# Some Connectedness and Related Property of Hyperspace with Vietoris Topology

Meili Zhang*
Dept. of Basic
Dalian Naval Academy
Dalian, China
*Corresponding author

Bo Deng
Dept. of Basic
Dalian Institute of science and technology
Dalian, China

Yue Yang
Dept. of Basic
Dalian Naval Academy
Dalian, China

Pilin Che
Dept. of math
Dalian 44 middle school
Dalian, China

***Abstract*—For a Hausdorff space $X$, we denote by $2^X$ the collection of all closed subsets of $X$. In this paper, we discuss the connectedness and locally connectedness of hyperspace $2^X$ endowed with the vietoris topology. Further path connectedness is investigated. The results generalize some theorems of E. Micheal.**

***Keywords-connectedness; locally connectedness; path connectedness; vietoris topology; hyperspace***

## I. Introduction

There are many different compatible topologies on hyperspace $2^X$. Among these topologies, it is well known that finite topology is an important topology. It is called Vietoris topology.

In 1951, E. Michael [1] made a systematic discussion on hyperspace properties with the finite topology. In this paper, the connectedness and related properties of hyperspace $2^X$ with Vietoris topology are discussed. The results improve some theorems of E. Micheal.

**Definition 1.1** Let $X$ be topology space. By $2^X$ we denote the family of nonempty closed subset of $X$, and then $\{<U>| U\in T\}\cup\{<X,V>| V\in T\}$ is a sub base to a topology $T_V$ in $2^X$.

$T_V$ is called the finite topology in $2^X$ or Vietoris topology.

Obviously, $\{<U_1,U_2\cdots,U_n>| U_i\in T, i\le n, n\in N\}$ is a base of Vietoris topology, where

$$<U_1,U_2\cdots,U_n>=\left\{E\in 2^X \mid E\in\bigcup_{i=1}^{n}U_i, E\cap U_i\neq\varnothing, \forall i\le n\right\}$$

$Z(X)=\{E \mid E\subset X, E \textit{ is a nonempty compact in } X\}$;

For simplicity, we denote by

$_n(X)=\{E\in 2^X \mid E$ has n elements in $X$ *at most*$\}$;

$(X)=\{E\in 2^X \mid E$ has finite elements in $X\}$.

## II. Connectedness of Hyperspace

**Proposition 2.1** Let $X$ be topology space, then $(X)$ is dense in $2^X$.

Proof. For given any $U\in T, U\neq\varnothing$, we have $U$ contains the finite subset n $(X)$, and $(X)=\bigcup_{n=1}^{\infty}$ n $(X)$, thus $<U>\cap(X)\neq\varnothing$. Similarly, suppose $U_1,U_2\cdots,U_n$ are nonempty open sets, $x_k\in U_k,(1\le k\le n)$, then $\{x_1,x_2,\cdots,x_n\}\in<X,U_1>\bigcap<X,U_2>\ \bigcap\cdots<X,U_n>\bigcap$ n$(X)\neq\varnothing$.

**Lemma 1** Let $X$ be topology space, we define a mapping $i: X\to 2^X$, $i(x)=\{x\}$, and then $i$ is continuous mapping.

Proof. Suppose $U\in T, U\neq\varnothing$, then

$$i^{-1}(<U>)=\{x\in X \mid i(x)\in U\}=\{x\in X \mid \{x\}\in U\}=U.$$

If $U_1,U_2\cdots,U_n\in T$, $i^{-1}(<U>)$

$$=\left\{x\in X \mid i(x)\in\bigcap_{i=1}^{n}<X,U_i>\neq\varnothing, 1\le i\le n\right\}$$

$$=\{x\in X \mid x\in U_i, 1\le i\le n\}=\bigcap_{i=1}^{n}U_i.$$

**Proposition 2.2** Let $X$ be topology space, a natural mapping $P_r: X^n\to\mathcal{F}\text{n}(X)$, we define $P_r((x_1,\cdots,x_n))$

 

$=\{x_1,\cdots,x_n\}$, then $P_r$ is continuous mapping.

Proof. For given any $U\in T, U\neq\varnothing$, we have

$$P_r^{-1}(<U>)=\{(x_1,x_2,\cdots,x_n)\mid (x_1,x_2,\cdots,x_n)\in U^n\}$$

$$=U^n,\ P_r^{-1}(<X,U>)=\bigcup_{i=1}^{n}X_1\times X_2\times\cdots\times X_{i-1}\times U\times X_{i+1}\times X_n,$$

where $X_i=X, 1\le i\le n,$ then $P_r$ is a continuous mapping.

**Lemma 2 [2]** Let $X$ be topology space, suppose $A\subset X$ is a closed (or an open)set, then $\{E\in 2^X \mid E\subset A\}$ is a closed(or an open)set in $2^X$.

