\omega |x_1'(t)|dt$$
$$\ge \frac{1}{2}\ln\frac{\beta^{2l}}{\bar{c}\omega}\left(\bar{d}\omega - \ln\prod_{k=1}^{q}(1+d_k)\right) - 2\bar{r}\omega - \ln\prod_{k=1}^{q}(1+c_k)$$
$$:= m_1,$$

then we can derive

$$|x_1(t)| \le \max\{m_1, M_1\} := A_1.$$

From (5), we also have

$$\bar{e}e^{x_2(\eta_2)} \ge \int_0^\omega e(t)e^{x_2(t)}dt$$
$$= \int_0^\omega \frac{c(t)e^{2x_1(t)}}{\beta^2(t)+e^{2x_1(t)}}dt - \bar{d}\omega + \ln\prod_{k=1}^{q}(1+d_k)$$
$$\ge \frac{c^l e^{2m_1}}{\beta^{2u}+e^{2m_1}}\omega + \ln\prod_{k=1}^{q}(1+d_k) - \bar{d}\omega.$$

It follows that

$$x_2(\eta_2) \ge \ln\frac{1}{\bar{e}}\left(\frac{c^l e^{2m_1}}{\beta^{2u}+e^{2m_1}}\omega + \ln\prod_{k=1}^{q}(1+d_k) - \bar{d}\omega\right)$$
$$:= m_2.$$

So, for $\forall t\in[0,\omega]$, we have

$$x_2(t) \ge x_2(\eta_2) - \int_0^\omega |x_2'(t)|dt$$
$$\ge \ln\frac{1}{\bar{e}}\left(\frac{c^l e^{2m_1}}{\beta^{2u}+e^{2m_1}}\omega + \ln\prod_{k=1}^{q}(1+d_k) - \bar{d}\omega\right)$$
$$-2\bar{d}\omega - \ln\prod_{k=1}^{q}(1+d_k)$$
$$:= m_2$$

$$x_2(t) \le x_2(\xi_2) + \int_0^\omega |x_2'(t)|dt$$
$$\le \ln\frac{(\bar{c}-\bar{d})\omega + \ln\prod_{k=1}^{q}(1+d_k)}{\bar{e}\omega} + 2\bar{d}\omega + \ln\prod_{k=1}^{q}(1+d_k)$$
$$:= M_2,$$

then we can derive

$$|x_2(t)| \le \max\{m_2, M_2\} := A_2.$$

Obviously, A_1, A_2 are independent of λ. Choosing $A > A_1 + A_2$, $\Omega = \{x\in X : \|x\|_c < r\}$, then N is L-compact on $\bar{\Omega}$. So, for $\forall x = (\bar{x}_1, \bar{x}_2)^T \in \partial\Omega\cap\operatorname{Ker}L$, we have $QNx \ne 0$. Let $J: \operatorname{Im}Q \to X$, $(d,0,...,0)\to d$. When $x\in\Omega\cap\operatorname{Ker}L$, in view of the assumptions in Mawhin's continuation theorem [10], one obtains, $\deg\{JQN, \Omega\cap\operatorname{Ker}L, 0\} \ne 0$. By now we have proved that Ω satisfies all the requirements in Mawhin's continuation theorem (Lemma 1). Hence, (2) has at least one ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ in $\operatorname{Dom}L\cap\bar{\Omega}$. Set $y_1^*(t) = \exp\{x_1^*(t)\}$, $y_2^*(t) = \exp\{x_2^*(t)\}$, then $y^*(t) = (y_1^*(t), y_2^*(t))^T$ is an ω-periodic solution of (1). The proof is complete.

III. AN ILLUSTRATIVE EXAMPLE

In system (1), we take

$r(t) = 2 - 0.5\sin t$, $a(t) = 0.2 - 0.1\cos t$,

$b(t) = 0.8 + 0.1\sin t$, $c(t) = 2 + 0.4\sin t$,

$d(t) = 1 - 0.2\cos t$, $e(t) = 0.7 + 0.1\cos t$,

$\beta(t) = 2 - \sin t$.

