

# Some Notes on Abundant Quasi-Ideals of Ample Semigroups

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**Abstract.** The aim of this paper is to study ample semigroups by using natural partial orders on abundant semigroups. After giving some properties and characterizations of abundant quasi-ideals on ample semigroups, we prove that there exists an idempotent-separating good congruence  $\rho$  on an abundant quasi-ideal  $M$  of an arbitrary ample semigroup  $S$ , which can be extended to an idempotent-separating good congruence on  $S$ . In this paper, we develop an approach to the structure of ample semigroups inspired by the semigroup algebra theory of Mario Petrich for inverse semigroups.

## Introduction

The concepts of the natural partial orders on an abundant semigroup were introduced by Lawson [5] in 1987. After that, various classes of abundant semigroups are researched (see, [4, 7]) by considering the natural partial orders on abundant semigroup. In this paper, we shall study the IC-abundant semigroups by considering their natural partial orders and (bi-)order ideals. In particular, we will consider abundant quasi-ideals on ample semigroups. Some properties on IC-abundant semigroups and ample semigroups will be described in terms of their natural partial orders and their abundant quasi-ideals. We shall proceed as follows: section 2 provides some known results. In section 3, we give some characterizations of the abundant quasi-ideals on IC-abundant semigroups (e.g., IC-quasi-adequate semigroups). The last section we consider the abundant quasi-ideals and idempotent-separating good congruence extensions on ample semigroups.

## Preliminaries

Here we provide some known results used repeatedly in the sequel. At first, we recall some basic facts about the relation  $L^*$  and  $R^*$ .

**Lemma II.1**<sup>[1]</sup> Let  $S$  be a semigroup and  $a, b \in S$ . Then the following statements are equivalent: (1)  $aL^*b$  ( $aR^*b$ ); (2) for all  $x, y \in S^1$   $ax = ay$  ( $xa = ya$ ) if and only if  $bx = by$  ( $xb = yb$ ).

**Corollary II.1**<sup>[1]</sup> Let  $S$  be a semigroup and  $a, e = e^2 \in S$ . Then  $aL^*e$  ( $aR^*e$ ) if and only if  $a = ae$  ( $a = ea$ ) and for all  $x, y \in S^1$   $ax = ay$  ( $xa = ya$ ) implies  $bx = by$  ( $xb = yb$ ).

Evidently,  $L^*$  is a right congruence while  $R^*$  is a left congruence. For convenience, we denote by  $a^+ [a^*]$  a typical idempotent  $R^*$ -related [ $L^*$ -related] to  $a$ .  $L_a^*$  ( $R_a^*$ ) denotes the  $L^*$  class ( $R^*$  class) containing  $a$ . And  $E(T)$  denotes the set of idempotents of  $T$ ;  $Reg(T)$  denotes the set of regular elements of  $T$ . We denote by  $V(a)$  the set of all inverses of  $a$ . A semigroup  $S$  is called *abundant*<sup>[3]</sup> if and only if each  $L^*$  class and each  $R^*$  class contains at least one idempotent. An abundant semigroup  $S$  is called *quasi-adequate*<sup>[2]</sup> if its set of idempotents constitutes a subsemigroup (i.e., its set of idempotents is a band). Moreover, a quasi-adequate semigroup is called *adequate*<sup>[6]</sup> if its bands of idempotents is a semilattice (i.e., the idempotents commute). An abundant semigroup  $S$  is called *ample*<sup>[6]</sup>, if for all  $e^2 = e, a \in S$ ,  $ae = (ae)^+a$  and  $ea = a(ea)^*$ . Following [1], an abundant semigroup  $S$  is called *idempotent-connected*, for short, *IC*, provided for each  $a \in S$  and for some  $a^+, a^*$ , there exists a bijection  $\theta: \langle a^+ \rangle \rightarrow \langle a^* \rangle$  such that  $xa = a(x\theta)$  for all  $x \in \langle a^+ \rangle$ , where  $\langle a^+ \rangle$  [resp.  $\langle a^* \rangle$ ] is the subsemigroup of  $S$  generated by the set  $\{y \in E(S) : y = ya^+ = a^+y\}$  [resp.  $\{y \in E(S) : y = ya^* = a^*y\}$ ]. In fact, an ample semigroup is an IC-adequate semigroup and vice versa. Evidently, for an arbitrary adequate semigroup  $S$ , its each  $L^*$  class and each  $R^*$  class contains exactly one idempotent (i.e.,  $S$  is both  $L^*$ - and  $R^*$ -unipotent).