**Corollary 1** Let $X$ be topology space, suppose $A\subset X$ is a closed set, and then $\{E\in 2^X \mid E\cap A\neq\varnothing\}$ is closed in $2^X$.

Proof. Since $A$ is closed in $X$, $X\setminus A=B$ is open in $X$. By Lemma 2, $\{E\in 2^X \mid E\subset B\}=\{E\in 2^X \mid E\subset X\setminus A\}$

$=\{E\in 2^X \mid E\cap A=\varnothing\}$ is open in $2^X$.

It follows that

$\{E\in 2^X \mid E\cap A\neq\varnothing\}=2^X\setminus\{E\in 2^X \mid E\cap A=\varnothing\}$ is closed in $2^X$.

**Proposition 2.3** $X$ is a connected topology space if and only if $(X)$ is connected.

Proof. Let $P_r: X^n\to$ n $(X)$ be natural mapping, that is, $P_r((x_1,\cdots,x_n))=\{x_1,\cdots,x_n\}$. According to Lemma 1, $P_r$ is a continuous mapping. As X is connected, $X^n$ is connected. So n $(X)$ is connected.

Since $(X)=\bigcup_{n=1}^{\infty}$ n $(X)$, $(X)$ is connected.

**Proposition 2.4** [3] $X$ is a connected topology space if and only if $2^X$ is connected.

Proof. Suppose $X$ is connected, by [1], $X^n$ is connected, $n=1,2,\cdots$. According to Proposition 2.2,

$P_r: X^n\to$ n $(X)$ is a continuous mapping, and $P_r(X^n)=$ n $(X)$, then n $(X)$ is connected, $n=1,2,\cdots$.

$(X)=\bigcup_{n=1}^{\infty}$ n $(X)$ and $\bigcap_{n=1}^{\infty}$ n $(X)$ =1 $(X)\neq\varnothing$, then $(X)$ is connected. Therefore the closure of $(X)$ is connected in $2^X$.

Suppose $2^X$ is connected, and $X=\bigcap_{E\in 2^X}E$ is not connected, there exists nonempty sets $A, B$ which is open and closed sets, such that $X=A\cup B$ and $A\cap B=\varnothing$, hence $2^X=\{E\in 2^X \mid E\cap A\neq\varnothing\}\cup\{E\in 2^X \mid E\cap A=\varnothing\}$.

Since $A$ is closed, $\{E\in 2^X \mid E\cap A\neq\varnothing\}$ is closed in $2^X$. Since $B$ is closed, $\{E\in 2^X \mid E\cap A=\varnothing\}$ $=\{E\in 2^X \mid$

$E\subset B\}$ is closed in $2^X$.

Obviously,

$\{E\in 2^X \mid E\cap A\neq\varnothing\}\cap\{E\in 2^X \mid E\cap A=\varnothing\}=\varnothing$, $\{E\in 2^X \mid E\cap A\neq\varnothing\}\neq\varnothing$, $\{E\in 2^X \mid E\cap A=\varnothing\}=\varnothing$, thus $2^X$ is not connected. This is contraction. Therefore $X$ is connected.

**Lemma 3** Suppose $U_i\subset X, i=1,2,\cdots,n$ is connected,

$<U_1,U_2\cdots,U_n>$ is connected.

Proof. Since $U_i, i=1,2,\cdots,n$ is connected, $(U_i)$ is connected (Theorem 4.10 of [1]).

$(U_1)\times(U_2)\times\cdots\times(U_n)$ is connected, and

$(X)\cap <U_1,U_2\cdots,U_n>$ is under continuous mapping image of $(U_1)\times(U_2)\times\cdots\times(U_n)$, then

$(X)\cap<U_1,U_2\cdots,U_n>$ is connected,

$(X)\cap<U_1,U_2\cdots,U_n>\subset<\overline{U}_1,\overline{U}_2,\cdots,\overline{U}_n>$, then $<U_1,U_2\cdots,U_n>$ is connected.

**Proposition 2.5** Suppose $X$ is locally connected topology space, then $2^X$ is locally connected.

Proof. Suppose $X$ is a locally connected topology space and $E\in 2^X$, there exists a neighborhood $\nu$ of $E$ in $2^X$, we can find the connected open sets $U_1,U_2\cdots,U_n\in T$ such that $E\in<U_1,U_2\cdots,U_n>\subset\nu$, hence $2^X$ is locally connected.

Suppose $X$ is locally connected topology space, $x\in U\subset X$, there exists a connected neighborhood $\beta$ of $\{x\}$, such that $\beta\subset<U>$, so $V=\bigcup_{A\in\beta}A$ is a neighborhood of $X$ , $V \subset U$ and $\beta$ are connected. Therefore $\{x\} \in \beta$ and $\{x\}$ are connected.