If $c_k = 0.3$, $d_k = 1.1$, $T = \pi/2$, $q = 4$, then all conditions of Theorem 1 are satisfied, system (1) has a unique 2π-periodic solution with four pulses (see Fig.1-Fig.3, we take $(y_1(0), y_2(0))^T = (1,1)^T$). We find the occurrence of sudden changes in the figures of the time-series and phase portrait. The influence of pulse is obvious.

If $T = 2$, then $c_{k+q} = c_k$, $d_{k+q} = d_k$ is not satisfied. Periodic oscillation of system (1) will be destroyed by impulsive effect. Numeric results show that system (1) has Gui chaotic strange attractor (see Fig. 4) [4-7]. In Fig. 4, we take $(y_1(0), y_2(0))^T = (1,1)^T$. Every solution of system (1) will finally tend to the Gui chaotic strange attractor.

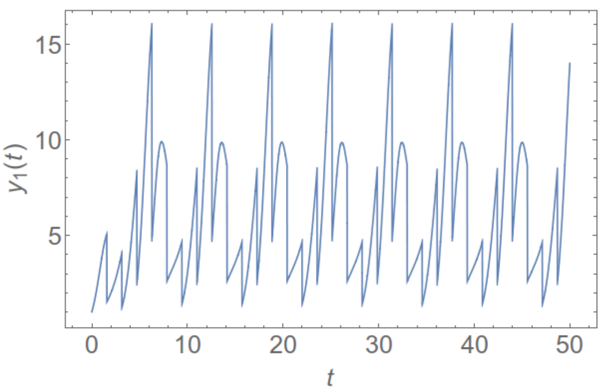

FIGURE I. TIME-SERIES OF $y_1(t)$ EVOLVED IN SYSTEM (1)

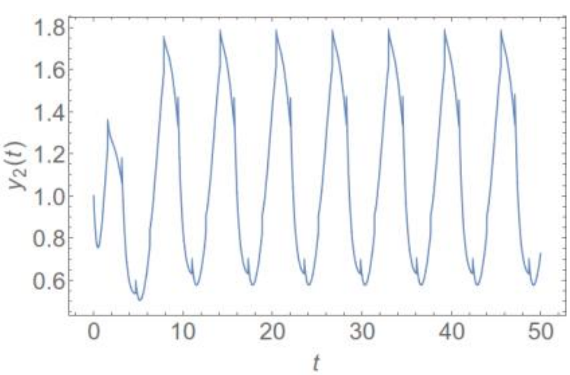

FIGURE II. TIME-SERIES OF $y_2(t)$ EVOLVED IN SYSTEM (1)

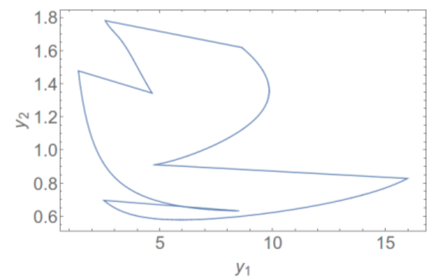

FIGURE III. PHASE PORTRAIT OF PERIODIC SOLUTIONS OF SYSTEM (1)

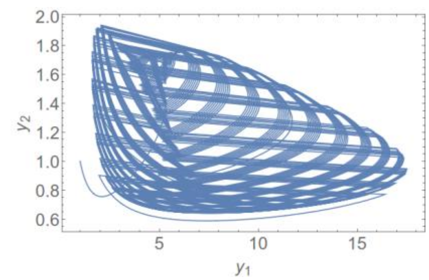

FIGURE IV. PHASE PORTRAIT OF GUI STRANGE ATTRACTOR OF SYSTEM (1)

ACKNOWLEDGMENT

This work is supported jointly by the Natural Sciences Foundation of China under Grant No. 60963025, Natural Sciences Foundation of Hainan Province under Grant No.613166.

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