**Definition II.1**<sup>[5]</sup> Let  $S$  be an abundant semigroup. We define three relations on  $S$ , as follows:

- (1)  $a \leq_r b \Leftrightarrow R_a^* \leq R_b^*$ , and there exists an idempotent  $f \in R_a^*$  such that  $a = fb$ ;
- (2)  $a \leq_l b \Leftrightarrow L_a^* \leq L_b^*$ , and there exists an idempotent  $e \in L_a^*$  such that  $a = be$ ;
- (3)  $\leq = \leq_r \cap \leq_l$ , i.e.  $a \leq b \Leftrightarrow$  there exist idempotents  $e, f$  such that  $a = eb = bf$ .

**Lemma II .2** <sup>[5]</sup> Let  $S$  be an abundant semigroup and  $a, b \in S$ . Then  $a \leq_r b$  if and only if there exists  $e \in R_a^* \cap E(S), f \in R_b^* \cap E(S)$  such that  $a = eb$  and  $e \leq f$ .

**Lemma II .3** <sup>[5]</sup> Let  $S$  be an abundant semigroup. Then  $S$  is IC if and only if  $\leq_r = \leq = \leq_l$ .

**Lemma II .4** <sup>[6]</sup> Let  $S$  be an ample semigroup,  $a, b \in S$  and  $a \leq b$ . Then for all  $c \in S$   $ac \leq bc, ca \leq cb, a^+ \leq b^+$  and  $a^* \leq b^*$ .

We recall from [1] that a semigroup homomorphism  $\Phi : S \rightarrow T$  is a *good homomorphism*, if for all elements  $a, b$  of  $S$ ,  $aL^*(S)b$  implies  $a\Phi L^*(T)b\Phi$  and  $aR^*(S)b$  implies  $a\Phi R^*(T)b\Phi$ . We say that a congruence  $\rho$  on a semigroup  $S$  is a *good congruence* if the natural homomorphism from  $S$  onto  $S/\rho$  is good. As defined in [9], a congruence  $\rho$  on a semigroup  $S$  is an *idempotent-separating congruence*, if for all  $e, f \in E(S)$  the equality  $e\rho = f\rho$  implies that  $e = f$ . From [1], we quote

**Lemma II .5** <sup>[1]</sup> Let  $\rho$  be a congruence on an abundant semigroup  $S$  and  $a \in S$ . Then  $\rho$  is good if and only if  $a\rho L^*(S/\rho)a^*\rho$  and  $a\rho R^*(S/\rho)a^+\rho$ .

**Lemma II .6** <sup>[4]</sup> Let  $\rho$  be a good congruence on a quasi-adequate semigroup  $S$ . Then  $\rho$  is idempotent-separating if and only if  $\rho \subseteq H^*$ .

Let  $M$  be a subsemigroup of a semigroup  $S$ . We say that  $M$  is a *quasi-ideal* of  $S$  if  $MSM \subseteq M$ . A quasi-ideal  $M$  is called *abundant*, if  $M$  is an abundant semigroup. For an arbitrary semigroup  $S$ , a subset  $F$  of the set of idempotents  $E(S)$  is called *order ideal* of  $E(S)$ , if for all  $f \in F, e \in E(S), e \leq f \Rightarrow e \in F$ . In particular, if  $\forall e, f \in F, S(e, f) \neq \emptyset$ , then we call  $F$  a *bi-order ideal* of  $E(S)$ , where  $S(e, f) = \{g \in E(S) : ef = egf, ge = g = fg\}$ .

### Properties of abundant quasi-ideals on IC-quasi-adequate semigroups

The aim of this section is to give some properties and characterizations of the natural partials on IC-quasi-adequate semigroups.

**Proposition III. 1** Let  $S$  be an IC quasi-adequate semigroup and  $Q$  a quasi-ideal of  $S$ .  $AQ$  denotes the set of abundant elements of  $S$ . Then the following statements are true:

- (1)  $E(Q)$  is a bi-order ideal of  $E(S)$ ;
- (2)  $AQ$  is an order ideal of  $S$ ;
- (3)  $AQ$  is an abundant quasi-ideal of  $S$ ;
- (4)  $AQ$  is an IC-quasi-adequate semigroup.