**Lemma 4** [4] Suppose $\beta$ is an open (closed)in $2^X$ , then $\bigcup_{E \in \beta} E$ is an open(closed)in $X$ .

**Lemma 5** Suppose $U$ is a connected component, $U$ is a connected closed set.

Proof. Since $U$ is a connected component in $X$ , $U$ is connected [1]. $\overline{U}$ is a connected, $U \subset \overline{U}$ , $U$ is a component in $X$ , then $U$ is a maximum connected set, $U = \overline{U}$ , thus $U$ is a connected closed set.

**Lemma 6** Suppose $X$ is a locally connected topology space, then $U$ is a connected component in $X$ and $U$ is an open set.

Proof. Suppose $P \in U$ , $X$ is a locally connected, $P$ belong to an open connected set $G_P$ at least, $U$ is a component which contain $P$ , then $P \in G_P \subset U$ and $U = \cup\{G_P \mid P \in U\}$, thus $U$ is open set as it is the union of open sets.

**Proposition 2.6** Suppose $X$ is a locally connected topology space, $U$ is a connected component in $X$ is and only if $\{E \in 2^X \mid E \subset U\}$ is a connected component in $2^X$ .

Proof. Since $X$ is locally connected, $U$ is a connected component in. By Lemma 5 and Lemma 6, $U$ is an open and closed set in $X$ . According to Lemma 2, it follows that $\{E \in 2^X \mid E \subset U\}$ is an open and closed set in $2^X$ . Hence $\{E \in 2^X \mid E \subset U\}$ is a connected component in $2^X$ .

Suppose $\{E \in 2^X \mid E \subset U\}$ is a connected component in $2^X$ . Since X is locally connected, by Corollary 2, we have $2^X$ is locally connected, so $\{E \in 2^X \mid E \subset U\}$ is an open and closed set in $2^X$ . We have $U \in 2^X$ :

In fact, $U_1 = \bigcup\{E \in 2^X \mid E \subset U\}$ . If $U \neq U_1$ , there exists $x \in U \setminus U_1$ such that $\{x\} \subset U$ and $\{x\} \in 2^X$ , hence

$x \in U_1$, this is contraction. So $U = \bigcup\{E \in 2^X \mid E \subset U\}$.

As $\{E \in 2^X \mid E \subset U\}$ is closed in $2^X$ , by Lemma 3, $U$ is closed in $X$ and $U \in 2^X$ .

Since $\{E \in 2^X \mid E \subset U\}$ is an open and closed set in $2^X$ , by Lemma 3, it follows that $U$ is an open and closed set in $X$ , hence $U$ is a connected component in $X$ .

**Lemma 7** [4] Let $X, Y$ be topology space and $X$ is path connected, $f: X \to Y$ is continuous mapping, then $f(X)$ is path connected.

**Proposition 2.7** Let $X$ be topology space, then $(X)$ is path connected.

Proof. Since $X$ is path connected, $X^n$ is path connected. We define a natural mapping $P_r: X^n \to$ n $(X)$ . By Proposition 2.2, $P_r$ is a continuous mapping.

By Lemma 7, n $(X) = P(X^n)$ [5] is path connected, we have 1⊂2⊂⋯⊂n⊂⋯.

As $(X) = \bigcup_{n=1}^{\infty}$ n $(X)$ , $\forall E_1, E_2 \in$ $(X)$ , there exists $n, m \in N$ , $E_1 \in$ n, $E_2 \in$ m.

Assume $n \leq m$ , $E_1 \in$ n⊂ m. Since m is path connected [6]-[9], there exists a path from $E_1$ to $E_2$ , then $X$ is path connected.

ACKNOWLEDGMENT

This work was supported by the Dalian Naval Academy for Basic Research.

REFERENCES

[1] E. Michael, "Topologies on spaces of subsets", Trans. Amer. Math. Soc , vol. 71, no. 1, pp. 152-182, 1951.

[2] J. C. Janusz, "Retractions and contractibility in hyperspaces", Topology and its Applications, no. 154, pp. 333-338, 2007.

[3] Meili Zhang, "Connectivity of hyperspace", Journal of Liaoning Technical University, no. 26, pp. 331-333, 2007.

[4] R. Engelking, "General Topology", Warszawa: RWN-polish Scientific Publishers, 1997.

[5] Meili Zhang, "Connectivity of hyperspace with Vietoris-topology", Journal of Wuhan University of Science and Technology, no. 31, pp.145-146, 2008.

[6] C. Costantini, Ĺ. Holá, P. Vitolo, "Tightness, character and related properties of hyperspace topologies", Topology Appl., no. 142, pp. 245–292, 2004.

[7] R. Lowen, P. Wuyts, "The Vietoris hyperspace structure for approach spaces", Acta Mathematica Hungarica, no. 139, pp. 286-302,2013.

[8] N.C. Esty, "CL(R) is simply connected under the Vietoris topology", Applied general topology, no. 8, pp. 259-265, 2007.

[9] V. Gutev, "Baire spaces and Vietoris hyperspaces", Proc. Amer. Math. Soc., no. 135,pp. 299-303,2007.