**Proof.** (1) Let  $e \in E(Q), f \in E(S)$  such that  $f \leq e$ . Then  $f = ef = fe = efe \in QSQ \subseteq Q$ . Hence  $f \in E(Q)$ . Suppose that  $f, g \in E(Q)$ . Since  $S$  is a quasi-adequate semigroup, we have that  $fg$  is an idempotent element of  $S$ . Let  $x$  be an inverse element of  $fg$ . We can see that  $gxf \in S(e, f) \cap E(Q)$ .

Therefore,  $E(Q)$  is a bi-order ideal of  $E(S)$ .

(2) We first prove that  $AQ$  is an abundant subsemigroup of  $S$ . To see it, let  $a, b \in AQ$ . Then there are  $a^+, b^* \in E(Q)$  such that  $a R^*(Q) a^+$  and  $b L^*(Q) b^*$ . Note that  $ab L^*(S) (ab)^*$ . Again since  $L^*$  is a right congruence, we have  $ab L^*(S) (ab)^* b^*$ . Obviously,  $(ab)^* b^* \in E(S), (ab)^* b^* \leq_l b^*$ . Note that  $S$  is IC. By Lemma II. 3, we have that  $(ab)^* b^* \leq b^*$ . Thus  $(ab)^* b^* \in E(Q)$ . Therefore,  $ab L^*(Q) (ab)^* b^*$ . Similarly,  $ab R^*(Q) a^+(ab)^+$ . This means that  $AQ$  is an abundant subsemigroup of  $S$ .

Next, we prove that  $AQ$  is an order ideal of  $S$ . To see it, let  $x \in S, a \in AQ$  such that  $x \leq a$ . Then  $x = ea = af$ , where  $e \in E(R_x^*), f \in E(L_x^*)$ . Hence  $x = a^+ea$ . Again since  $Q$  is a quasi-ideal of  $S$ , we have  $x \in Q$ . On the other hand, since  $R^*$  is a left congruence, we get that  $x = a^+x R^*(S) a^+e$ . Obviously,  $a^+e \leq_r a^+$ . Note that  $S$  is IC. By Lemma II. 3, we have that  $a^+e \leq a^+$ . By (1), we have  $a^+e \in E(Q)$  and  $x R^*(Q) a^+e$ . Similarly,  $x L^*(Q) fa^*$ , where  $fa^* \in E(Q)$ . Therefore,  $x \in AQ$ . That is,  $AQ$  is an order ideal of  $S$ .

(3) Let  $a, b \in AQ, x \in S$ . Then  $axb \in Q$ . On the other hand,  $axb L^*(S) (axb)^*$ . Since  $L^*$  is a right congruence, we have that  $axb L^*(S) (axb)^* b^*$ . Obviously,  $(axb)^* b^* \leq_l b^*$ . Note that  $S$  is IC. By Lemma II. 3, we get  $(axb)^* b^* \leq b^*$ . By (1),  $(axb)^* b^* \in E(Q)$ . Therefore,  $axb L^*(Q) (axb)^* b^*$ .

Similarly,  $axb R^*(Q) a^+(axb)^+$  and  $a^+(axb)^+ \in E(Q)$ . Therefore,  $axb \in AQ$ . This means that  $AQ$  is an abundant quasi-ideal of  $S$ .

(4) It follows from (3) and the definition of IC.

**Theorem III.1** Let  $S$  be an IC-quasi-adequate semigroup and  $F$  be a bi-order ideal of  $E(S)$ . Put  $M = \{a \in S : a^* \in F \cap L_a^*, a^+ \in F \cap R_a^*\}$ . Then  $M$  is an abundant quasi-ideal of  $S$  with  $F = E(M)$ , and  $M = \bigcup \{uSv : u \in F, v \in F\}$ .

**Proof.** Let  $F$  be a bi-order ideal of  $E(S)$ . Then for all  $a, b \in M, x \in S$ ,  $axb L^*(S) (axb)^*$ . Since  $L^*$  is a right congruence, we have that  $axb L^*(S) (axb)^* b^*$ . Obviously,  $(axb)^* b^* \leq_l b^*$ . Note that  $S$  is IC. By Lemma II. 3, we have that  $(axb)^* b^* \leq b^*$ , and so  $(axb)^* b^* \in F$ . Similarly, we can prove that  $axb R^*(S) a^+(axb)^+$ , and  $a^+, a^+(axb)^+ \in F$ . Therefore,  $axb \in M$ . That is,  $MSM \subseteq M$ . Obviously,  $M$  is abundant. This means that  $M$  is an abundant quasi-ideal of  $S$ .

Next, we prove that  $F = E(M)$ . It is clear that  $F \subseteq E(M)$ . We only need to prove  $F \supseteq E(M)$ . To see it, let  $e \in E(M)$ . Then there exist idempotents  $e^*, e^+ \in F$  such that  $e^+ R^*(M) e L^*(M) e^*$ . But  $S$  is an ample semigroup. We have  $e = e^+ = e^* \in F$ . This gives that  $F = E(M)$ .

Now, we prove that  $M = \bigcup \{uSv : u \in F, v \in F\}$ . We only need to prove inverse inclusion. Let  $a \in uSv$ , where  $u, v \in F$ . Then there is  $x \in S$  such that  $a = uxv$ . Note that  $ux L^*(S) (ux)^*$ . We have that  $ux v L^*(S) (ux)^* v$  since  $L^*$  is a right congruence. Obviously,  $(ux)^* v \leq_l v$ . Again since  $S$  is IC, by Lemma II. 3,  $(ux)^* v \leq v$ , and so  $(ux)^* v \in F$ . Similarly,  $ux v R^*(S) u(xv)^+ \in F$ . Therefore,  $uxv \in M$ . This completes the proof.

### Abundant quasi-ideals on an ample semigroup

In this section, we will consider abundant quasi-ideals on ample semigroups.

Let  $R$  be a relation on a semigroup  $S$ ,  $a, b \in S$  and  $N$  denote the set of all non-negative integers. We define a relation as follows:

$$(a, b) \in R^\# \Leftrightarrow a = b \text{ or } a = z_1 \rightarrow z_2 \rightarrow \cdots \rightarrow z_{n-1} \rightarrow z_n = b \quad (n \in N)$$

Where  $z_i \rightarrow z_{i+1}$  denotes  $z_i = u_i a_i v_i, z_{i+1} = u_i b_i v_i$ , and  $(a_i, b_i) \in \rho$  ( $1 \leq i \leq n$ ). We call the above relation a Sequence of elementary  $R$ -transitions connecting  $a$  to  $b$  [8].

**Theorem IV.1** Let  $S$  be an ample semigroup. Then the following statements are true:

- (1) for all  $e, f \in E(S)$ ,  $eSf$  is a quasi-ideal of  $S$ ;
- (2) if  $eSf$  is an abundant quasi-ideal of  $S$  for all  $e, f \in E(S)$ , then  $eSf$  is an ample semigroup, and  $a^* = a^* e, a^+ = f a^+$  for all  $a \in eSf$ ;
- (3) if  $eSf$  is an abundant quasi-ideal of  $S$  for all  $e, f \in E(S)$ , then a congruence  $\rho$  on  $eSf$  can be extended to a congruence on  $S$ .

**Proof.** (1) This is trivial. (2) Let  $eSf$  be an abundant quasi-ideal of  $S$  for all  $e, f \in E(S)$ . Then, by Proposition III.1, it is clear that  $eSf$  is ample. Let  $a \in eSf$ . Then  $a L^*(eSf) a^*$ , and  $a R^*(eSf) a^+$ . Hence  $a^* = e a^* = a^* e, a^+ = a^+ f = f a^+$ .

(3) Let  $eSf$  be an abundant quasi-ideal of  $S$  for all  $e, f \in E(S)$  and let  $\rho$  be a congruence on  $eSf$ . Denote  $\rho^\#$  a congruence on  $S$  generated by  $\rho$ . We only need to prove  $\rho = \rho^\# \cap eSf \times eSf$ . Obviously,  $\rho \subseteq \rho^\# \cap eSf \times eSf$ . Conversely, let  $(x, y) \in \rho^\# \cap eSf \times eSf$ . Then either  $x = y$  or for some  $n$  in  $N$  there is a sequence  $x = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n = y$  such that  $x_i = u_i a_i v_i, x_{i+1} = u_i b_i v_i$ , where  $u_i, v_i \in S^1, (a_i, b_i) \in \rho$  ( $1 \leq i \leq n$ ). Again since  $x = exf, y = eyf$ , we have

$$x = ex_1 f \rightarrow ex_2 f \rightarrow \cdots \rightarrow ex_n f = y$$

By (2),  $ex_i f = eu_i a_i v_i f = eu_i a_i^+ a_i a_i^* v_i f = eu_i f a_i^+ a_i a_i^* ev_i f = (eu_i f) a_i (ev_i f)$ . Similarly, we have  $ex_{i+1} f = eu_i b_i v_i f = (eu_i f) b_i (ev_i f)$ . But  $(a_i, b_i) \in \rho$  and  $eu_i f, ev_i f \in eSf$ . Thus  $(ex_i f, ex_{i+1} f) \in \rho$ ,  $(1 \leq i \leq n)$ . Therefore,  $(x, y) \in \rho$ . The proof is completed.

**Theorem IV.2** Let  $M$  be an abundant quasi-ideal on an ample semigroup  $S$  and  $\rho$  an idempotent-separating good congruence on  $M$ . Then the following statements are true:

- (1)  $a\rho b \Rightarrow a^* = b^*, a^+ = b^+ (\forall a, b \in M)$ ;
- (2) if  $\rho = 1_M$ , then  $\rho^\#$  is an idempotent-separating good congruence on  $S$ ;
- (3) if  $\rho^\#$  is an idempotent-separating good congruence on  $S$ , then  $\rho = \rho^\# \cap M \times M$ .

**Proof.** (1) Note that  $S$  is an ample semigroup. By Proposition III.1,  $M$  is an ample subsemigroup of  $S$ . Let  $a, b \in M$  such that  $a\rho b$ . Then, by Lemma II.6,  $\rho \subseteq H^*(M) \subseteq L^*(M)$ . Hence  $a L^*(M) b$ , and so  $a^* = b^*$ . Similarly,  $a^+ = b^+$ . (2) It is clear. (3) Firstly, we note that  $(s, t) \in \rho^\#$  for all  $s, t \in S$  if and only if either  $s = t$  or there exists a sequence  $s = s_1 \rightarrow s_2 \rightarrow \dots \rightarrow s_n = t$  ( $n \in \mathbb{N}$ ), where  $s_i \rightarrow s_{i+1}$  means that  $s_i = u_i a_i v_i, s_{i+1} = u_i b_i v_i$  and  $(a_i, b_i) \in \rho$  ( $1 \leq i \leq n$ ). Suppose that  $s_i \in M$ , we prove that  $s_{i+1} \in M$  and  $s_i, s_{i+1}$  are  $\rho$  related. Since  $\rho$  and  $\rho^\#$  are idempotent-separating good congruences on  $M$  and  $(a_i, b_i) \in \rho$ , we have that  $s_{i+1}^* = s_i^*, s_{i+1}^+ = s_i^+$ ,  $a_i^* = b_i^*$  and  $a_i^+ = b_i^+$ . Hence  $s_{i+1} = s_i^+ s_{i+1} s_i^* = s_i^+ u_i a_i^+ b_i a_i^* v_i s_i^* = (s_i^+ u_i a_i^+) b_i (a_i^* v_i s_i^*)$ . Since  $M$  is an abundant quasi-ideal of  $S$ , we have that  $w_i = (s_i^+ u_i a_i^+) \in MSM \subseteq M$ ,  $z_i = (a_i^* v_i s_i^*) \in MSM \subseteq M$ . Thus  $s_{i+1} \in M$ . But  $(a_i, b_i) \in \rho$ , we have  $(w_i a_i z_i, w_i b_i z_i) \in \rho$ . That is  $(s_i, s_{i+1}) \in \rho$ . Therefore, if  $s \in M$  and  $(s, t) \in \rho^\#$ , then  $t \in M$  and  $(s, t) \in \rho$ . That is,  $\rho^\# \cap M \times M \subseteq \rho$ . The reverse containment is clear. Therefore,  $\rho = \rho^\# \cap M \times M$ .

## Conclusions

In this paper, we investigate abundant quasi-ideals on IC-abundant semigroups and give some properties and characterizations of abundant quasi-ideals on ample semigroups by using the natural partial orders on abundant semigroup.

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